AN INVERSE BOUNDARY VALUE PROBLEM FOR DIRAC OPERATORS WITH SMALL POTENTIALS

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1. Introduction

In this paper we show that scalar potential and magnetic field are uniquely determined by a Dirichlet to Dirichlet map, which is defined through solutions of a Dirichlet boundary value problem for a Dirac operator on a bounded domain in $\mathbb{R}^3$.

We consider a Dirac operator

$$\begin{aligned} L_{\vec{a},q} &= L_0 + (-\alpha\vec{a}(x) + q(x)) = \alpha D + \beta + (-\alpha\vec{a}(x) + q(x)) \\ &= \sum_{j=1}^3 \alpha_j (D_j - a_j(x)) + \beta + q(x), \end{aligned} \quad (1.1)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $D_j = -i\partial/\partial x_j$, and α_j , $j = 1, 2, 3$, β are Dirac matrices, i.e. 4×4 Hermitian matrices satisfying the anti-commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I, \quad j, k = 1, 2, 3, 4, \quad (1.2)$$

with the convention $\alpha_4 = \beta$. Let

$$P_{\pm} = (I \pm \beta)/2 \quad (1.3)$$

be orthogonal projections on $\mathbb{C}^4$. We assume that the scalar potential q is of the form

$$q(x) = q^+(x)P_+ + q^-(x)P_-,$$

where $q^{\pm}(x)$ are real valued functions, and that the vector potential $\vec{a} = (a_1, a_2, a_3)$ is an $\mathbb{R}^3$ -valued function.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a connected smooth boundary Γ . When $\vec{a}, q \in L^{\infty}(\Omega)$, $L_{\vec{a},q}$ is a selfadjoint operator on $(L^2(\Omega))^4$ with the domain

$$D(L_{\vec{a},q}) = \{u \in (L^2(\Omega))^4 \mid P_+ u \in (H_0^1(\Omega))^4, P_- u \in \mathcal{H}\},$$

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where

$$\mathcal{H} = \overline{(H^1(\Omega))^4}^{\|\cdot\| + \|\alpha D \cdot\|},$$

with $L^2(\Omega)$ -norm $\|\cdot\|$. If $1 \in \rho(L_{\vec{a},q})$ (resolvent set of $L_{\vec{a},q}$), then for any boundary value $f \in P_+(H^{1/2}(\Gamma))^4$ there exists a unique solution u with $P_+u \in (H^1(\Omega))^4$ and $P_-u \in \mathcal{H}$, of the Dirichlet boundary value problem

$$\begin{cases} (\alpha(D - \vec{a}(x)) + \beta + q(x))u(x) = u(x), & \text{in } \Omega, \\ P_+u|_\Gamma = f \in P_+(H^{1/2}(\Gamma))^4. \end{cases} \quad (1.4)$$

The unique solvability can be seen in the following way. For $f \in P_+(H^{1/2}(\Gamma))^4$, there exists $\varphi \in (H^1(\Omega))^4$ such that $P_+\varphi|_\Gamma = f$. We seek a solution of the form $u = \varphi + h$, $h \in D(L_{\vec{a},q})$. If $(\alpha(D - \vec{a}) + \beta + q)u = u$, then $(L_{\vec{a},q} - 1)h = -(\alpha(D - \vec{a}) + \beta + q - 1)\varphi \in (L^2(\Omega))^4$, so $h = -(L_{\vec{a},q} - 1)^{-1}(\alpha(D - \vec{a}) + \beta + q - 1)\varphi$, if $1 \in \rho(L_{\vec{a},q})$. The uniqueness follows since $P_+u \in (H^1(\Omega))^4$, $P_-u \in \mathcal{H}$ and $P_+u|_\Gamma = 0$ imply $u \in D(L_{\vec{a},q})$, so $(L_{\vec{a},q} - 1)u = 0$ implies $u = 0$.

Here we define a Dirichlet to Dirichlet (D-D) map $\Lambda_{\vec{a},q}$, which maps $P_+(H^{1/2}(\Gamma))^4$ into $(H^{-1/2}(\Gamma))^4$ as follows.

Definition. For any $f \in P_+(H^{1/2}(\Gamma))^4$

$$\Lambda_{\vec{a},q}f = i\alpha NP_-u|_\Gamma \in (H^{-1/2}(\Gamma))^4,$$

where u is the unique solution of (1.4) and N is the unit outer normal on Γ .

We note that the D-D map $\Lambda_{\vec{a},q}$ is invariant under a gauge transformation in the vector potential: if $p \in W^{1,\infty}(\Omega)$ and $p|_\Gamma = 0$, then $\Lambda_{\vec{a}+\nabla p,q} = \Lambda_{\vec{a},q}$.

The aim of this paper is to show that $\Lambda_{\vec{a},q}$ determines $\text{rot } \vec{a}$ and q uniquely under the assumptions that potentials $\vec{a}, q$ are small. We put $W_\Omega^{1,\infty} = \{u \in W^{1,\infty}(\mathbb{R}^3) \mid \text{supp } u \subset \overline{\Omega}\}$ and $C_\Omega^2 = \{u \in C^2(\mathbb{R}^3) \mid \text{supp } u \subset \overline{\Omega}\}$. Our main theorems are the following.

THEOREM 1. Let $\vec{a}_j \in C_\Omega^2$ and $q_j \in W^{1,\infty}(\Omega)$, $j = 1, 2$. Assume that $1 \in \rho(L_{\vec{a}_j,q_j})$, $j = 1, 2$. Then there exists a constant $\varepsilon > 0$ such that if $\|\text{rot } \vec{a}_j\|_{L^\infty(\Omega)} \leq \varepsilon$, $j = 1, 2$, and $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, then $\text{rot } \vec{a}_1 = \text{rot } \vec{a}_2$ and $q_1 = q_2$ in Ω .

THEOREM 2. Let $\vec{a}_j \in W_\Omega^{1,\infty}$ and $q_j \in W^{1,\infty}(\Omega)$, $j = 1, 2$. Assume that $1 \in \rho(L_{\vec{a}_j,q_j})$, $j = 1, 2$. Then there exists a constant $\varepsilon > 0$ such that if $\|\vec{a}_j\|_{W^{1,\infty}(\Omega)} + \|q_j\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$, $j = 1, 2$, and $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, then $\text{rot } \vec{a}_1 = \text{rot } \vec{a}_2$ and $q_1 = q_2$ in Ω .

For Schrödinger operators on $\Omega \subset \mathbb{R}^n$, $n \geq 3$, a Dirichlet–Neumann map can be defined and the map determines $\text{rot } \vec{a}$ and q uniquely under some assumptions for potentials. In [2], the case where $\vec{a} \equiv 0$ and $q \in L^\infty$ was treated. In [4], the case where $\vec{a} \in W_\Omega^{2,\infty}$, $q \in L^\infty$ and $\|\text{rot } \vec{a}\|_{L^\infty}$ is small was treated, and in [3], the two cases where $\vec{a}, q \in C^\infty(\bar{\Omega})$ and $\vec{a} \in C_0^\infty(\bar{\Omega})$ and $q \in L^\infty$ without the smallness assumption for vector potentials were treated.

Our theorems are proved in a similar way to [4] and [1]: we construct a family of exponentially growing functions in a certain direction satisfying the equation $L_{\vec{a},q}u = u$ and investigate the asymptotic behavior of the functions as the exponential growth rate increases.

We give the proofs of Theorems 2 and 3 in Sections 2 and 3, respectively.

2. Proof of Theorem 1

For $a \in \mathbb{C}^3$, let $\alpha a = \sum_{j=1}^3 \alpha_j a_j$. Throughout this paper we will often use the following equalities: for $a, b \in \mathbb{C}^3$,

$$\alpha a \alpha b + \alpha b \alpha a = 2abI_4, \quad \text{in particular } (\alpha a)^2 = a^2I_4, \quad (2.1a)$$

$$\alpha a P_\pm = P_\mp \alpha a, \quad (2.1b)$$

which follow by (1.2) and (1.3). We omit ‘ $\rightarrow$ ’ of the vector $\vec{a}$ in Sections 2 and 3.

LEMMA 2.1. *If u_j satisfies $L_{a_j,q_j}u_j = u_j$, $u_j^+ \in (H^1(\Omega))^4$, $u_j^- \in \mathcal{H}$, $j = 1, 2$, then*

$$_{H^{1/2}(\Gamma)} \langle \overline{u_2^+}, (\Lambda_{a_1,q_1} - \Lambda_{a_2,q_2})u_1^+ \rangle_{H^{-1/2}(\Gamma)} = \int_\Omega \overline{u_2} \cdot (V_1 - V_2)u_1 dx,$$

where $u_j^\pm = P_\pm u_j$ and $V_j = -\alpha a_j + q_j$, $j = 1, 2$.

