AN ANALOGUE OF PASCHKE'S THEOREM FOR ACTIONS OF COMPACT KAC ALGEBRAS

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1. Introduction

Based on Rieffel's work [8] about Morita equivalence for von Neumann algebras, Paschke proved in [7] that for a faithful action of a compact abelian group, the crossed product is a factor if and only if the fixed point algebra is a factor, and he also showed a certain relation between the crossed product and the fixed point algebra for a (not necessarily abelian) compact group action. Since this work, much progress on compact group actions has been made, and one of the important classes of compact group actions is that of the so-called minimal actions. Wassermann [12] proved the existence of a minimal action for a compact Lie group. Recently, Izumi *et al* [4] proved Galois correspondences for minimal actions of compact groups and compact Kac algebras, and Ueda [11] showed the existence of minimal actions of $SU_q(n)$ and compact Kac algebras. However, two different definitions of minimality for actions of compact Kac algebras have been used, and it has been shown by Izumi *et al* [4] that the minimality in their sense implies that of Wassermann. (Ueda [11] used the definition of minimality of Wassermann.)

Motivated by Paschke's result and works on minimal actions of Wassermann, Izumi *et al* and Ueda, we shall extend Paschke's theorem to actions of compact Kac algebras and show that the minimality in the sense of Wassermann implies Izumi *et al* as a corollary; namely, it will be shown that the two kinds of definitions of minimality are equivalent.

After the completion of this work, the author learned of preprints of Yamanouchi [13] and Ng [6]. In the first paper, our main result was proved by another method, and in the second paper, it was shown that for a certain action of a compact quantum group on a C^* -algebra, the reduced crossed product and the fixed point algebra are Morita equivalent.

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2. Results

Throughout this paper, $\mathbf{K} = (M, \Gamma, \kappa, \varphi)$ denotes a compact Kac algebra with the Haar state φ , which is represented on the standard Hilbert space $H_\varphi = L^2(M, \varphi)$ and we let α be an action of $\mathbf{K}$ on a von Neumann algebra A represented on a Hilbert space H . Then the crossed product $A \rtimes_\alpha \mathbf{K}$ is defined to be the von Neumann algebra generated by $\alpha(A)$ and $1 \otimes \hat{M}'$. Here $\hat{M}'$ is the von Neumann algebra generated by the range of the Fourier representation λ^σ of the opposite $\mathbf{K}^\sigma = (M, \sigma \circ \Gamma, \kappa, \varphi)$. Let A^α be the fixed point subalgebra under α and let $E : A \rightarrow A^\alpha$ be the faithful normal conditional expectation given by

$$E(x) = (1 \otimes \varphi)(\alpha(x)), \quad x \in A.$$

For definitions and fundamental facts on Kac algebras and their actions, we refer to [1–3, 9, 10].

LEMMA 1. *With the above notation, we have, for $x \in A$, $\omega \in M_*$, $\xi \in H$,*

$$(1 \otimes \lambda^\sigma(\omega))\alpha(x)(\xi \otimes \Lambda_\varphi(1)) = \alpha((1 \otimes \omega \circ \kappa)(\alpha(x)))(\xi \otimes \Lambda_\varphi(1)).$$

Proof. Notice that the commutant of the dual $\hat{\mathbf{K}}' = (\hat{M}', \hat{\Gamma}', \hat{\kappa}', \hat{\varphi}')$ is the dual of the opposite $\mathbf{K}^\sigma = (M, \sigma \circ \Gamma, \kappa, \varphi)$, so it is well-known from Enock and Schwartz [3] that

$$\lambda^\sigma(\omega) = (\omega \circ \kappa \otimes 1)(W^\sigma), \quad \omega \in M_*,$$

where W^σ denotes the fundamental operator of $\mathbf{K}^\sigma$. As in Landstad *et al* [5], we compute, for $x \in A$, $\omega \in M_*$,

$$\begin{aligned} (1 \otimes \lambda^\sigma(\omega))\alpha(x) &= (1 \otimes (\omega \circ \kappa \otimes 1)(W^\sigma))\alpha(x) \\ &= (1 \otimes \omega \circ \kappa \otimes 1)((1 \otimes W^\sigma)\alpha(x)_{13}) \\ &= (1 \otimes \omega \circ \kappa \otimes 1)((1 \otimes W^\sigma)\alpha(x)_{13}(1 \otimes W^\sigma)^*(1 \otimes W^\sigma)) \\ &= (1 \otimes \omega \circ \kappa \otimes 1)((1 \otimes \sigma)((1 \otimes \Gamma)(\alpha(x)))(1 \otimes W^\sigma)) \\ &= (1 \otimes \omega \circ \kappa \otimes 1)((1 \otimes \sigma)((\alpha \otimes 1)(\alpha(x)))(1 \otimes W^\sigma)). \end{aligned}$$

