MINIMAL SURFACES WITH PLANAR BOUNDARY CURVES

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1. Introduction

In 1956, Shiffman [Sh] proved that any compact minimal annulus with two convex boundary curves (respectively circles) in parallel planes is foliated by convex planar curves (respectively circles) in the intermediate planes. In 1978, Meeks [M] conjectured that the assumption the minimal surface is an annulus is unnecessary; that is, he conjectured that any compact connected minimal surface with two planar convex boundary curves in parallel planes must be an annulus.

Partial results have been proven in the direction of this conjecture. Schoen [Sc1] proved the Meeks conjecture in the case where the two boundary curves share a pair of reflectional symmetries in planes perpendicular to the planes containing the boundary curves. Another interesting result related to the Meeks conjecture was proved by Meeks and White [MW2, Theorem 1.2]. Recall that a minimal surface M is called *stable* if, with respect to any normal variation that vanishes on ∂M , the second derivative of the area functional is positive. The minimal surface is *unstable* if there exists such a variation with negative second derivative for the area functional, and it is *almost-stable* if the second derivative is nonnegative for all such variations and is zero for some nontrivial variation. Also recall that a subset of $\mathbb{R}^3$ is called *extremal* if it is contained in the boundary of its convex hull. The result of Meeks and White is that if Γ is an extremal pair of smooth disjoint convex curves in distinct planes, then exactly one of the following holds:

- (1) Γ is not the boundary of any connected compact minimal surface, with or without branch points.
- (2) Γ is the boundary of exactly one minimal annulus and this annulus is almost-stable. In this case, Γ bounds no other connected branched minimal surface.
- (3) Γ is the boundary of exactly two minimal annuli, one stable and one unstable.

Other partial results toward the Meeks conjecture have been proven by Meeks and White for stable surfaces [MW1, MW2]. They proved the conjecture for stable and almost-stable minimal surfaces that have two convex boundary curves lying in

parallel planes such that:

- (1) the two boundary curves have a common plane of reflective symmetry perpendicular to the planes containing them; or
- (2) the two boundary curves are reflected into each other by a plane parallel to the planes containing them.

The first of these two conditions has been extended by Meeks and White to boundary curves lying in nonparallel planes, but still forming an extremal set. The second of these two conditions is generalized to nonparallel planes by Theorem 2.1 in the next section.

In Section 3 we consider a more general setting: compact connected minimal surfaces, with a pair of boundary curves (not necessarily convex) in distinct planes, that have least-area amongst all orientable surfaces with the same boundary. When the planes containing these two boundary curves are either parallel or ‘sufficiently close’ to parallel, and when the boundary curves themselves are ‘sufficiently close’ to each other, one can draw specific conclusions about the geometry and topology of the surfaces, as in Theorems 3.1 and 3.2. (These theorems are formal statements about surfaces that can be physically realized by experimentation with wire frames and soap films.)

In the final section we state two well-known results on the existence of compact minimal surfaces with planar boundary curves in parallel planes. These results are easily proved using the maximum principle for minimal surfaces, but do not appear elsewhere in the literature. A corollary of these results is a generalization of a result of Nitsche [N1]: let M be any compact minimal annulus with two planar boundary curves of diameters d_1 and d_2 in parallel planes P_1 and P_2 ; if the distance between P_1 and P_2 is h , then the inequality $h \leq \frac{3}{2} \max\{d_1, d_2\}$ is satisfied. The corollary we prove here does not assume that the minimal surface M is an annulus and has the strengthened conclusion $h \leq \max\{d_1, d_2\}$. We also include a similar result for nonminimal constant mean curvature surfaces.

2. Topological uniqueness in the stable case

Meeks and White [MW1] proved the following, which we generalize in Theorem 2.1.

Let α be a smooth convex plane curve in $\mathbb{R}^3$, let P_0 be a plane parallel to the plane containing α , and let $\text{Ref} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a reflection in the plane P_0 . If Σ is a connected stable or almost-stable compact minimal surface with boundary $\alpha \cup \text{Ref}(\alpha)$, then Σ is an embedded annulus.

Theorem 2.1 is similar to the above result, but we assume α does not lie in a plane parallel to P_0 . We assume only that α lies on one side of P_0 , thus α and $\text{Ref}(\alpha)$ do not lie in parallel planes. In order to state this precisely, we introduce some notation.

Consider all closed half-planes in $\mathbb{R}^3$ with boundary the x_1 -axis. Let $\omega(0)$ be the half-plane containing the positive x_2 -axis, and let $\omega(\theta)$ be the half-plane making an angle of θ with $\omega(0)$. (Thus the half-plane containing the positive x_3 -axis is $\omega(\pi/2)$.) Let R_θ be a rotation of angle θ about the x_1 -axis.

Definition 2.1. The wedge between $\omega(\gamma)$ and $\omega(\beta)$ of angle $\beta - \gamma$ in $\mathbb{R}^3$ is $W_{\gamma,\beta} = \cup_{(\gamma \leq \alpha \leq \beta)} \omega(\alpha)$.

THEOREM 2.1. *If $M \subseteq W_{\gamma,\beta}$ is a stable or almost-stable compact connected minimal surface with boundary $\partial M = C \cup R_{\beta-\gamma}(C)$, where C is a strictly convex curve contained in $\omega(\gamma)$, then M is an embedded annulus.*

We now describe the natural free boundary problem for a wedge. Suppose C is a strictly convex Jordan curve in $\omega(\beta)$ and M is a compact minimal surface such that the boundary ∂M consists of C and a nonempty collection of immersed curves in the x_1x_2 -plane. We may assume that $0 \leq \beta \leq \pi/2$, for if not, by reflections through the x_1x_3 -plane and x_1x_2 -plane, we have a congruent problem where this is so. If M is orthogonal to the x_1x_2 -plane along $\partial M \cap \omega(0)$, then M is called a solution of the free boundary value problem for C and $W_{0,\beta}$. By the maximum principle (see, for example, [Sc1]), if M is such a solution, then the portion of ∂M lying in the x_1x_2 -plane actually lies in $\omega(0)$. If, with respect to any normal variation of M that vanishes on C , the second derivative of the area functional is positive, then M is a *stable* solution to the free boundary value problem. Similarly, we can define when M is *unstable* or *almost-stable* [MW1, MW2].