Proof. By $L_{a_j,q_j}u_j = u_j$, $j = 1, 2$, and integration by parts, we have

$$\begin{aligned} & \int_\Omega \overline{u_2} \cdot (V_1 - V_2)u_1 dx \\ &= \int_\Omega (\overline{L_0 u_2} \cdot u_1 - \overline{u_2} \cdot L_0 u_1) dx \\ &= i \int_\Gamma \overline{\alpha N u_2} \cdot u_1 dS \\ &= i \int_\Gamma (\overline{u_2^+} \cdot \alpha N u_1^- + \overline{\alpha N u_2^-} \cdot u_1^+) dS \quad (\text{by (2.1b)}) \\ &= {}_{H^{1/2}(\Gamma)} \langle \overline{u_2^+}, \Lambda_{a_1,q_1} u_1^+ \rangle_{H^{-1/2}(\Gamma)} - {}_{H^{-1/2}(\Gamma)} \langle \overline{\Lambda_{a_2,q_2} u_2^+}, u_1^+ \rangle_{H^{1/2}(\Gamma)}. \end{aligned}$$

Moreover, it follows that for any $f, g \in P_+(H^{1/2}(\Gamma))^4$, ${}_{H^{-1/2}(\Gamma)}\langle \overline{\Lambda_{a_2, q_2} f}, g \rangle_{H^{1/2}(\Gamma)} = {}_{H^{1/2}(\Gamma)}\langle \tilde{f}, \Lambda_{a_2, q_2} g \rangle_{H^{-1/2}(\Gamma)}$, because this is the case $V_1 = V_2$ in the calculation above. Hence we get Lemma 2.1. $\square$

From this lemma if $\Lambda_{a_1, q_1} = \Lambda_{a_2, q_2}$, then we have

$$\int_{\Omega} \overline{u_2} \cdot (V_1 - V_2) u_1 \, dx = 0, \quad (2.2)$$

which plays a key role in the proofs of Theorems 1 and 2.

For any $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3$, let $\zeta^2 = \sum_{j=1}^3 \zeta_j^2$. We put $Z = \{\zeta \in \mathbb{C}^3 \mid \zeta^2 = 0, |\zeta| > 1\}$ and $\Pi_{\zeta} = P_+ + \alpha\zeta/2$. Then $(\alpha\zeta + \beta)\Pi_{\zeta} = \Pi_{\zeta}$ for $\zeta \in Z$ by (1.2), (1.3) and (2.1a).

Assume that $a \in C_{\Omega}^2$ and $q \in W^{1,\infty}(\Omega)$. We look for a solution of $(L_{a,q} - I)u = 0$ of the form: for $\zeta \in Z$, with (4×4) -matrix-valued functions u_{ζ}, v_{ζ} ,

$$u_{\zeta}(x) = e^{i\zeta x + \varphi_{\zeta}(x)} (\Pi_{\zeta} + v_{\zeta}(x)). \quad (2.3)$$

Here $\varphi_{\zeta}(x)$ satisfies an equation

$$\zeta \cdot (a(x) - D\varphi_{\zeta}(x)) = 0 \quad (2.4)$$

and we take a solution of (2.4) of the form

$$\varphi_{\zeta}(x) = \mathcal{F}^{-1} \left(\frac{\zeta \hat{a}(\xi)}{\zeta \xi} \right). \quad (2.5)$$

LEMMA 2.2. *There exists a constant C independent of $\zeta \in Z$ such that*

$$\begin{aligned} \|\varphi_{\zeta}\|_{W^{2,\infty}(\mathbb{R}^3)} &\leq C \|a\|_{W^{2,\infty}(\Omega)}, \\ \|a - D\varphi_{\zeta}\|_{L^{\infty}(\mathbb{R}^3)} &\leq C \|\text{rot } a\|_{L^{\infty}(\Omega)}. \end{aligned}$$

If $\zeta, \zeta_0 \in Z$, $\zeta \rightarrow \zeta_0$, then $\varphi_{\zeta} \rightarrow \varphi_{\zeta_0}$ in $W^{2,\infty}(\mathbb{R}^3)$.

Proof. The above two estimates are the same as (2.7) and (2.8) in [4]. The latter follows from the fact that for any $f \in C_0(\mathbb{R}^3)$ and $\zeta = \eta + i\gamma \in Z$, with $|\eta| = |\gamma| = 1$,

$$\begin{aligned} (L_{\zeta} f)(x) &\equiv \mathcal{F}^{-1} \left(\frac{\hat{f}(\xi)}{\zeta \xi} \right) \\ &= \frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - \eta y_1 - \gamma y_2)}{y_1 + iy_2} \, dy_1 \, dy_2 \rightarrow (L_{\zeta_0} f)(x) \text{ in } L^{\infty}(\mathbb{R}^3) \end{aligned}$$

(cf. [4]). $\square$

We denote by $L^{2,\delta}$, $\delta \in \mathbb{R}$, the weighted L^2 space

$$L^{2,\delta} = \{f \in L^2_{\text{loc}}(\mathbb{R}^3) \mid (1 + |x|^2)^{\delta/2} f(x) \in L^2(\mathbb{R}^3)\}$$

with norm

$$\|f\|_\delta = \|(1 + |x|^2)^{\delta/2} f(x)\|_{L^2},$$

and denote by $H^{\alpha,\delta}$, $\alpha \geq 0$, $\delta \in \mathbb{R}$, the weighted Sobolev space

$$H^{\alpha,\delta} = \{f \in L^{2,\delta} \mid (1 + |x|^2)^{\delta/2} f(x) \in H^\alpha(\mathbb{R}^3)\}$$

with norm

$$\|f\|_{\alpha,\delta} = \|(1 + |x|^2)^{\delta/2} f(x)\|_{H^\alpha}.$$

Substituting (2.3) in $(L_{a,q} - I)u = 0$ and using $(\alpha\zeta + \beta)\Pi_\zeta = \Pi_\zeta$, we get an equation for $v = v_\zeta$:

$$\begin{aligned} (\alpha(D + \zeta) - 2P_- + Q_\zeta)v &= -Q_\zeta \Pi_\zeta, \\ \text{with } Q_\zeta &= -\alpha(a - D\varphi_\zeta) + q. \end{aligned} \quad (2.6)$$

Let $\tilde{q} \in W_0^{1,\infty}(\mathbb{R}^3) = \{q \in W^{1,\infty}(\mathbb{R}^3); q \text{ has compact support}\}$ be an extension of $q \in W^{1,\infty}(\Omega)$ and let $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi = 1$ in a neighborhood of $\overline{\Omega}$ and put

$$b_\zeta \equiv \chi(a - D\varphi_\zeta) \quad \text{and} \quad \tilde{Q}_\zeta \equiv -\alpha b_\zeta + \tilde{q} \in W_0^{1,\infty}(\mathbb{R}^3).$$

Then $\zeta b_\zeta = 0$ by (2.4).

We replace Q_ζ by $\tilde{Q}_\zeta$ in (2.6) and consider the following equation for $v_\zeta \in L^{2,-s}$, $1/2 < s < 1$:

$$(\alpha(D + \zeta) - 2P_- + \tilde{Q}_\zeta)v_\zeta = -\tilde{Q}_\zeta \Pi_\zeta, \quad \text{in } \mathbb{R}^3. \quad (2.7)$$

In what follows we always assume $1/2 < s < 1$.

PROPOSITION 2.3. *For $\zeta \in Z$ we put*

$$(g_\zeta f)(x) \equiv \mathcal{F}^{-1} \left(\frac{\hat{f}(\xi)}{\xi^2 + 2\zeta\xi} \right) (x) \quad \text{for } f \in \mathcal{S}(\mathbb{R}^3).$$

Then g_ζ extends to a bounded operator in $B(L^{2,s}, H^{2,-s})$ and

$$\|g_\zeta f\|_{\alpha,-s} \leq C|\zeta|^{\alpha-1} \|f\|_s, \quad 0 \leq \alpha \leq 2.$$

Proof. We refer to the proof of Theorem 1.1 in [6].

PROPOSITION 2.4. *Let $\zeta \in Z$ and $f \in L^{2,s}$. $(\alpha(D + \zeta) - 2P_-)u = f$ has a unique solution $u \in L^{2,-s}$ and*

$$u = (\alpha(D + \zeta) + 2P_+)g_\zeta f.$$

Hence by Proposition 2.3 we have $u \in H^{1,-s}$.