Furthermore, since $\lambda^\sigma(\varphi)$ is the projection from H_φ onto $\mathbf{C}\Lambda_\varphi(1)$, we have, for $y \in A$, $m \in M$, $\xi \in H$,

$$\begin{aligned} &(1 \otimes \omega \circ \kappa \otimes 1)((1 \otimes \sigma)((\alpha \otimes 1)(y \otimes m))(1 \otimes W^\sigma))(\xi \otimes \Lambda_\varphi(1)) \\ &= (1 \otimes \omega \circ \kappa \otimes 1)(\alpha(y)_{13}(1 \otimes m \otimes 1)(1 \otimes W^\sigma))(\xi \otimes \Lambda_\varphi(1)) \end{aligned}$$

$$\begin{aligned}
&= \alpha(y)(1 \otimes ((\kappa(m) \cdot \omega) \circ \kappa \otimes 1)(W^\sigma))(\xi \otimes \Lambda_\varphi(1)) \\
&= \alpha(y)(1 \otimes \lambda^\sigma(\kappa(m) \cdot \omega))(\xi \otimes \Lambda_\varphi(1)) \\
&= \alpha(y)(1 \otimes \lambda^\sigma(\kappa(m) \cdot \omega))(1 \otimes \lambda^\sigma(\varphi))(\xi \otimes \Lambda_\varphi(1)) \\
&= \omega(\kappa(m))\alpha(y)(\xi \otimes \Lambda_\varphi(1)) \\
&= (\alpha \otimes \omega \circ \kappa)(y \otimes m)(\xi \otimes \Lambda_\varphi(1)) \\
&= \alpha((1 \otimes \omega \circ \kappa)(y \otimes m))(\xi \otimes \Lambda_\varphi(1)).
\end{aligned}$$

Hence we have the conclusion. $\square$

LEMMA 2. Let $\alpha(A)(H \otimes \Lambda_\varphi(1))$ be the closed subspace of $H \otimes H_\varphi$ generated by $\alpha(x)(\xi \otimes \Lambda_\varphi(1))$, $x \in A$, $\xi \in H$, and let p_α be the orthogonal projection of $H \otimes H_\varphi$ onto $\alpha(A)(H \otimes \Lambda_\varphi(1))$. Then p_α is the central support of $1 \otimes \lambda^\sigma(\varphi)$ in $A \times_\alpha \mathbf{K}$. In particular, p_α belongs to the center of $A \times_\alpha \mathbf{K}$.

Proof. We shall show that $\alpha(A)(H \otimes \Lambda_\varphi(1))$ is equal to the closed subspace $(A \times_\alpha \mathbf{K})(1 \otimes \lambda^\sigma(\varphi))(H \otimes H_\varphi)$.

Thanks to a computation in Landstad *et al* [5] (see also the proof of Lemma 1), we first note that any element in $A \times_\alpha \mathbf{K}$ can be approximated by a finite linear combination of elements of the form $\alpha(x)(1 \otimes \lambda^\sigma(\omega))(x \in A, \omega \in M_*)$. Since $\lambda^\sigma(\varphi)$ is the projection from H_φ onto $\mathbf{C}\Lambda_\varphi(1)$, we get the assertion. $\square$

Following Paschke [7], we define an A^α -valued inner product on A (viewed as a right A^α -module) by

$$\langle x, y \rangle = E(y^*x), \quad x, y \in A$$

and we let $\bar{A}$ be the set of all bounded linear A^α -module maps from A into A^α and $L(\bar{A})$ be the set of all bounded A^α -module maps from $\bar{A}$ into $\bar{A}$. Setting $K = \bar{A} \otimes_{A^\alpha} H$, the relative tensor product of $\bar{A}$ and H over A^α , we denote by $\theta : L(\bar{A}) \rightarrow B(K)$ the faithful normal representation defined by

$$(\theta(T))(\tau \otimes_{A^\alpha} \xi) = T(\tau) \otimes_{A^\alpha} \xi, \quad T \in L(\bar{A}), \tau \in \bar{A}, \xi \in H.$$

LEMMA 3. With the above notation, there exists a surjective isometry $V : K \rightarrow p_\alpha(H \otimes H_\varphi)$ such that

$$V(x \otimes_{A^\alpha} \xi) = \alpha(x)(\xi \otimes \Lambda_\varphi(1)), \quad x \in A, \xi \in H.$$