COROLLARY 2.1. *Suppose that M is a stable or almost-stable solution to the free boundary value problem for C and $W_{0,\beta}$; then M is an embedded annulus.*

The proof of Theorem 2.1 requires the use of the Jacobi operator and Jacobi fields. Following the notation of Choe [C], three types of Killing vector fields in $\mathbb{R}^3$ are ϕ_n , ϕ_l , and ϕ_p , where these three vector fields are the variation vector fields produced by translating in the direction of the unit vector n , rotating around a straight line l , and homothetically expanding from a point p , respectively. Meeks and White [MW1] were interested in ϕ_n , and here we are interested in ϕ_p .

Let S be a smooth immersed surface in $\mathbb{R}^3$. The *horizon* of S with respect to ϕ_p , denoted by $H(S; \phi_p)$, is the set of all points of S at which ϕ_p is a tangent vector of S . A connected subset D of S is called a *visible* set with respect to ϕ_p if D is disjoint from $H(S; \phi_p)$. If M is a minimal surface in $\mathbb{R}^3$, then $\phi_n(M^\perp)$, $\phi_l(M^\perp)$, and $\phi_p(M^\perp)$ (the projection of ϕ_n , ϕ_l , and ϕ_p onto the normal bundle of M , respectively) are Jacobi fields [C]. If M has a strictly proper open connected subset D such that $\partial D \subseteq H(M; \phi_p)$, then it follows from the Smale index theorem (see, for example, [C, p. 199]) that D is almost-stable or unstable, and hence M is unstable. We use this fact in the following proof.

Proof of Theorem 2.1. By a rotation of $\mathbb{R}^3$ if necessary, we may assume that $M \subseteq W_{-\beta, \beta}$, with $\partial M = C \cup R_{-2\beta}(C)$, $C \subseteq \omega(\beta)$, $R_{-2\beta}(C) \subseteq \omega(-\beta)$, and $0 < \beta \leq \pi/2$. In fact $\beta < \pi/2$, since otherwise (by the maximum principle) M would be a pair of disks, which are not connected. The orthogonal projection Λ from $\mathbb{R}^3$ to the x_1x_2 -plane maps C and $R_{-2\beta}(C)$ to a strictly convex smooth Jordan curve $\Lambda(C)$ in $\omega(0)$. By Theorem 2 of [Sc1], M is embedded and $M = \{(x_1, x_2, \pm u(x_1, x_2)) \mid (x_1, x_2) \in \Omega\}$, where Ω is a compact region in $\omega(0)$, and $u(x_1, x_2)$ is a nonnegative function defined on Ω . Furthermore, since $M_0^+ = M \cap \{x_3 \geq 0\}$ and $M_0^- = M \cap \{x_3 \leq 0\}$ have locally bounded slope in their interiors [Sc1], and since the maximum principle immediately implies that the tangent planes of M along C and $R_{-2\beta}(C)$ can never be vertical, the normal vectors of M are horizontal only on $M \cap \omega(0)$.

Assume M is not an annulus. Then $\partial\Omega = \Lambda(C) \cup C_1 \cup \dots \cup C_k$, where $k > 1$, and C_i is a curve in the compact region of $\omega(0)$ bounded by $\Lambda(C)$ for all i . Note that the curves C_i are also planar geodesics in M . We now check that each C_i must be strictly convex. If not, then there exists an i and a $q \in C_i$ such that the Gaussian curvature of M at q is zero. Thus in a small neighborhood of q , $M \cap T_q(M)$ is a set of at least three curves crossing at q . This implies M_0^+ cannot be a graph, contradicting Theorem 2 of [Sc1].

There must be at least one point in the interior of M_0^+ at which the Gaussian curvature K vanishes. This follows from an argument identical to the argument given in the third paragraph of the proof of Theorem 3.1 in [MW1], so we do not repeat it here. Thus there exists a point $q \in \text{Int}(\Omega)$ such that the curvature K at $(q, u(q))$ vanishes. Let p be a point on the x_1 -axis which is also contained in $T_{(q, u(q))}(M)$. By translating if necessary, we may assume that p is the origin $\vec{0}$. Thus $(q, u(q)) \in H(M; \phi_{\vec{0}})$. (If $T_{(q, u(q))}(M)$ does not intersect the x_1 -axis, then we can replace $\phi_{\vec{0}}$ with ϕ_n , $n = \vec{e}_1$, and the remaining arguments of this proof

follow through.)

Since the Gauss map G is a conformal map with branch points, $H(M; \phi_0)$ consists of smooth curve segments whose endpoints meet in even numbers at isolated points in the interior of M . In particular, at least four such curves meet at the point $(q, u(q))$, because it is a branch point of G . Note also that $H(M; \phi_0)$ intersects each of C , $R_{-2\beta}(C)$, and each C_i at exactly two points, as these are strictly convex planar curves. Let $Z = \Lambda(H(M; \phi_0))$. Since Z is homeomorphic to $H(M_0^+; \phi_0)$, it has the same structure. Hence Z meets each C_i and also $\Lambda(C)$ exactly twice.