Proof. By (2.1a, b) we have $(\alpha(D + \zeta) - 2P_-)(\alpha(D + \zeta) + 2P_+) = D^2 + 2\zeta D$ and apply Proposition 2.1 in [5] to get the proposition. $\square$

Since $\tilde{Q}_\zeta \in W_0^{1,\infty}(\mathbb{R}^3)$ and by Proposition 2.3, $(\alpha(D + \zeta) + 2P_+)g_\zeta \tilde{Q}_\zeta$ is a compact operator on $H^{1,-s}$.

PROPOSITION 2.5. *There exist $\varepsilon > 0$ and $r > 1$ such that if $\|\text{rot } a\|_{L^\infty(\Omega)} < \varepsilon$ and $|\zeta| > r$, $\zeta \in Z$, then (2.7) has a unique solution $v_\zeta \in H^{1,-s}$ and*

$$v_\zeta = -(I + F_\zeta g_\zeta M_\zeta)^{-1} F_\zeta g_\zeta M_\zeta \Pi_\zeta. \quad (2.8)$$

Here M_ζ is a multiplication operator

$$M_\zeta \equiv \alpha D(-\alpha b_\zeta + \tilde{q}) + (b_\zeta^2 - \tilde{q}^+ \tilde{q}^- + 2\tilde{q}^+)I_4, \quad (2.9)$$

and $F_\zeta \equiv (I - g_\zeta 2b_\zeta D)^{-1}$ is a bounded operator on $H^{1,-s}$.

Proof. First we show the uniqueness. By Proposition 2.4, (2.7) for $v_\zeta \in L^{2,-s}$ is equivalent to

$$v_\zeta = -(\alpha(D + \zeta) + 2P_+)g_\zeta(\tilde{Q}_\zeta v_\zeta + \tilde{Q}_\zeta \Pi_\zeta). \quad (2.10)$$

Hence $v_\zeta \in H^{1,-s}$ by Proposition 2.3. By the Fredholm alternative it suffices to show that

$$(I + (\alpha(D + \zeta) + 2P_+)g_\zeta \tilde{Q}_\zeta)v = 0, \quad v \in H^{1,-s} \quad (2.11)$$

implies $v = 0$. (2.11) yields that

$$(\alpha(D + \zeta) - 2P_-)v = -\tilde{Q}_\zeta v, \quad (2.12)$$

$$(D^2 + 2\zeta D)v = -(\alpha(D + \zeta) + 2P_+)\tilde{Q}_\zeta v. \quad (2.13)$$

By (2.1a, b), (2.4) and (2.12), the right-hand side of (2.13) equals

$$-(\alpha(D + \zeta) + 2P_+)\tilde{Q}_\zeta v = 2b_\zeta Dv - M_\zeta v.$$

So from (2.13) we get

$$(D^2 + 2\zeta D)v = 2b_\zeta Dv - M_\zeta v. \quad (2.14)$$

The right-hand side of (2.14) is in $L^{2,s}$ since $v \in H^{1,-s}$. Therefore (2.14) is equivalent to (cf. [5, Proposition 2.1])

$$(I - g_\zeta 2b_\zeta D)v = -g_\zeta M_\zeta v. \quad (2.15)$$

From Lemma 2.2 and Proposition 2.3, it follows that

$$\|g_\zeta 2b_\zeta Dw\|_{1,-s} \leq C \|\operatorname{rot} a\|_{L^\infty} \|w\|_{1,-s}, \quad w \in H^{1,-s},$$

and hence when $\|\operatorname{rot} a\|_{L^\infty}$ is sufficiently small, there exists a bounded inverse $F_\zeta = (I - g_\zeta 2b_\zeta D)^{-1}$ on $H^{1,-s}$. Here we prepare a lemma.

LEMMA 2.6. *For any $f \in L^{2,s}$ we have*

$$\|F_\zeta g_\zeta f\|_{-s} \leq C |\zeta|^{-1} \|f\|_s.$$

The proof of this lemma is given after the proof of Proposition 2.5.

Since $\|M_\zeta\|_{L^\infty} \leq C$ (C is independent of ζ), we have

$$\|v\|_{-s} = \|F_\zeta g_\zeta M_\zeta v\|_{-s} \leq C |\zeta|^{-1} \|M_\zeta v\|_s \leq C |\zeta|^{-1} \|v\|_{-s}$$

and therefore $v = 0$ for $|\zeta|$ large.

Next we show (2.8). By (2.7) we have

$$(D^2 + 2\zeta D)v_\zeta = -(\alpha(D + \zeta) + 2P_+)(\tilde{Q}_\zeta v_\zeta + \tilde{Q}_\zeta \Pi_\zeta).$$

We calculate the right-hand side of this, using (2.7) again, to obtain

$$(D^2 + 2\zeta D)v_\zeta = 2b_\zeta Dv_\zeta - M_\zeta v_\zeta - M_\zeta \Pi_\zeta \in L^{2,s}.$$

This is equivalent to

$$(I - g_\zeta 2b_\zeta D)v_\zeta = -g_\zeta M_\zeta v_\zeta - g_\zeta M_\zeta \Pi_\zeta,$$

hence

$$(I + F_\zeta g_\zeta M_\zeta)v_\zeta = -F_\zeta g_\zeta M_\zeta \Pi_\zeta.$$

From Lemma 2.6 it follows that

$$\|F_\zeta g_\zeta M_\zeta w\|_{-s} \leq C|\zeta|^{-1} \|M_\zeta w\|_s \leq C|\zeta|^{-1} \|w\|_{-s}, \quad w \in L^{2,-s}.$$

So for $|\zeta|$ large, $I + F_\zeta g_\zeta M_\zeta$ has a bounded inverse on $L^{2,-s}$. Therefore we obtain

$$v_\zeta = -(I + F_\zeta g_\zeta M_\zeta)^{-1} F_\zeta g_\zeta M_\zeta \Pi_\zeta$$

and have proved Proposition 2.5. $\square$

Proof of Lemma 2.6. From the boundedness of F_ζ and Proposition 2.3, it follows that

$$\|F_\zeta g_\zeta f\|_{1,-s} \leq C \|g_\zeta f\|_{1,-s} \leq C \|f\|_s.$$

From this and Proposition 2.3, putting $w = F_\zeta g_\zeta f$, we have

$$\begin{aligned} \|w\|_{-s} &\leq \|g_\zeta f\|_{-s} + \|g_\zeta 2b_\zeta Dw\|_{-s} \\ &\leq C|\zeta|^{-1} \|f\|_s + C|\zeta|^{-1} \|w\|_{1,-s} \leq C|\zeta|^{-1} \|f\|_s \end{aligned}$$

and hence Lemma 2.6. $\square$

We take a sequence $\{\zeta(\lambda) = \lambda(\omega(\lambda) + i\gamma)\}_{\lambda>1} \subset Z$ such that $\omega(\lambda)$, γ and η are unit vectors in $\mathbb{R}^3$ and $\omega(\lambda) \perp \gamma$, $\eta \perp \gamma$, and $\omega(\lambda) \rightarrow \eta$ as $\lambda \rightarrow \infty$ and we put $\zeta_0 \equiv \lim_{\lambda \rightarrow \infty} \lambda^{-1} \zeta(\lambda) = \eta + i\gamma$.

PROPOSITION 2.7. *For the solution $v_\zeta = v_{\zeta(\lambda)}$ of (2.7) in Proposition 2.5 we have*

$$v_\zeta \rightarrow -N_{\zeta_0} M_{\zeta_0} \frac{\alpha \zeta_0}{2}, \quad \text{in } L^{2,-s} \quad \text{as } \lambda \rightarrow \infty,$$

where $N_{\zeta_0} \in B(L^{2,s}, L^{2,-s})$ is defined by

$$(N_{\zeta_0} f)(x) \equiv \mathcal{F}^{-1} \left(\frac{\hat{f}(\xi)}{2\zeta_0 \xi} \right) (x).$$

Proof. From the proof of Proposition 2.5 we have $(I + F_\zeta g_\zeta M_\zeta)^{-1} \rightarrow I$ ($\lambda \rightarrow \infty$) in the operator norm on $L^{2,-s}$. So by (2.8) we have only to show

$$F_\zeta g_\zeta M_\zeta \Pi_\zeta \rightarrow N_{\zeta_0} M_{\zeta_0} \frac{\alpha \zeta_0}{2}, \quad \text{in } L^{2,-s} \quad (\lambda \rightarrow \infty). \quad (2.16)$$

First, by Lemma 2.2 we have

$$\lambda^{-1} M_\zeta \Pi_\zeta \rightarrow M_{\zeta_0} \frac{\alpha \zeta_0}{2}, \quad \text{in } L^{2,s} \quad (\lambda \rightarrow \infty). \quad (2.17)$$