Proof. For $x_1, x_2, \dots, x_n \in A$ and $\xi_1, \xi_2, \dots, \xi_n \in H$, we have

$$\begin{aligned}
 \left\| \sum_{i=1}^n x_i \otimes_{A^\alpha} \xi_i \right\|^2 &= \sum_{i,j=1}^n (x_i \otimes_{A^\alpha} \xi_i \mid x_j \otimes_{A^\alpha} \xi_j) \\
 &= \sum_{i,j=1}^n (E(x_j^* x_i) \xi_i \mid \xi_j) \\
 &= \sum_{i,j=1}^n ((1 \otimes \varphi)(\alpha(x_j^* x_i)) \xi_i \mid \xi_j) \\
 &= \sum_{i,j=1}^n (\alpha(x_j^* x_i)(\xi_i \otimes \Lambda_\varphi(1)) \mid \xi_j \otimes \Lambda_\varphi(1)) \\
 &= \left\| \sum_{i=1}^n \alpha(x_i)(\xi_i \otimes \Lambda_\varphi(1)) \right\|^2.
 \end{aligned}$$

By Paschke [7], $\{x \otimes_{A^\alpha} \xi \mid x \in A, \xi \in H\}$ is total in K . So we conclude the result. $\square$

THEOREM 4. Let $\rho : L(\bar{A}) \rightarrow B(p_\alpha(H \otimes H_\varphi))$ be the faithful normal representation given by

$$\rho(T) = V\theta(T)V^*, \quad T \in L(\bar{A}).$$

Then we have

$$\rho(L(\bar{A})) = p_\alpha(A \rtimes_\alpha \mathbf{K})p_\alpha.$$

Proof. For $x, y \in A$, we have

$$\langle xy, xy \rangle = E(y^* x^* xy) \leq \|x\|^2 E(y^* y) = \|x\|^2 \langle y, y \rangle.$$

Hence, for a fixed $x \in A$, the map $y \in A \rightarrow xy \in A$ is a bounded A^α -module map, so it extends uniquely to a map $m_x \in L(\bar{A})$. Then it is easy to check that

$$\rho(m_x)\alpha(y)(\xi \otimes \Lambda_\varphi(1)) = \alpha(x)\alpha(y)(\xi \otimes \Lambda_\varphi(1))$$

and

$$\rho(m_x) = p_\alpha \alpha(x) p_\alpha, \quad x \in A.$$

For $\omega \in M_*$, we define a map $k_\omega : A \rightarrow A$ by

$$k_\omega(x) = (1 \otimes \omega \circ \kappa)(\alpha(x)), \quad x \in A.$$

Let F be the set of all states ω on M with the property that the Radon–Nikodym derivative $d\omega \circ \kappa/d\varphi$ of $\omega \circ \kappa$ with respect to φ is in M . For $\omega \in F$, setting $h = d\omega \circ \kappa/d\varphi (\in M_+)$, we have

$$\begin{aligned}
 \langle k_\omega(x), k_\omega(x) \rangle &= E((1 \otimes \omega \circ \kappa)(\alpha(x))^*(1 \otimes \omega \circ \kappa)(\alpha(x))) \\
 &\leq E((1 \otimes \omega \circ \kappa)(\alpha(x^*x))) \\
 &= (1 \otimes \varphi \otimes \omega \circ \kappa)((\alpha \otimes 1)(\alpha(x^*x))) \\
 &\leq \|h\|(1 \otimes \varphi \otimes \varphi)((\alpha \otimes 1)(\alpha(x^*x))) \\
 &= \|h\|(1 \otimes \varphi \otimes \varphi)((1 \otimes \Gamma)(\alpha(x^*x))) \\
 &= \|h\|(1 \otimes \varphi)(\alpha(x^*x)) \\
 &= \|h\|\langle x, x \rangle.
 \end{aligned}$$

Hence k_ω is a bounded A^α -module map. (We denote again by k_ω its extension.) By Lemma 1, we get

$$\rho(k_\omega)\alpha(x)(\xi \otimes \Lambda_\varphi(1)) = (1 \otimes \lambda^\sigma(\omega))\alpha(x)(\xi \otimes \Lambda_\varphi(1))$$

and

$$\rho(k_\omega) = p_\alpha(1 \otimes \lambda^\sigma(\omega))p_\alpha, \quad \omega \in F.$$

Since F is total in M_* , we conclude that $p_\alpha(A \times_\alpha \mathbf{K})p_\alpha \subseteq \rho(L(\bar{A}))$.