Now form a topological space $\hat{\Omega}$ from Ω by identifying each C_i to a point, and note that $\hat{\Omega}$ is topologically a disk. The corresponding set $\hat{Z}$ (of Z) in $\hat{\Omega}$ is a graph in which each vertex (except for the two vertices on $\Lambda(C) = \partial\hat{\Omega}$) has an even number of edges. Furthermore, the vertex q has at least four edges. It follows that $\hat{\Omega} - \hat{Z}$ contains at least one connected component $\hat{U}$ whose closure does not intersect $\Lambda(C) = \partial\hat{\Omega}$. Let $\mathcal{U}$ be the corresponding region in Ω , and let $\tilde{\mathcal{U}} = \{(x_1, x_2, x_3) \in M \mid (x_1, x_2) \in \mathcal{U}\}$. Then $\partial\tilde{\mathcal{U}} \subseteq H(M, \phi_0)$, and therefore $\tilde{\mathcal{U}}$ is an almost-stable or unstable proper subset of M , so M must be unstable (by the Smale index theorem). This contradiction implies that $k = 1$ and M is an annulus. $\square$

Proof of Corollary 2.1. Let M satisfy the conditions of Corollary 2.1, and let $\text{Ref}(M)$ be the reflection of M across $\omega(0)$. It follows from the Schwarz reflection principle (see, for example, [O1, Lemma 7.3]) that $M \cup \text{Ref}(M)$ is a smooth minimal surface which satisfies the conditions of Theorem 2.1 and therefore is an embedded annulus. By the arguments above, ∂M intersects $\omega(0)$ along a single curve (since $k = 1$), and M is a graph over $\omega(0)$, so M itself must be an embedded annulus. $\square$

Remark. It is not possible to generalize the Meeks and White result or Theorem 2.1 to the case where a boundary curve has a one-to-one perpendicular projection to a strictly convex planar curve, but is not itself a planar curve. A counterexample can be constructed as follows. Let the x_1x_2 -plane be the plane in which the strictly convex projection lies. Consider the two line segments connecting the points $(0, 0, 1)$ and $(N, 0, 1)$, $(0, \delta, 1)$ and $(N, \delta, 1)$, respectively, for some large N . Connect them at the ends with two horizontal semicircles of radius $\delta/2$. (The semicircles are $\{x_1 \leq 0\} \cap \{x_3 = 1\} \cap \{x_1^2 + (x_2 - \delta/2)^2 = (\delta/2)^2\}$ and $\{x_1 \geq N\} \cap \{x_3 = 1\} \cap \{(x_1 - N)^2 + (x_2 - \delta/2)^2 = (\delta/2)^2\}$.) A small perturbation gives us a smooth strictly convex curve in the plane $\{x_3 = 1\}$, which is symmetric with respect to reflection in the plane $\{x_2 = \delta/2\}$. Along each of the two almost straight segments deform the third coordinate of the curve by making n smooth vertical dips almost to the x_1x_2 -plane, so that the curve remains symmetric with

respect to the plane $\{x_2 = \delta/2\}$. Calling this resulting loop α , it is easy to imagine a stable embedded minimal surface of genus $n - 1$ with boundary $\alpha \cup \text{Ref}(\alpha)$, where Ref is reflection through the x_1x_2 -plane (see Figure 1).

FIGURE 1. A counterexample with nonplanar boundary curves.

3. Topological uniqueness for least-area surfaces

In this section, we extend our considerations to include non-convex boundary curves.

Let α and β be two C^2 planar Jordan curves (not necessarily convex) in parallel planes P_0 and P_1 , respectively. Without loss of generality, we may assume $P_0 = \{x_3 = 0\}$ and $P_1 = \{x_3 = 1\}$. Let $\alpha(t)$ be a one-parameter family of planar curves (lying in planes $P_t = \{x_3 = t\}$ parallel to P_0) that are vertical translations of the curve $\alpha = \alpha(0)$ in the direction of the plane P_1 at constant unit speed (making $(\partial/\partial t)\alpha(t)$ a constant vector) so that $\alpha(1)$ and β both lie in P_1 . Consider the points of β where $\alpha(1)$ and β intersect. Assume these intersections are transverse, and that there are a finite number n of them. We will call these points $\{p_1, \dots, p_n\}$ on β the *crossing points* of α and β . Since $\alpha(1)$ intersects β transversely, every vertex in the graph $\alpha(1) \cup \beta$ has exactly four edges emanating from it. Furthermore, to each of the connected components of $P_1 \setminus \{\alpha(1) \cup \beta\}$ we can uniquely assign either a plus sign or a minus sign so that the following statement is true: no two adjacent components have the same sign, and the single component which is not compact has a minus sign (see Figure 2). Define s to be the number of components with a plus sign. Define $\mathcal{W}$ to be the union of the open components of $P_1 \setminus \{\alpha(1) \cup \beta\}$ which are assigned a plus sign.

For any set $B \subseteq P_1$, let $C(B) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2, 1) \in B\}$ be the cylindrical domain in $\mathbb{R}^3$ over the set B . Let $B_{\epsilon, P_1}(p)$ be an open ϵ -ball in P_1 about a point $p \in P_1$.

FIGURE 2. The least-area surface $M(t)$ for t close to 1.

THEOREM 3.1. *Suppose that $\mathcal{W}$ is the unique least-area set in P_1 with boundary equal to $\alpha(1) \cup \beta$. Let $\{p_1, \dots, p_n\} = \alpha(1) \cap \beta$ be the crossing points. Then there exists $\epsilon > 0$ and $t_0 \in (0, 1)$ so that for all $t \in [t_0, 1)$, there is an orientable minimal surface $M(t) \subset \mathbb{R}^3$ with boundary $\alpha(t) \cup \beta$ having the following properties:*

- (1) $M(t)$ has least-area amongst all orientable surfaces with the same boundary;
- (2) $M(t)$ is embedded;
- (3) $M(t) \setminus C(\cup_{i=1}^n B_{\epsilon, P_1}(p_i))$ is a graph over the set $\mathcal{W} \subset P_1$;
- (4) for each i , $M(t) \cap C(B_{\epsilon, P_1}(p_i))$ is homeomorphic to a disk with total absolute curvature less than 2π ;
- (5) $M(t)$ has genus $(n - s)/2$.

Remark. The assumption that $M(t)$ is orientable is natural, since $\alpha(t) \cup \beta$ is extremal and therefore any compact non-orientable minimal surface with this boundary cannot be embedded. Any surface which is not embedded cannot be least-area, since one can easily decrease area by adding topology and desingularizing at intersection points.