Next, we verify for any $f \in L^{2,s}$

$$\lambda F_\zeta g_\zeta f \rightarrow N_{\zeta_0} f \quad \text{in } L^{2,-s} \quad (\lambda \rightarrow \infty). \quad (2.18)$$

By Lemma 2.6 it suffices to check (2.18) for $f \in C_0^\infty(\mathbb{R}^3)$. Put $w_\zeta = \lambda F_\zeta g_\zeta f$. From Proposition 2.3 it follows that

$$\|w_\zeta\|_{1,-s} \leq \|F_\zeta\|_{B(H^{1,-s}, H^{1,-s})} \|\lambda g_\zeta f\|_{1,-s} \leq C \|f\|_{1,s}$$

and hence

$$\|g_\zeta 2b_\zeta D w_\zeta\|_{-s} \leq \|g_\zeta\|_{B(L^{2,s}, L^{2,-s})} \|2b_\zeta D w_\zeta\|_s \leq C \lambda^{-1} \|w_\zeta\|_{1,-s} \rightarrow 0 \quad (\lambda \rightarrow \infty). \quad (2.19)$$

Moreover, by Theorem 4.6 in [1] we have for any $f \in L^{2,s}$

$$\lambda g_\zeta f \rightarrow N_{\zeta_0} f \quad \text{in } L^{2,-s} \quad (\lambda \rightarrow \infty). \quad (2.20)$$

(2.19) and (2.20) yield (2.18), and (2.17) and (2.18) yield (2.16). $\square$

Fix $k \neq 0, \eta, \gamma \in \mathbb{R}^3$ such that $k \cdot \eta = k \cdot \gamma = \eta \cdot \gamma = 0, |\eta| = |\gamma| = 1$. We define $\{\zeta_j(\lambda)\}_{\lambda > 1} \subset Z, j = 1, 2$, by

$$\begin{cases} \zeta_1 = \zeta_1(\lambda) = \lambda(\omega_1(\lambda) + i\gamma), & \omega_1(\lambda) = \left(1 - \frac{k^2}{4\lambda^2}\right)^{1/2} \eta - \frac{k}{2\lambda}, \\ \zeta_2 = \zeta_2(\lambda) = \lambda(\omega_2(\lambda) - i\gamma), & \omega_2(\lambda) = \left(1 - \frac{k^2}{4\lambda^2}\right)^{1/2} \eta + \frac{k}{2\lambda}. \end{cases} \quad (2.21)$$

Then we have $\zeta_1^2 = \zeta_2^2 = 0, \overline{\zeta_2} - \zeta_1 = k$ and $\zeta_1/\lambda, \overline{\zeta_2}/\lambda \rightarrow \zeta_0 \equiv \eta + i\gamma (\lambda \rightarrow \infty)$.

For $\zeta_j = \zeta_j(\lambda), j = 1, 2$, we substitute the solution u_{ζ_j} of the form (2.3) of $(L_{a_j, q_j} - I)u_{\zeta_j} = 0, j = 1, 2$, for u_j of (2.2) to get

$$K(\lambda) \equiv \int_{\Omega} e^{-ikx + \varphi_1 + \overline{\varphi_2}} (\Pi_{\zeta_2} + v_{\zeta_2})^* (V_1 - V_2) (\Pi_{\zeta_1} + v_{\zeta_1}) dx = 0.$$

Here A^* denotes the adjoint matrix of A and we put

$$\varphi_j = \mathcal{F}^{-1} \left(\frac{\zeta_j \hat{a}_j(\xi)}{\zeta_j \xi} \right), \quad j = 1, 2.$$

First we show $\text{rot } a_1 = \text{rot } a_2$. By Proposition 2.7 we have

$$\lambda^{-2} K(\lambda) \sim \lambda^{-2} \int_{\Omega} e^{-ikx + \varphi_1 + \overline{\varphi_2}} \Pi_{\xi_2}^* (V_1 - V_2) \Pi_{\xi_1} dx, \quad (2.22)$$

where $A \sim B$ means $A - B = o(1)(\lambda \rightarrow \infty)$. Using $(\Pi_{\xi_2})^* = \Pi_{\bar{\xi}_2}$ and $\lambda^{-1} \Pi_{\xi_1}, \lambda^{-1} \Pi_{\bar{\xi}_2} \rightarrow \alpha \zeta_0 / 2$ and

$$\varphi_1 + \overline{\varphi_2} \rightarrow \psi \equiv \mathcal{F}^{-1} \left(\frac{\zeta_0((\hat{a}_1 - \hat{a}_2)(\xi))}{\zeta_0 \xi} \right), \quad (2.23)$$

we get by (2.22)

$$\begin{aligned} \lambda^{-2} K(\lambda) &\sim \int_{\Omega} e^{-ikx + \psi} \frac{\alpha \zeta_0}{2} (V_1 - V_2) \frac{\alpha \zeta_0}{2} dx \\ &= -\frac{\alpha \zeta_0}{2} \int_{\Omega} e^{-ikx + \psi} \zeta_0 (a_1 - a_2) dx. \end{aligned}$$

Since the left-hand side of this equals zero and $\alpha \zeta_0 \neq 0$, it follows that

$$\int_{\Omega} e^{-ikx + \psi} \zeta_0 (a_1 - a_2) dx = 0.$$

This yields $\text{rot } a_1 = \text{rot } a_2$ by arguments in [4, §4].

Next we show $q_1 = q_2$. Since $\text{rot } a_1 = \text{rot } a_2$ and Γ is connected, there exists $p \in C^1(\mathbb{R}^3)$ such that $a_1 - a_2 = \nabla p$ and $p|_{\Gamma} = 0$. Hence by the gauge invariance, $\Lambda_{a_1, q_1} = \Lambda_{a_2, q_2}$ implies $\Lambda_{a_1, q_1} = \Lambda_{a_1, q_2}$. So we may assume $a_1 = a_2 = a \in C_{\Omega}^2$ to prove $q_1 = q_2$.

PROPOSITION 2.8.

$$P_{\pm} \lambda^{-1} K(\lambda) P_{\pm} \rightarrow \frac{1}{4} \alpha k \int_{\Omega} e^{-ikx} P_{\mp}(q_1 - q_2) P_{\mp} dx \alpha \zeta_0, \quad (\lambda \rightarrow \infty). \quad (2.24)$$

Once this is proved, by replacing ζ_0 by $\bar{\zeta}_0$ in the right-hand side of (2.24) and adding this to (2.24), we have

$$\alpha k \int_{\Omega} e^{-ikx} P_{\mp}(q_1 - q_2) P_{\mp} dx \alpha \eta = 0.$$

We multiply this by αk from the left and by $\alpha \eta$ from the right to obtain

$$k^2 \int_{\Omega} e^{-ikx} P_{\mp}(q_1 - q_2) P_{\mp} dx = 0.$$

Hence this implies $q_1 = q_2$.

Proof of Proposition 2.8. Put $q \equiv q_1 - q_2$. Since $\varphi_1 + \bar{\varphi}_2 \rightarrow 0$ ($\lambda \rightarrow \infty$), it follows that

$$\lambda^{-1} K(\lambda) \sim \int_{\Omega} e^{-ikx} (L_1(\lambda) + L_2(\lambda) + L_3(\lambda) + L_4(\lambda)) dx, \quad (2.25)$$

with

$$\begin{aligned} L_1(\lambda) &= \lambda^{-1} \Pi_{\bar{\zeta}_2} q \Pi_{\zeta_1}, & L_2(\lambda) &= \lambda^{-1} \Pi_{\bar{\zeta}_2} q v_{\zeta_1}, \\ L_3(\lambda) &= \lambda^{-1} v_{\zeta_2}^* q \Pi_{\zeta_1}, & L_4(\lambda) &= \lambda^{-1} v_{\zeta_2}^* q v_{\zeta_1}. \end{aligned}$$