To show the reverse inclusion, we first consider the rank-one operator $1 \otimes 1^c \in L(\bar{A})$. Then, for $x \in A$, $\xi \in H$,

$$\begin{aligned}
 \rho(1 \otimes 1^c)\alpha(x)(\xi \otimes \Lambda_\varphi(1)) &= V(E(x) \otimes_{A^\alpha} \xi) \\
 &= V(1 \otimes_{A^\alpha} E(x)\xi) \\
 &= E(x)\xi \otimes \Lambda_\varphi(1).
 \end{aligned}$$

Therefore, $\rho(1 \otimes 1^c)$ is the projection from $p_\alpha(H \otimes H_\varphi)$ onto $H \otimes \Lambda_\varphi(1)$, i.e.

$$\rho(1 \otimes 1^c) = p_\alpha(1 \otimes \lambda^\sigma(\varphi))p_\alpha \in p_\alpha(A \times_\alpha \mathbf{K})p_\alpha.$$

For $x, y, z \in A$ and $\xi \in H$,

$$\begin{aligned}
 \rho(x \otimes y^c)\alpha(z)(\xi \otimes \Lambda_\varphi(1)) &= V(xE(y^*z) \otimes_{A^\alpha} \xi) \\
 &= V(x \otimes_{A^\alpha} E(y^*z)\xi) \\
 &= \alpha(x)(E(y^*z)\xi \otimes \Lambda_\varphi(1)) \\
 &= \alpha(x)\rho(1 \otimes 1^c)\alpha(y)^*\alpha(z)(\xi \otimes \Lambda_\varphi(1)).
 \end{aligned}$$

We have

$$\rho(x \otimes y^c) = p_\alpha(\alpha(x)\rho(1 \otimes 1^c)\alpha(y)^*)p_\alpha \in p_\alpha(A \times_\alpha \mathbf{K})p_\alpha.$$

By Paschke [7], the linear span of rank-one operators $x \otimes y^c$ ($x, y \in A$) is dense in $L(\bar{A})$. We get $\rho(L(\bar{A})) \subseteq p_\alpha(A \times_\alpha \mathbf{K})p_\alpha$. $\square$

COROLLARY 5. *The following conditions are equivalent:*

- (i) $A \times_\alpha \mathbf{K}$ is a factor;
- (ii) A^α is a factor and $p_\alpha = 1$.

Proof. The assertion follows from Rieffel's results [8] on Morita equivalence for von Neumann algebras. (See also Paschke [7].) $\square$

PROPOSITION 6. *If $p_\alpha = 1$, i.e. $\alpha(A)(H \otimes \Lambda_\varphi(1)) = H \otimes H_\varphi$, then the action α is faithful in the sense of Izumi et al [4]; that is, the linear span of $\{(\psi \otimes 1)(\alpha(x)) \mid x \in A, \psi \in A_*\}$ is σ -weakly dense in M .*

Proof. We assume that $\omega \in M_*$ satisfies

$$\omega((\psi \otimes 1)(\alpha(x))) = 0, \quad \forall x \in A, \quad \forall \psi \in A_*.$$

We have to show that $\omega = 0$. We have

$$(1 \otimes \omega)(\alpha(x)) = 0, \quad \forall x \in A.$$

By Lemma 1,

$$(1 \otimes \lambda^\sigma(\omega \circ \kappa))\alpha(x)(\xi \otimes \Lambda_\varphi(1)) = \alpha((1 \otimes \omega)(\alpha(x)))(\xi \otimes \Lambda_\varphi(1)) = 0.$$

Since $p_\alpha = 1$, we have $\lambda^\sigma(\omega \circ \kappa) = 0$. Thanks to the faithfulness of λ^σ , we obtain $\omega = 0$. $\square$

COROLLARY 7. *The following conditions are equivalent:*

- (i) *minimality in the sense of Wassermann:* $(A^\alpha)' \cap A = \mathbf{C}$ and $A \times_\alpha \mathbf{K}$ is a factor;
- (ii) *minimality of Izumi et al:* $(A^\alpha)' \cap A = \mathbf{C}$ and the linear span of $\{(\psi \otimes 1)(\alpha(x)) \mid x \in A, \psi \in A_*\}$ is σ -weakly dense in M .

Proof. The implication (i) $\rightarrow$ (ii) follows from Corollary 5 and Proposition 6. The other implication has been proved by Izumi et al [4]. $\square$

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