Remark. The assumption that $\mathcal{W}$ is the least-area set in P_1 with boundary equal to $\alpha(1) \cup \beta$ is really necessary. For α and β for which this assumption does not hold, it follows from the results of [HS] and [D] that there exists an embedded minimal surface with the same boundary and with area strictly less than that of the $M(t)$ described above. Again, embeddedness of this new surface with lesser area implies that it is also orientable.

Remark. With further arguments, one can show that in a small vertical neighborhood of each crossing point, $M(t)$ is ‘approximately helicoidal’ in the following sense:

for each crossing point $p_j \in P_1$, there exists a portion $S \subset \mathbb{R}^3$ of a helicoid bounded by a pair of infinite lines and with total absolute curvature less than 2π , and there exist homotheties ϕ_t centered at p_j for each $t \in (0, 1)$, such that for any sequence $t_i \in (0, 1)$ with $t_i \rightarrow 1$, $\{\phi_{t_i}(M(t_i))\}_{i=1}^\infty$ converges to S . (We say that a sequence of surfaces $\{S_i\}_{i=1}^\infty$ converges as $i \rightarrow \infty$ to a surface S in $\mathbb{R}^3$ if, for any compact region B of $\mathbb{R}^3$, there exists an integer N_B such that for $i > N_B$, $S_i \cap B$ is a normal graph over S , and $\{S_i \cap B\}_{i=1}^\infty$ converges to S in the C^1 -norm.)

Before giving the proof of Theorem 3.1, we give a useful preliminary lemma. For an oriented surface $\mathcal{M} \subset \mathbb{R}^3$, let $\text{dist}_{\mathcal{M}}(A, B)$ be the distance in $\mathcal{M}$ between two sets $A \subset \mathcal{M}$ and $B \subset \mathcal{M}$. Let $\partial\mathcal{M}$ be the boundary of $\mathcal{M}$. For each point $q \in \mathcal{M}$, let K_q be the Gaussian curvature of $\mathcal{M}$ at q , and let $\vec{N}_q$ be the oriented unit normal vector of $\mathcal{M}$ at q . Let $\vec{e}_3 = (0, 0, 1)$, and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^3$.

LEMMA 3.1. *Let $\mathcal{M}$ be an oriented minimal surface lying in the closed region $\{t \leq x_3 \leq 1\}$ between the two distinct planes P_t and P_1 , and consider a point $p \in \mathcal{M}$. Suppose that there exists a positive constant $\mathcal{A}$ such that $\text{dist}_{\mathcal{M}}(p, \partial\mathcal{M}) \geq 1/\sqrt{\mathcal{A}}$ and $|K_q| < \mathcal{A}$ for all $q \in \mathcal{M}$. Then, for $\mathcal{T} := 1 - 2\sqrt{\mathcal{A}}(1 - t)$,*

$$|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > \mathcal{T}.$$

Proof. Note that $\mathcal{T} < 1$, and that the result is obvious if $\mathcal{T} < 0$, so we assume $\mathcal{T} \in [0, 1)$. Suppose that $|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 \leq \mathcal{T}$. Then there exists a unit vector $\vec{T} \in T_p\mathcal{M}$ so that $\langle \vec{T}, \vec{e}_3 \rangle \geq \sqrt{1 - \mathcal{T}}$, and there exists a unit speed geodesic $\gamma(s) \subset \mathcal{M}$, $s \in [0, 1/\sqrt{\mathcal{A}}]$ so that $\gamma(0) = p$ and $\gamma'(0) = \vec{T}$, where $\prime = \partial/\partial s$. Let $k_g(s)$ be the geodesic curvature of $\gamma(s)$. Since $|K_q| < \mathcal{A}$ and $\mathcal{M}$ is minimal, $|k_g(s)| < \sqrt{\mathcal{A}}$ for all $s \in [0, 1/\sqrt{\mathcal{A}}]$. Thus $|\gamma''(s)| < \sqrt{\mathcal{A}}$. Writing $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ in terms of coordinates in $\mathbb{R}^3$, we have $|\gamma_3''(s)| < \sqrt{\mathcal{A}}$. Then for $s \in [0, 1/\sqrt{\mathcal{A}}]$,

$$|\gamma_3'(s) - \gamma_3'(0)| = \left| \int_0^s \gamma_3'' \right| \leq \int_0^s |\gamma_3''| < \sqrt{\mathcal{A}} \cdot s,$$

and thus $\gamma_3'(s) > \gamma_3'(0) - \sqrt{\mathcal{A}} \cdot s$. Therefore

$$\int_0^{\gamma_3'(0)/\sqrt{\mathcal{A}}} \gamma_3'(s) ds > \int_0^{\gamma_3'(0)/\sqrt{\mathcal{A}}} (\gamma_3'(0) - \sqrt{\mathcal{A}} \cdot s) ds = \gamma_3'(0) \cdot \frac{\gamma_3'(0)}{\sqrt{\mathcal{A}}} - \frac{1}{2} \sqrt{\mathcal{A}} \left(\frac{\gamma_3'(0)}{\sqrt{\mathcal{A}}} \right)^2,$$

and so

$$\gamma_3 \left(\frac{\gamma_3'(0)}{\sqrt{\mathcal{A}}} \right) - \gamma_3(0) > \frac{(\gamma_3'(0))^2}{2\sqrt{\mathcal{A}}} \geq \frac{1 - \mathcal{T}}{2\sqrt{\mathcal{A}}} = 1 - t,$$

since $\gamma'_3(0) = \langle \vec{T}, \vec{e}_3 \rangle \geq \sqrt{1 - T}$. Hence the vertical change $\gamma_3(\gamma'_3(0)/\sqrt{A}) - \gamma_3(0)$ of the geodesic $\gamma(s) \subset \mathcal{M}$ is greater than $1 - t$, and since $1 - t$ is the distance between the planes P_t and P_1 , $\mathcal{M}$ cannot lie between P_t and P_1 . This contradiction proves the lemma. $\square$

Proof of Theorem 3.1. It was shown in [HS] that there exists an orientable minimal surface $M(t)$ with boundary $\alpha(t) \cup \beta$ that has least-area amongst all orientable surfaces with the same boundary, and that $M(t)$ is embedded, with finite genus, and with bounded curvature. $M(t)$ cannot have any interior branch points [O2], and since its boundary is extremal, it cannot have any boundary branch points [N2, section 366].