We investigate the limit of each $L_j(\lambda)$, $1 \leq j \leq 4$. For $L_1(\lambda)$ we have

$$\begin{aligned} L_1(\lambda) &= \lambda^{-1} \left(P_+ + \frac{\alpha \bar{\zeta}_2}{2} \right) q \left(P_+ + \frac{\alpha \zeta_1}{2} \right) \\ &\sim \frac{\alpha \zeta_0}{2} q P_+ + P_+ q \frac{\alpha \zeta_0}{2} + \lambda^{-1} \frac{\alpha \bar{\zeta}_2}{2} q \frac{\alpha \zeta_1}{2} \\ &\sim \frac{\alpha \zeta_0}{2} q^+ + \frac{1}{4} \alpha k q \alpha \zeta_0, \end{aligned} \quad (2.26)$$

where we have used $\bar{\zeta}_2 = \zeta_1 + k$ and $(\alpha \zeta_1)^2 = \zeta_1^2 = 0$ (by (2.1a)) in the last step. For $L_2(\lambda)$ we have by Proposition 2.7

$$L_2(\lambda) \sim -\frac{\alpha \zeta_0}{2} q N_{\zeta_0} M_{1,\zeta_0} \frac{\alpha \zeta_0}{2}, \quad (2.27)$$

with

$$M_{1,\zeta_0} = \alpha D(-\alpha b_{\zeta_0} + \tilde{q}_1) + (b_{\zeta_0}^2 - \tilde{q}_1^+ \tilde{q}_1^- + 2\tilde{q}_1^+) I.$$

Using $\tilde{q} \alpha \zeta_0 = \alpha \zeta_0 \tilde{q}^I$ with $\tilde{q}^I \equiv \tilde{q}^+ P_- + \tilde{q}^- P_+$ (by (2.1b)), we have

right-hand side of (2.27)

$$\begin{aligned} &= -\frac{\alpha \zeta_0}{2} q N_{\zeta_0} (\alpha D(-\alpha b_{\zeta_0} + \tilde{q}_1)) \frac{\alpha \zeta_0}{2} \\ &= -\frac{\alpha \zeta_0}{2} q N_{\zeta_0} \left(\alpha D \frac{\alpha \zeta_0}{2} (\alpha b_{\zeta_0} + \tilde{q}_1^I) \right) \\ &= -\frac{\alpha \zeta_0}{2} q N_{\zeta_0} \left[-\frac{\alpha \zeta_0}{2} \alpha D(\alpha b_{\zeta_0} + \tilde{q}_1^I) + \zeta_0 D(\alpha b_{\zeta_0} + \tilde{q}_1^I) \right] \\ &= -\frac{\alpha \zeta_0}{4} q (\alpha b_{\zeta_0} + \tilde{q}_1^I). \end{aligned} \quad (2.28)$$

Here we have used $(\alpha\zeta_0)^2 = \zeta_0^2 = 0$ in the first and fourth equalities, $\zeta_0 b_{\zeta_0} = 0$ in the second equality, (2.1a) in the second and third equalities and $N_{\zeta_0}(\zeta_0 Df) = f/2$ ($f \in H^{1,s}$) in the fourth equality. Similarly, for $L_3(\lambda)$ we have

$$L_3(\lambda) \sim -(\alpha b_{\zeta_0} + \tilde{q}_2^I) q \frac{\alpha \zeta_0}{4}. \quad (2.29)$$

$L_4(\lambda)$ tends to zero clearly. So by (2.25), (2.26), (2.28), (2.29) and $\alpha \zeta_0 q \alpha b_{\zeta_0} + \alpha b_{\zeta_0} q \alpha \zeta_0 = 0$, we get

$$\lambda^{-1} K(\lambda) \sim \int_{\Omega} e^{-ikx} \left(\frac{\alpha \zeta_0}{2} q^+ + \frac{1}{4} \alpha k q \alpha \zeta_0 - \frac{\alpha \zeta_0}{4} q \tilde{q}_1^I - \tilde{q}_2^I q \frac{\alpha \zeta_0}{4} \right) dx.$$

This and $P_{\pm} \alpha \zeta_0 P_{\pm} = 0$ yield

$$P_{\pm} \lambda^{-1} K(\lambda) P_{\pm} \sim \frac{1}{4} P_{\pm} \int_{\Omega} e^{-ikx} \alpha k q \alpha \zeta_0 dx P_{\pm},$$

and hence by (2.1b) we have Proposition 2.8. $\square$

3. Proof of Theorem 2

We can see Theorem 2 as a corollary of the following two theorems.

THEOREM 2(A). Let $\vec{a}_j \in W_{\Omega}^{1,\infty}$ and $q_j \in L^{\infty}(\Omega)$, $j = 1, 2$. Assume that $1 \in \rho(L_{\vec{a}_j, q_j})$, $j = 1, 2$. Then there exists a constant $\varepsilon > 0$ such that if $\|\text{rot } \vec{a}_j\|_{L^{\infty}(\Omega)} + \|q_j\|_{L^{\infty}(\Omega)} \leq \varepsilon$, $j = 1, 2$, and $\Lambda_{\vec{a}_1, q_1} = \Lambda_{\vec{a}_2, q_2}$, then $\text{rot } \vec{a}_1 = \text{rot } \vec{a}_2$ in Ω .

THEOREM 2(B). Let $\vec{a} \in W^{1,\infty}(\Omega)$ and $q_j \in W^{1,\infty}(\Omega)$, $j = 1, 2$. Assume that $1 \in \rho(L_{\vec{a}, q_j})$, $j = 1, 2$. Then there exists a constant $\varepsilon > 0$ such that if $\|\vec{a}\|_{W^{1,\infty}(\Omega)} + \|q_j\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$, $j = 1, 2$, and $\Lambda_{\vec{a}, q_1} = \Lambda_{\vec{a}, q_2}$, then $q_1 = q_2$ in Ω .

Proof of Theorem 2(A). Assume $a \in W_{\Omega}^{1,\infty}$ and $q \in L^{\infty}(\Omega)$. Let $\{\zeta(\lambda)\}_{\lambda>1} \subset Z$ be a sequence as defined just before Proposition 2.7 and recall $\lambda^{-1}\zeta(\lambda) \rightarrow \zeta_0$ ($\lambda \rightarrow \infty$).

For $\zeta = \zeta(\lambda)$ we construct a solution of $(L_{a,q} - I)u = 0$ of the form

$$u_{\zeta}(x) = e^{i\zeta x + \varphi_{\zeta_0}(x)} (I + v_{\zeta}(x)) \Pi_{\zeta}, \quad (3.1)$$

with

$$\varphi_{\zeta_0}(x) = \mathcal{F}^{-1} \left(\frac{\zeta_0 \hat{a}(\xi)}{\zeta_0 \xi} \right) (x)$$

and then v_ζ satisfies

$$(\alpha(D + \zeta) - 2P_- + Q_{\zeta_0})v_\zeta \Pi_\zeta = -Q_{\zeta_0} \Pi_\zeta, \quad (3.2)$$

with

$$Q_{\zeta_0} = -\alpha(a - D\varphi_{\zeta_0}) + q.$$

Let $\tilde{q} \in L^\infty(\mathbb{R}^3)$ be an extension of q by zero outside Ω . Let $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\chi = 1$ in a neighborhood of $\overline{\Omega}$ and put $\tilde{Q}_{\zeta_0} = -\alpha\chi(a - D\varphi_{\zeta_0}) + \tilde{q}$. Then $\tilde{Q}_{\zeta_0}$ is in $L^\infty(\mathbb{R}^3)$ and has compact support. Replacing Q_{ζ_0} by $\tilde{Q}_{\zeta_0}$ and removing Π_ζ in (3.2) we consider the equation for $v_\zeta \in L^{2,-s}$:

$$(\alpha(D + \zeta) - 2P_- + \tilde{Q}_{\zeta_0})v_\zeta = -\tilde{Q}_{\zeta_0} \quad \text{in } \mathbb{R}^3. \quad (3.3)$$

PROPOSITION 3.1. *If $\|\text{rot } a\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)}$ is sufficiently small, there exists a unique solution $v_\zeta \in H^{1,-s}$ of (3.3). Moreover, it follows that*

$$v_\zeta \rightarrow \tilde{v}_{\zeta_0} \equiv -N_{\zeta_0}(\alpha\zeta_0\tilde{Q}_{\zeta_0}) \quad \text{in } L^{2,-s} (\lambda \rightarrow \infty) \quad (3.4)$$

and

$$\tilde{v}_{\zeta_0}\alpha\zeta_0 = 0, \quad (3.5)$$

with N_{ζ_0} given in Proposition 2.7.

Proof. First we show the existence. By hypothesis $v_\zeta \in L^{2,-s}$ and Proposition 2.4, (3.3) is equivalent to

$$v_\zeta + (\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}_{\zeta_0}v_\zeta = -(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}_{\zeta_0}. \quad (3.6)$$

By Proposition 2.3 and Lemma 2.2, we have for any $w \in L^{2,-s}$

$$\begin{aligned} \|(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}_{\zeta_0}w\|_{-s} &\leq \|\alpha Dg_\zeta\tilde{Q}_{\zeta_0}w\|_{-s} + \|(\alpha\zeta + 2P_+)g_\zeta\tilde{Q}_{\zeta_0}w\|_{-s} \\ &\leq \|g_\zeta\tilde{Q}_{\zeta_0}w\|_{1,-s} + C|\zeta|\|g_\zeta\tilde{Q}_{\zeta_0}w\|_{-s} \\ &\leq \|\tilde{Q}_{\zeta_0}w\|_s + C|\zeta|C|\zeta|^{-1}\|\tilde{Q}_{\zeta_0}w\|_s \\ &\leq C(\|\text{rot } a\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)})\|w\|_{-s}. \end{aligned}$$

Thus if $\|\text{rot } a\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)}$ is sufficiently small, (3.3) has a unique solution $v_\zeta \in L^{2,-s}$ and setting $A_\zeta \equiv (\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}_{\zeta_0} \in B(L^{2,-s})$, we can write

$$v_\zeta = -(I + A_\zeta)^{-1}(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}_{\zeta_0},$$

and v_ζ is in $H^{1,-s}$ by (3.6).

Next we show (3.4). By Theorem 4.6 in [1], the following two assertions hold:

$$\begin{aligned} A_\zeta w &\rightarrow \tilde{A}_{\zeta_0} w \equiv N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}_{\zeta_0} w) \quad \text{in } L^{2,-s} \text{ for any } w \in L^{2,-s}(\lambda \rightarrow \infty). \\ (\alpha(D + \zeta) + 2P_+)g_\zeta \tilde{Q}_{\zeta_0} &\rightarrow N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}_{\zeta_0}) \quad \text{in } L^{2,-s}(\lambda \rightarrow \infty). \end{aligned}$$

Hence

$$v_\zeta \rightarrow -(I + \tilde{A}_{\zeta_0})^{-1} N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}_{\zeta_0}) \quad \text{in } L^{2,-s}(\lambda \rightarrow \infty). \quad (3.7)$$

By $\alpha \zeta_0 \tilde{Q}_{\zeta_0} \alpha \zeta_0 = 0$ we have $\tilde{A}_{\zeta_0} N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}_{\zeta_0}) = 0$. This and (3.7) yield (3.4). (3.5) follows from (3.4) and $\alpha \zeta_0 \tilde{Q}_{\zeta_0} \alpha \zeta_0 = 0$. $\square$

We return to the proof of Theorem 2(A). For $\zeta_j = \zeta_j(\lambda)$, $j = 1, 2$, given in (2.21), we substitute the solution u_{ζ_j} of $(L_{a_j, q_j} - I)u_{\zeta_j} = 0$, $j = 1, 2$, for u_j of (2.2) to get

$$K(\lambda) \equiv \int_{\Omega} e^{-ikx+\psi} ((I + v_{\zeta_2})\Pi_{\zeta_2})^* (V_1 - V_2)(I + v_{\zeta_1})\Pi_{\zeta_1} dx = 0.$$

Here ψ is given in (2.23). By Proposition 3.1 we have

$$\lambda^{-2} K(\lambda) \sim \int_{\Omega} e^{-ikx+\psi} \left((I + \tilde{v}_{\zeta_0}) \frac{\alpha \zeta_0}{2} \right)^* (V_1 - V_2)(I + \tilde{v}_{\zeta_0}) \frac{\alpha \zeta_0}{2} dx$$

and so by (3.5)

$$\lambda^{-2} K(\lambda) \sim \int_{\Omega} e^{-ikx+\psi} \frac{\alpha \zeta_0}{2} (V_1 - V_2) \frac{\alpha \zeta_0}{2} dx.$$

Thus in the same way as in the previous section we get

$$\int_{\Omega} e^{-ikx+\psi} \zeta_0 (a_1 - a_2) dx = 0$$

and so $\text{rot } a_1 = \text{rot } a_2$. $\square$

Proof of Theorem 2(B). Assume $a \in W^{1,\infty}(\Omega)$ and $q \in W^{1,\infty}(\Omega)$. For $\zeta = \zeta(\lambda)$ as defined just before Proposition 2.7, we construct a solution of $(L_{a,q} - I)u_\zeta = 0$ of the form

$$u_\zeta(x) = e^{i\zeta x} (I + v_\zeta(x)) \Pi_\zeta. \quad (3.8)$$

Let $\tilde{a}, \tilde{q} \in W_0^{1,\infty}(\mathbb{R}^3)$ be extensions of $a, q \in W^{1,\infty}(\Omega)$ and put $\tilde{Q} = -\alpha \tilde{a} + \tilde{q} \in W_0^{1,\infty}(\mathbb{R}^3)$. We consider the equation for $v_\zeta \in L^{2,-s}$:

$$(\alpha(D + \zeta) - 2P_- + \tilde{Q})v_\zeta = -\tilde{Q} \quad \text{in } \mathbb{R}^3. \quad (3.9)$$

PROPOSITION 3.2. *If $\|\tilde{a}\|_{W^{1,\infty}} + \|\tilde{q}\|_{W^{1,\infty}}$ is sufficiently small, (3.9) has a unique solution $v_\zeta \in H^{1,-s}$ and*

$$v_\zeta \rightarrow \tilde{v}_{\zeta_0} \equiv -\alpha\zeta_0(I - B_{\zeta_0})^{-1}N_{\zeta_0}(\tilde{Q}) \quad \text{in } H^{1,-s}(\lambda \rightarrow \infty), \quad (3.10)$$

where $B_{\zeta_0} \in B(H^{1,-s})$ is defined by

$$B_{\zeta_0}w \equiv \mathcal{F}^{-1}\left(\frac{\mathcal{F}(\zeta_0\tilde{a}w)(\xi)}{\zeta_0\xi}\right) \quad \text{for } w \in H^{1,-s}. \quad (3.11)$$

Moreover, putting

$$v_{\zeta_0}^\bullet \equiv (I - B_{\zeta_0})^{-1}\varphi_{\zeta_0} \quad \text{with } \varphi_{\zeta_0} \equiv \mathcal{F}^{-1}\left(\frac{\zeta_0\hat{\tilde{a}}(\xi)}{\zeta_0\xi}\right), \quad (3.12)$$

we have

$$\tilde{v}_{\zeta_0}\alpha\zeta_0 = v_{\zeta_0}^\bullet\alpha\zeta_0, \quad (3.13)$$

and

$$1 + v_{\zeta_0}^\bullet = e^{\varphi_{\zeta_0}}. \quad (3.14)$$

We note that $v_{\zeta_0}^\bullet$ is a scalar function.

Proof. By Proposition 2.4, (3.9) is equivalent to

$$v_\zeta + (\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}v_\zeta = -(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}. \quad (3.15)$$

We have for any $w \in H^{1,-s}$,

$$\begin{aligned} \|(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}w\|_{1,-s} &\leq C\|\tilde{Q}w\|_{1,s} \\ &\leq C(\|\tilde{a}\|_{W^{1,\infty}} + \|\tilde{q}\|_{W^{1,\infty}})\|w\|_{1,-s}. \end{aligned}$$

Thus if $\|\tilde{a}\|_{W^{1,\infty}} + \|\tilde{q}\|_{W^{1,\infty}}$ is small enough, (3.9) has a unique solution $v_\zeta \in H^{1,-s}$ and we can write with, $C_\zeta \equiv (\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q} \in B(H^{1,-s})$,

$$v_\zeta = -(I + C_\zeta)^{-1}(\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q}.$$

To get (3.10) note that

$$\begin{aligned} C_\zeta w &\rightarrow \tilde{C}_{\zeta_0}w \equiv N_{\zeta_0}(\alpha\zeta_0\tilde{Q}w) \quad \text{in } H^{1,-s} \text{ for any } w \in H^{1,-s} (\lambda \rightarrow \infty). \\ (\alpha(D + \zeta) + 2P_+)g_\zeta\tilde{Q} &\rightarrow N_{\zeta_0}(\alpha\zeta_0\tilde{Q}) \quad \text{in } H^{1,-s} (\lambda \rightarrow \infty). \end{aligned}$$

Thus it follows that

$$v_\zeta \rightarrow -(I + \tilde{C}_{\zeta_0})^{-1} N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}) \quad \text{in } H^{1,-s}(\lambda \rightarrow \infty).$$

Using

$$\begin{aligned} \alpha \zeta_0 \tilde{Q} \alpha \zeta_0 &= \alpha \zeta_0 (-\alpha \tilde{a} + \tilde{q}) \alpha \zeta_0 \\ &= -2\zeta_0 \tilde{a} \alpha \zeta_0 + \alpha \tilde{a} (\alpha \zeta_0)^2 + \tilde{q}^l (\alpha \zeta_0)^2 \quad (\text{by (2.1a)}) \\ &= -2\zeta_0 \tilde{a} \alpha \zeta_0, \end{aligned} \tag{3.16}$$

we get for any integer $n \geq 0$

$$(-\tilde{C}_{\zeta_0})^n N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}) = (B_{\zeta_0})^n N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}),$$

and thus (3.10). (3.13) holds by (3.10) and (3.16).