We divide the proof into steps. We fix ϵ to be a small positive number such that $1/\epsilon$ is much greater than the maximum planar curvature of $\alpha \cup \beta$. (In Steps 3 and 6 we will add further constraints on ϵ .) Let $N_{\epsilon, P_1}(\alpha(1) \cup \beta) := \{p \in P_1 \mid \text{dist}(p, \alpha(1) \cup \beta) < \epsilon\}$ be the ϵ neighborhood of $\alpha(1) \cup \beta$ in P_1 .

Step 1. $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ is a collection of graphs over P_1 for t sufficiently close to 1. In fact, for any $y \in (0, 1)$, there exists a $t_y \in (0, 1)$ such that for all $t \in [t_y, 1)$, $|\langle \vec{N}_p, \vec{e}_3 \rangle| > y$ for all $p \in M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$.

By Corollary 4 of [Sc2] there exists a universal constant $c \geq 1$ such that $|K_q| < c/r^2$, where K_q is the Gaussian curvature at some point q of a stable minimal surface and r is the distance from q to the boundary of the surface. Thus for all $q \in M(t) \setminus C(N_{\epsilon/2, P_1}(\alpha(1) \cup \beta))$, $|K_q| < 4c/\epsilon^2$. We now apply Lemma 3.1 with $\mathcal{M} = M(t) \setminus C(N_{\epsilon/2, P_1}(\alpha(1) \cup \beta))$ and $\mathcal{A} = 4c/\epsilon^2$. Let p be any point in $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$. Note that $\text{dist}_{\mathcal{M}}(p, \partial\mathcal{M}) \geq \epsilon/2 = \sqrt{c}/\sqrt{A} \geq 1/\sqrt{A}$. Hence by Lemma 3.1,

$$|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > 1 - 2\frac{2\sqrt{c}}{\epsilon}(1 - t).$$

So for any $t \in [1 - \epsilon/4\sqrt{c}, 1)$, we have $|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > 0$. Thus for $t \in [1 - \epsilon/4\sqrt{c}, 1)$ the normal vector is never horizontal on $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$, implying that $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ is a collection of graphs. Furthermore, for any $y \in (0, 1)$, let $t_y = 1 - (1 - y^2)(\epsilon/(4\sqrt{c}))$, and then $|\langle \vec{N}_p, \vec{e}_3 \rangle| > y$ for all $p \in M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ if $t \in [t_y, 1)$. Step 1 is shown.

Step 2. $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ is a single graph over P_1 for t sufficiently close to 1.

Suppose that for some point $q \in P_1 \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)$, $C(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ contains two graphs. These two graphs are bounded by a pair of curves C_1 and C_2 in the boundary $\partial(C(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)))$; $C_1 \cup C_2$ must lie between P_t and P_1 , since $M(t)$ itself lies between P_t and P_1 . Note that this pair of graphs has area at least $2 \cdot \text{Area}(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta))$. Since $M(t)$ is least-area, this pair of graphs must be least-area with respect to its boundary $C_1 \cup C_2$. However, $C_1 \cup C_2$ also bounds an annulus that has area less than $(1-t) \text{Length}(\partial(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)))$, and $(1-t) \text{Length}(\partial(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta))) < 2 \text{Area}(B_{\epsilon/2, P_1}(q) \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta))$, if t is sufficiently close to 1. This contradiction implies Step 2.

Step 3. $C(\mathcal{W} \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)) \cap M(t) = M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$.

Let $\tilde{\mathcal{W}}$ be the union of open components of $P_1 \setminus \{\alpha(1) \cup \beta\}$ in P_1 whose intersections with the projection of $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ are nonempty. Clearly, the boundary of $\tilde{\mathcal{W}}$ in P_1 equals $\alpha(1) \cup \beta$, thus $\text{Area}(\tilde{\mathcal{W}}) > \text{Area}(\mathcal{W})$ if $\tilde{\mathcal{W}} \neq \mathcal{W}$, by assumption. Suppose $\tilde{\mathcal{W}} \neq \mathcal{W}$.

Since $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ is a graph over $\tilde{\mathcal{W}} \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)$, we have $\text{Area}(M(t)) \geq \text{Area}(\tilde{\mathcal{W}} \setminus N_{\epsilon, P_1}(\alpha(1) \cup \beta)) > \text{Area}(\tilde{\mathcal{W}}) - 2\epsilon(\text{Length}(\alpha) + \text{Length}(\beta))$. (The final strict inequality follows from the fact that $1/\epsilon$ is much greater than the maximum planar curvature of $\alpha \cup \beta$.) Choosing ϵ smaller if necessary, we may assume $2\epsilon(\text{Length}(\alpha) + \text{Length}(\beta)) < (1/2)(\text{Area}(\tilde{\mathcal{W}}) - \text{Area}(\mathcal{W}))$. Therefore

$$\text{Area}(M(t)) > \text{Area}(\tilde{\mathcal{W}}) - \frac{1}{2}(\text{Area}(\tilde{\mathcal{W}}) - \text{Area}(\mathcal{W})).$$

For any $\delta > 0$, there exists a $t_\delta \in (0, 1)$ so that for all $t \in [t_\delta, 1)$, there is an orientable surface $\check{M}(t)$ bounded by $\alpha(t) \cup \beta$ satisfying $\text{Area}(\check{M}(t)) < \text{Area}(\mathcal{W}) + \delta$: just consider a surface that is a graph away from crossing points over each open component of $P_1 \setminus (\alpha(1) \cup \beta)$ assigned a plus sign, and connect these graphs by small disks twisted by approximately π radians at each crossing point. Choosing $\delta = (1/2)(\text{Area}(\tilde{\mathcal{W}}) - \text{Area}(\mathcal{W}))$, and choosing $t \geq t_\delta$, we have

$$\text{Area}(M(t)) > \frac{1}{2}(\text{Area}(\tilde{\mathcal{W}}) + \text{Area}(\mathcal{W})) > \text{Area}(\check{M}(t)).$$

But $M(t)$ is least-area, so this contradiction implies $\tilde{\mathcal{W}} = \mathcal{W}$. This shows Step 3.

To prepare for Step 4, consider $N_{\epsilon, P_1}(\alpha(1) \cap \beta) := \{p \in P_1 \mid \text{dist}(p, \alpha(1) \cap \beta) < \epsilon\}$, the ϵ neighborhood in P_1 of the crossing points.

Step 4. Along the curves $(\alpha(t) \cup \beta) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$ the normal vector to $M(t)$ must become arbitrarily close to vertical, uniformly as $t \rightarrow 1$.

To show this, for t sufficiently close to 1 we consider a compact connected piece of a catenoid $\mathcal{C}(r_1, r_2, t)$ with the following properties:

- $\mathcal{C} = \mathcal{C}(r_1, r_2, t)$ has vertical axis and is bounded by circles $P_t \cap \mathcal{C}$ and $P_1 \cap \mathcal{C}$, of radii r_2 and r_1 respectively, in the planes P_t and P_1 ;
- r_2 is much smaller than r_1 , and r_1 is much smaller than ϵ ;
- The normal vector to $\mathcal{C}$ is close to vertical everywhere on $\mathcal{C}$.

Choosing r_2 and r_1 small enough and choosing t close enough to 1, $\mathcal{C}$ can be placed in the vertical cylinder over any open component of $P_1 \setminus (\alpha(1) \cup \beta)$ assigned a minus sign, so that $\mathcal{C} \cap M(t) = \emptyset$, by Step 3. We can then translate $\mathcal{C}$ horizontally until it makes first contact with $M(t)$ at any given point $p \in \alpha(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$. (The maximum principle implies that first contact cannot be at an interior point of $M(t)$. See Figure 3.) By the maximum principle, the normal vector of $M(t)$ at p must be even more vertical than it is at the same point on $\mathcal{C}$. Hence, the normal vector of $M(t)$ at p is close to vertical. Reflecting $\mathcal{C}$ through the plane $P_{(1+t)/2}$, the same argument shows that the normal vector of $M(t)$ at any point of $\beta \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$ must also be close to vertical.

Additionally, we have just shown that the projection of $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$ to P_1 lies entirely inside $\mathcal{W}$, for otherwise $\mathcal{C}$ (or its reflection through $P_{(1+t)/2}$) could be translated in such a way as to make first contact at an interior point.

At that first boundary point $p \in \partial M(t)$ of contact, the catenoid $\mathcal{C}$ (or its reflection through $P_{(1+t)/2}$) makes a small angle with the horizontal plane containing the boundary curve. Among all catenoid pieces of the type $\mathcal{C}(r_1, r_2, t)$ contacting $M(t)$ only at p , there is a greatest lower bound $\theta(p, t)$ for this angle, and for all boundary points $p \in (\alpha(t) \cup \beta) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$, the lower bound $\theta(p, t)$ approaches zero as t approaches 1 (see Figure 3). Let

$$\theta_0(t) = \max\{\theta(p, t) : p \in (\alpha(t) \cup \beta) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))\}.$$

Since $(\alpha(t) \cup \beta) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cap \beta))$ is compact, $\lim_{t \rightarrow 1} \theta_0(t) = 0$.

Step 5. $M(t) \setminus C(N_{2\epsilon, P_1}(\alpha(1) \cap \beta))$ is a graph over $\mathcal{W}$, for t sufficiently close to 1. In fact, for any $y \in (0, 1)$, there exists a $t_y \in (0, 1)$ such that if $t \in [t_y, 1)$, then $|\langle \vec{N}_p, \vec{e}_3 \rangle| > y \forall p \in M(t) \setminus C(N_{2\epsilon, P_1}(\alpha(1) \cap \beta))$.

Since we have already proved this for $M(t) \setminus C(N_{\epsilon, P_1}(\alpha(1) \cup \beta))$ in Steps 1 and 2, we may restrict our considerations here to any points $p \in M(t) \cap C(N_{\epsilon, P_1}(\alpha(1) \cup \beta) \setminus N_{2\epsilon, P_1}(\alpha(1) \cap \beta))$. Let $\hat{p}$ be the vertical projection of p into P_1 , and let r be the distance of $\hat{p}$ to $\alpha(1) \cup \beta$. (Note that $r \leq \epsilon$.) Assume for the moment that the closest point in $\alpha(t) \cup \beta$ to p is a point in β .

FIGURE 3. A pictorial view of $\theta(p, t)$.

Consider the bounded cylinder $\text{Cyl} = C(B_{r/2, P_1}(\hat{p})) \cap \{1 - 2r \tan \theta_0(t) \leq x_3 \leq 1\}$. Cyl is a finite solid cylinder of radius $r/2$ and height $2r \tan \theta_0(t)$ with two planar horizontal circular boundary disks. Let ∂Cyl be its boundary. It follows from the construction of the catenoid barriers in the previous step (see Figure 3) that $M(t) \cap \partial \text{Cyl}$ is contained entirely in the cylindrical part of ∂Cyl , and is disjoint from the upper and lower horizontal boundary disks. To avoid unnecessary complications near the crossing points, we have increased ϵ to 2ϵ .