To get (3.14) we have only to show that for any integer $n \geq 1$

$$(B_{\zeta_0})^{n-1} \varphi_{\zeta_0} = \varphi_{\zeta_0}^n / n!. \tag{3.17}$$

We prove this by induction. It is clear for $n = 1$. Assume that (3.17) is valid for $n \leq l$. Then, since $(B_{\zeta_0})^l \varphi_{\zeta_0} = B_{\zeta_0}(\varphi_{\zeta_0}^l) / l!$, it suffices to verify $(l+1)B_{\zeta_0}(\varphi_{\zeta_0}^l) = \varphi_{\zeta_0}^{l+1}$, $l \geq 1$. This follows from the following two facts:

$$\begin{aligned} \varphi_{\zeta_0}(x) &= \frac{i}{2\pi} \int_{\mathbb{R}^2} \frac{(\zeta_0 \tilde{a})(x - \eta y_1 - \gamma y_2)}{y_1 + i y_2} dy_1 dy_2 \quad (\zeta_0 = \eta + i\gamma), \\ \prod_{k=1}^{l+1} \frac{1}{z_k} &= \sum_{j=1}^{l+1} \frac{1}{z_j \prod_{k \neq j} (z_k - z_j)} \quad \text{for } z_k \in \mathbb{C}, 1 \leq k \leq l+1, \text{ and } z_k \neq z_j, k \neq j. \end{aligned}$$

□

For $\zeta_j = \zeta_j(\lambda)$ $j = 1, 2$, given in (2.21), we substitute the solution u_{ζ_j} of the form (3.8) of $(L_{\tilde{a}, q_j} - I)u_{\zeta_j} = 0$ for u_j of (2.2) to get

$$K(\lambda) \equiv \int_{\Omega} e^{-ikx} ((I + v_{\zeta_2})\Pi_{\zeta_2})^* (q_1 - q_2) (I + v_{\zeta_1})\Pi_{\zeta_1} dx = 0.$$

PROPOSITION 3.3.

$$P_{\pm} \lambda^{-1} K(\lambda) P_{\pm} \rightarrow \frac{1}{4} \alpha k \int_{\Omega} e^{-ikx} P_{\mp} (q_1 - q_2) P_{\mp} dx \alpha \zeta_0 \quad (\lambda \rightarrow \infty).$$

From this proposition we get $q_1 = q_2$ in the same way as in Section 2.

Proof. Put $q \equiv q_1 - q_2$ and write

$$\lambda^{-1} K(\lambda) = \int_{\Omega} e^{-ikx} (L_1(\lambda) + L_2(\lambda) + L_3(\lambda) + L_4(\lambda)) dx,$$

where

$$\begin{aligned} L_1(\lambda) &= \lambda^{-1} P_+(I + v_{\zeta_2}^*) q (I + v_{\zeta_1}) P_+, \\ L_2(\lambda) &= \lambda^{-1} P_+(I + v_{\zeta_2}^*) q (I + v_{\zeta_1}) \frac{\alpha \zeta_1}{2}, \\ L_3(\lambda) &= \lambda^{-1} \frac{\alpha \bar{\zeta}_2}{2} (I + v_{\zeta_2}^*) q (I + v_{\zeta_1}) P_+, \\ L_4(\lambda) &= \lambda^{-1} \frac{\alpha \bar{\zeta}_2}{2} (I + v_{\zeta_2}^*) q (I + v_{\zeta_1}) \frac{\alpha \zeta_1}{2}. \end{aligned}$$

We investigate the limit of each $L_j(\lambda)$, $1 \leq j \leq 4$. It is clear $L_1(\lambda) \rightarrow 0$. For $L_2(\lambda)$, by (3.13) we have

$$\begin{aligned} L_2(\lambda) &\sim P_+(I + (\tilde{v}_{\zeta_0}^*)^*) q (I + \tilde{v}_{\zeta_0}) \frac{\alpha \zeta_0}{2} \\ &= P_+(I + (\tilde{v}_{\zeta_0}^*)^*) \frac{\alpha \zeta_0}{2} q^I (I + v_{\zeta_0}^*) \\ &= P_+ \frac{\alpha \zeta_0}{2} q^I (I + v_{\zeta_0}^*), \end{aligned} \tag{3.18}$$

using $\alpha \bar{\zeta}_0 \tilde{v}_{\zeta_0} = 0$ (by (3.10)) in the second equality. Thus from $P_{\pm} \alpha \zeta_0 P_{\pm} = 0$ and $q^I P_{\pm} = P_{\pm} q^I$, it follows that

$$P_{\pm} L_2(\lambda) P_{\pm} \sim 0. \tag{3.19}$$

In the same way as above we get

$$P_{\pm} L_3(\lambda) P_{\pm} \sim 0. \tag{3.20}$$

We write $L_4(\lambda)$ as $L_4(\lambda) = \sum_{j=1}^4 M_j(\lambda)$, with

$$\begin{aligned} M_1(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 q \alpha \zeta_1, & M_2(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 q v_{\zeta_1} \alpha \zeta_1, \\ M_3(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 v_{\zeta_2}^* q \alpha \zeta_1, & M_4(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 v_{\zeta_2}^* q v_{\zeta_1} \alpha \zeta_1. \end{aligned}$$

For $M_1(\lambda)$, by $\bar{\zeta}_2 = \zeta_1 + k$ and $\alpha\zeta_1 q \alpha\zeta_1 = (\alpha\zeta_1)^2 q^I = 0$, we have

$$M_1(\lambda) = \frac{1}{4\lambda} \alpha(\zeta_1 + k) q \alpha\zeta_1 \sim \frac{1}{4} \alpha k q \alpha\zeta_0. \quad (3.21)$$

For $M_2(\lambda)$, writing $\tilde{Q}_j = -\alpha\tilde{a} + \tilde{q}_j$ $j = 1, 2$, by (3.15) we have

$$v_{\zeta_j} = -\mathcal{F}^{-1} \left[\frac{\alpha(\xi + \zeta_j) + 2P_+}{\xi^2 + 2\zeta_j\xi} \mathcal{F}(\tilde{Q}_j(v_{\zeta_j} + 1)) \right], \quad j = 1, 2. \quad (3.22)$$

We obtain by (2.20) for any $f \in H^{1,s}$,

$$\lambda \mathcal{F}^{-1} \left(\frac{\alpha\xi + 2P_+}{\xi^2 + 2\zeta_1\xi} \hat{f}(\xi) \right) \rightarrow \mathcal{F}^{-1} \left(\frac{\alpha\xi + 2P_+}{2\zeta_0\xi} \hat{f}(\xi) \right) \quad \text{in } L^{2,-s}(\lambda \rightarrow \infty) \quad (3.23)$$

and by Proposition 3.2

$$\tilde{Q}_1 v_{\zeta_1} \rightarrow \tilde{Q}_1 \tilde{v}_{\zeta_0} \quad \text{in } H^{1,s}(\lambda \rightarrow \infty). \quad (3.24)$$

So (3.22)–(3.24) yield that

$$\begin{aligned} v_{\zeta_1} \alpha\zeta_1 &\sim -\mathcal{F}^{-1} \left[\frac{\alpha\xi + 2P_+}{2\zeta_0\xi} \mathcal{F}(\tilde{Q}_1(\tilde{v}_{\zeta_0} + 1)) \right] \alpha\zeta_0 \\ &\quad -\mathcal{F}^{-1} \left[\frac{\alpha\zeta_1}{\xi^2 + 2\zeta_1\xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha\zeta_1. \end{aligned}$$

Thus by $\bar{\zeta}_2 = \zeta_1 + k$, (2.20) and (3.13), we have

$$\begin{aligned} M_2(\lambda) &\sim -\frac{1}{4} \alpha\zeta_0 q \mathcal{F}^{-1} \left[\frac{\alpha\xi + 2P_+}{2\zeta_0\xi} \mathcal{F}(\tilde{Q}_1(\tilde{v}_{\zeta_0} + 1)) \right] \alpha\zeta_0 \\ &\quad -\frac{1}{4\lambda} \alpha\bar{\zeta}_2 q \alpha\zeta_1 \mathcal{F}^{-1} \left[\frac{1}{\xi^2 + 2\zeta_1\xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha\zeta_1 \\ &\sim -\frac{1}{4} \alpha\zeta_0 q \mathcal{F}^{-1} \left[\frac{\alpha\xi + 2P_+}{2\zeta_0\xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_0}^* + 1)) \right] \alpha\zeta_0 \\ &\quad -\frac{1}{4} \alpha k q \alpha\zeta_0 N_{\zeta_0}(\tilde{Q}_1(v_{\zeta_0}^* + 1)) \alpha\zeta_0 \\ &\equiv M_{2,1} + M_{2,2}. \end{aligned}$$