Let $\mathcal{M} = \text{Cyl} \cap M(t)$. Then $\mathcal{M}$ lies between the planes $P_{1-2r \tan \theta_0(t)}$ and P_1 . Since for any point $q \in \mathcal{M}$, $\text{dist}_{M(t)}(q, \alpha(t) \cup \beta) \geq r/2$, there exists a universal constant $c \geq 1$ such that $|K_q| < 4c/r^2$ [Sc2], as in Step 1. Let $\mathcal{A} = 4c/(r^2)$, and thus $\text{dist}_{\mathcal{M}}(p, \partial \mathcal{M}) \geq r/2 = \sqrt{c}/\sqrt{\mathcal{A}} \geq 1/\sqrt{\mathcal{A}}$. Hence, we may apply Lemma 3.1 to conclude that

$$|\langle \vec{N}_p, \vec{e}_3 \rangle|^2 > 1 - 2 \frac{2\sqrt{c}}{r} 2r \tan \theta_0(t).$$

For any $y \in (0, 1)$, choose $t_y \in (0, 1)$ so that for all $t \in [t_y, 1)$,

$$\tan \theta_0(t) \leq \frac{1 - y^2}{8\sqrt{c}}.$$

Then, for $t \in [t_y, 1)$, $|\langle \vec{N}_p, \vec{e}_3 \rangle| > y \forall p \in M(t) \setminus C(N_{2\epsilon, P_1}(\alpha(1) \cap \beta))$.

When the closest point in $\alpha(t) \cup \beta$ to p is a point in $\alpha(t)$, we can reflect $M(t)$ across the plane $P_{(1+t)/2}$ and we have the same situation, but the roles of $\alpha(t)$ and β are reversed. So the above argument applies to this case as well. Step 5 is shown.

Step 6. For each crossing point p_i , $C(B_{2\epsilon, P_1}(p_i)) \cap M(t)$ is a disk with total curvature less than 2π , for t sufficiently close to 1.

We could have originally chosen ϵ small enough so that the two curve segments $\alpha(t) \cap C(B_{2\epsilon, P_1}(p_i))$ and $\beta \cap C(B_{2\epsilon, P_1}(p_i))$ have total curvature as close to zero as we wish. We may then choose t sufficiently close to 1 so that $\partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i))) \setminus \{\alpha(t) \cup \beta\}$ approximates arbitrarily closely a pair of circular arcs, each of less than π radians, and so that the exterior angles at the four singular points of $\partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$ are each arbitrarily close to $\pi/2$. (This follows from the fact, as shown in Steps 1 and 5, that for any given $y \in (0, 1)$ we have $|\langle \vec{N}_q, \vec{e}_3 \rangle| > y$ on $M(t) \setminus C(N_{2\epsilon, P_1}(\alpha(1) \cap \beta))$ for t sufficiently close to 1.) So for ϵ small enough and t sufficiently close to 1, $\partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$ has total curvature less than 4π .

If $M(t) \cap C(B_{2\epsilon, P_1}(p_i))$ is not a disk, then there exists a smooth Jordan curve $\sigma \subset \text{Int}(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$ with the following properties:

- σ is a smooth approximation to $\partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$ that lies on the boundary of a convex region in $\mathbb{R}^3$ and has total curvature less than 4π . (This property is possible because $\partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$ is extremal.)
- σ separates $M(t) \cap C(B_{2\epsilon, P_1}(p_i))$ into two components; one component $\mathcal{A}_1$ is bounded by σ , and the other component $\mathcal{A}_2$ is an annulus bounded by $\sigma \cup \partial(M(t) \cap C(B_{2\epsilon, P_1}(p_i)))$.
- The component $\mathcal{A}_1$ is not a disk.

However, Section 3 of [MY] then implies that $\mathcal{A}_1$ must be a disk. This contradiction implies that $M(t) \cap C(B_{2\epsilon, P_1}(p_i))$ is actually a disk. The Gauss–Bonnet theorem then implies that the total absolute curvature satisfies

$$\int_{M(t) \cap C(B_{2\epsilon, P_1}(p_i))} |K| dA < 2\pi.$$

This shows step 6.

Dividing our original choice for ϵ by 2, Step 6 implies the fourth item of the theorem.

Thus, $M(t)$ with t close to 1 is a graph away from crossing points over components of $P_1 \setminus \{\alpha(1) \cup \beta\}$ assigned a plus sign, and $M(t)$ is a disk in small vertical neighborhoods of each crossing point. One can then easily check that the Euler characteristic of $M(t)$ is $s - n$, and therefore the genus of $M(t)$ is $(n - s)/2$. $\square$

Just as the Meeks and White result was extended to the case of a wedge by Theorem 2.1 in the previous section, likewise Theorem 3.1 can be extended to the case of a wedge. This is done in Theorem 3.2, and the proof is essentially the same as for Theorem 3.1. Let β be a C^2 Jordan curve in the interior of the half-plane

$\omega(0)$, and let α be a C^2 Jordan curve in the interior of the half-plane $\omega(\pi/2)$. Let $\alpha(t) = R_{-\pi t/2}(\alpha)$ be the rotation about the x_1 -axis of α into the half-plane $\omega((\pi/2)(1-t))$. Suppose $\alpha(1)$ and β intersect transversely at a finite number of points. Call these n points the *crossing points* of α and β . Define s and $\mathcal{W}$ just as before.

THEOREM 3.2. *Suppose that $\mathcal{W}$ is the unique least-area set in $\omega(0)$ with boundary equal to $\alpha(1) \cup \beta$. Let $\{p_1, \dots, p_n\} = \alpha(1) \cap \beta$ be the crossing points. Then there exists $\epsilon > 0$ and $t_0 \in (0, 1)$ so that for all $t \in [t_0, 1)$, there is an orientable minimal surface $M(t) \subset \mathbb{R}^3$ with boundary $\alpha(t) \cup \beta$ having the following properties:*

- (1) *$M(t)$ has least area amongst all orientable surfaces with the same boundary;*
- (2) *$M(t)$ is embedded;*
- (3) *$M(t) \setminus C(\cup_{i=1}^n B_{\epsilon, \omega(0)}(p_i))$ is a graph over the set $\mathcal{W} \subset \omega(0)$;*
- (4) *for each i , $M(t) \cap C(B_{\epsilon, \omega(0)}(p_i))$ is homeomorphic to a disk with total absolute curvature less than 2π .*
- (5) *$M(t)$ has genus $(n-s)/2$.*

4. Non-existence results

The results we prove in this section are about existence of compact minimal and constant mean curvature surfaces with a given pair of boundary curves C_1, C_2 in parallel horizontal planes. When C_1 and C_2 are convex and the vertical projection of C_1 into the plane containing C_2 does not intersect C_2 , it is well known that such a minimal surface does not exist. For completeness we give the proof here.

LEMMA 4.1. *If C_1 and C_2 are convex curves in parallel planes and if the perpendicular projection of C_1 into the plane containing C_2 is disjoint from C_2 , then there does not exist a compact connected minimal surface with boundary $C_1 \cup C_2$.*

Proof. We may assume C_1 and C_2 are contained in planes parallel to the x_1x_2 -plane, and by hypothesis we may assume C_1 and C_2 lie on opposite sides of, and are disjoint from, the plane $\{x_1 = 0\}$. Suppose that there exists a compact connected minimal surface M with boundary $C_1 \cup C_2$, then let $\text{Ref}(M)$ be the reflection of M through the plane $\{x_1 = 0\}$. Translate $\text{Ref}(M)$ in the direction of the x_2 -axis until it is disjoint from M , then translate it back until the first moment when M and $\text{Ref}(M)$ intersect. This point of intersection must be in the interiors of both M and $\text{Ref}(M)$, and $M \neq \text{Ref}(M)$, contradicting the maximum principle. $\square$

We use the same kind of technique to prove the next lemma. A *slab* in $\mathbb{R}^3$ is a region lying between two parallel planes. A slab is called *vertical* if its two boundary planes are perpendicular to the x_1x_2 -plane.

LEMMA 4.2. *Suppose C_1 and C_2 are two planar curves in horizontal planes, and these two planes are of distance h_1 apart. If $C_1 \cup C_2$ is contained in a vertical slab of width $h_2 < h_1$, then there does not exist a compact connected minimal surface with boundary $C_1 \cup C_2$.*

Proof. We may assume that the boundary planes of the vertical slab are parallel to the x_1 -axis, and we may assume the x_1 -axis is equidistant from the two planes containing C_1 and C_2 and also from the boundary planes of the vertical slab. Suppose that there exists a compact connected minimal surface M with boundary $C_1 \cup C_2$. Let $R_{\pi/2}(M)$ be a rotation of M by $\pi/2$ radians about the x_1 -axis. Translate $R_{\pi/2}(M)$ in the direction of the x_1 -axis until it is disjoint from M , then translate it back until the first moment that M and $R_{\pi/2}(M)$ intersect. Since $h_2 < h_1$, this point of intersection must be in the interiors of both M and $R_{\pi/2}(M)$, and $M \neq R_{\pi/2}(M)$, contradicting the maximum principle. $\square$

COROLLARY 4.1. *If C_1 and C_2 are planar curves with diameters d_1 and d_2 in parallel planes, and these two planes are of distance h apart, and these two curves bound a compact connected minimal surface M , then $h \leq \max\{d_1, d_2\}$.*

Proof. We may assume C_1 and C_2 lie in horizontal planes of distance h apart. Let P_1 be a vertical plane that is tangent to both C_1 and C_2 , so that $C_1 \cup C_2$ lies entirely to one side of P_1 . There exists another plane P_2 parallel to P_1 such that $C_1 \cup C_2$ lies entirely within the vertical slab bounded by P_1 and P_2 , and the distance between P_1 and P_2 is at most $\max\{d_1, d_2\}$. The result follows from Lemma 4.2. $\square$

As stated in the introduction, the above corollary is a generalization of a result by Nitsche [N1], where it is assumed that M is an annulus and gives the weaker conclusion $h \leq \frac{3}{2} \max\{d_1, d_2\}$. Even our result here does not perhaps give the strongest possible result for an upper bound on h . For example, if M is an annular subregion of a catenoid with a vertical axis in $\mathbb{R}^3$, and the boundary ∂M is two circles (with radii r_1 and r_2 , respectively) in horizontal planes, then the two horizontal planes containing ∂M are of distance less than $\frac{2}{3}(r_1 + r_2)$ apart. One can easily see this by elementary considerations of the generating curve $y = \cosh(x)$ of the catenoid. Since $\frac{2}{3}(r_1 + r_2) \leq \frac{2}{3} \max\{d_1 = 2r_1, d_2 = 2r_2\}$, we know that Corollary 4.1 is not the strongest possible result in the case of this catenoid M .

Lemma 4.2 cannot be extended to surfaces of constant non-zero mean curvature, as the round cylinder shows, but something can be said about the possible values of the mean curvature in the case of nonminimal constant mean curvature surfaces.

PROPOSITION 4.1. *Let $C = \{x_1^2 + x_2^2 \leq r^2, 0 \leq x_3 \leq d\} \subset \mathbb{R}^3$, with $r < d/2$. If $\Sigma \subset C$ is an embedded constant mean curvature surface with mean curvature H (with respect to the inward pointing normal) and boundary in the planes $P_0 = \{x_3 = 0\}$ and $P_d = \{x_3 = d\}$, then*

$$H > \frac{d^2 - 12r^2}{2r(d^2 - 4r^2)} - \frac{32r^2d^2}{(d^2 - 4r^2)^3}.$$

Thus when d is large relative to r , the mean curvature H of Σ is bounded away from zero. Furthermore, the limiting value for this lower bound as $d \rightarrow \infty$ is equal to the mean curvature of a cylinder of radius r .

Proof. Consider the embedded annular surfaces in C which are surfaces of rotation about the x_3 -axis with boundary curves $\{x_1^2 + x_2^2 = r^2, x_3 = 0\}$ and $\{x_1^2 + x_2^2 = r^2, x_3 = d\}$ and with generating curves that are arcs of a circle. Among this 1-parameter family of surfaces the mean curvature (with respect to the inward pointing normal) is always larger than $((d^2 - 12r^2)/2r(d^2 - 4r^2)) - (32r^2d^2/(d^2 - 4r^2)^3)$. Shrinking this family of surfaces from a cylinder of radius r to the surface which makes first contact with Σ , we may apply the maximum principle [Sc1] to conclude the result. $\square$

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