For $M_{2,1}$ we get

$$\begin{aligned}
 P_{\pm} M_{2,1} P_{\pm} &= -\frac{1}{4} \alpha \zeta_0 q \mathcal{F}^{-1} \left[\frac{\alpha \xi P_{\pm} + 2P_{+} P_{\mp}}{2\zeta_0 \xi} \mathcal{F}((-P_{\pm} \alpha \tilde{a} + P_{\mp} \tilde{q}_1)(v_{\zeta_0}^{\bullet} + 1)) \right] \alpha \zeta_0 \quad (\text{by (2.1b)}) \\
 &= -\frac{1}{4} \alpha \zeta_0 q \mathcal{F}^{-1} \left[\frac{\alpha \xi P_{\pm}}{2\zeta_0 \xi} \mathcal{F}((-P_{\pm} \alpha \tilde{a})(v_{\zeta_0}^{\bullet} + 1)) \right] \alpha \zeta_0 \\
 &\quad - \frac{1}{4} \alpha \zeta_0 q \mathcal{F}^{-1} \left[\frac{2P_{+} P_{\mp}}{2\zeta_0 \xi} \mathcal{F}(P_{\mp} \tilde{q}_1(v_{\zeta_0}^{\bullet} + 1)) \right] \alpha \zeta_0 \\
 &= \frac{1}{4} P_{\pm} \alpha \zeta_0 q X_{\zeta_0} \alpha \zeta_0 P_{\pm}, \tag{3.25}
 \end{aligned}$$

where we have put

$$X_{\zeta_0} \equiv \mathcal{F}^{-1} \left[\frac{\alpha \xi}{2\zeta_0 \xi} \mathcal{F}(\alpha \tilde{a}(v_{\zeta_0}^{\bullet} + 1)) \right], \tag{3.26}$$

and used $(\alpha \zeta_0)^2 = 0$ in the third equality. For $M_{2,2}$

$$\begin{aligned}
 M_{2,2} &= -\frac{1}{4} \alpha k q N_{\zeta_0} (\alpha \zeta_0 (-\alpha \tilde{a} + \tilde{q}_1) \alpha \zeta_0 (v_{\zeta_0}^{\bullet} + 1)) \\
 &= \frac{1}{4} \alpha k q \alpha \zeta_0 \mathcal{F}^{-1} \left[\frac{1}{\zeta_0 \xi} \mathcal{F}(\zeta_0 \tilde{a}(v_{\zeta_0}^{\bullet} + 1)) \right] \quad (\text{by (2.1a)}) \\
 &= \frac{1}{4} \alpha k q \alpha \zeta_0 v_{\zeta_0}^{\bullet}, \tag{3.27}
 \end{aligned}$$

where we have used, in the third equality,

$$v_{\zeta_0}^{\bullet} = \mathcal{F}^{-1} \left[\frac{1}{\zeta_0 \xi} \mathcal{F}(\zeta_0 \tilde{a}(v_{\zeta_0}^{\bullet} + 1)) \right] \quad (\text{by (3.12)}). \tag{3.28}$$

Hence by (3.25) and (3.27)

$$P_{\pm} M_2(\lambda) P_{\pm} \sim \frac{1}{4} P_{\pm} (\alpha \zeta_0 q X_{\zeta_0} \alpha \zeta_0 + \alpha k q \alpha \zeta_0 v_{\zeta_0}^{\bullet}) P_{\pm}. \tag{3.29}$$

For $M_3(\lambda)$, a similar calculation follows:

$$P_{\pm} M_3(\lambda) P_{\pm} \sim \frac{1}{4} P_{\pm} (\alpha \zeta_0 Y_{\zeta_0} q \alpha \zeta_0 + \alpha k q \alpha \zeta_0 \overline{v_{\zeta_0}^{\bullet}}) P_{\pm}, \tag{3.30}$$

with

$$Y_{\zeta_0} \equiv \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^{\bullet}} + 1) \alpha \tilde{a}) \frac{\alpha \xi}{2\zeta_0 \xi} \right] \tag{3.31}$$

and

$$\overline{v_{\zeta_0}^\bullet} = -\mathcal{F}^{-1} \left[\frac{1}{\zeta_0 \xi} \mathcal{F}(\zeta_0 \tilde{a}(\overline{v_{\zeta_0}^\bullet} + 1)) \right]. \quad (3.32)$$

For $M_4(\lambda)$, by (3.22), we put $M_4(\lambda) = \sum_{j=1}^4 N_j(\lambda)$, with

$$\begin{aligned} N_1(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 \mathcal{F}^{-1} \left[\mathcal{F}((v_{\zeta_2}^* + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{\xi^2 - 2\bar{\zeta}_2 \xi} \right] q \\ &\quad \times \mathcal{F}^{-1} \left[\frac{\alpha \xi + 2P_+}{\xi^2 + 2\zeta_1 \xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha \zeta_1, \\ N_2(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 \mathcal{F}^{-1} \left[\mathcal{F}((v_{\zeta_2}^* + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{\xi^2 - 2\bar{\zeta}_2 \xi} \right] q \\ &\quad \times \mathcal{F}^{-1} \left[\frac{\alpha \zeta_1}{\xi^2 + 2\zeta_1 \xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha \zeta_1, \\ N_3(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 \mathcal{F}^{-1} \left[\mathcal{F}((v_{\zeta_2}^* + 1) \tilde{Q}_2) \frac{\alpha \bar{\zeta}_2}{\xi^2 - 2\bar{\zeta}_2 \xi} \right] q \\ &\quad \times \mathcal{F}^{-1} \left[\frac{\alpha \xi + 2P_+}{\xi^2 + 2\zeta_1 \xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha \zeta_1, \\ N_4(\lambda) &= \frac{1}{4\lambda} \alpha \bar{\zeta}_2 \mathcal{F}^{-1} \left[\mathcal{F}((v_{\zeta_2}^* + 1) \tilde{Q}_2) \frac{\alpha \bar{\zeta}_2}{\xi^2 - 2\bar{\zeta}_2 \xi} \right] q \\ &\quad \times \mathcal{F}^{-1} \left[\frac{\alpha \zeta_1}{\xi^2 + 2\zeta_1 \xi} \mathcal{F}(\tilde{Q}_1(v_{\zeta_1} + 1)) \right] \alpha \zeta_1. \end{aligned}$$

$N_1(\lambda)$ tends to zero. For $N_2(\lambda)$, from (3.23), (2.20), (3.13) and (3.28), it follows that

$$\begin{aligned} N_2(\lambda) &\sim \frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\tilde{v}_{\zeta_0}^* + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{-2\zeta_0 \xi} \right] q N_{\zeta_0}(\alpha \zeta_0 \tilde{Q}_1(\tilde{v}_{\zeta_0} + 1)) \alpha \zeta_0 \\ &= \frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^\bullet} + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{-2\zeta_0 \xi} \right] q N_{\zeta_0}(\alpha \zeta_0 (-\alpha \tilde{a})(v_{\zeta_0}^\bullet + 1)) \alpha \zeta_0 \\ &= -\frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^\bullet} + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{-2\zeta_0 \xi} \right] q \alpha \zeta_0 \\ &\quad \times \mathcal{F}^{-1} \left[\frac{1}{\zeta_0 \xi} \mathcal{F}(\zeta_0 \tilde{a}(v_{\zeta_0}^\bullet + 1)) \right] \\ &= -\frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^\bullet} + 1) \tilde{Q}_2) \frac{-\alpha \xi + 2P_+}{-2\zeta_0 \xi} \right] q \alpha \zeta_0 v_{\zeta_0}^\bullet. \end{aligned}$$

So by (2.1b) we get

$$\begin{aligned}
& P_{\pm} N_2(\lambda) P_{\pm} \\
& \sim -\frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^*} + 1)(-\alpha \tilde{a} P_{\pm} + \tilde{q}_2 P_{\mp})) \frac{-P_{\pm} \alpha \xi + 2P_{\mp} P_{+}}{-2\zeta_0 \xi} \right] q \alpha \zeta_0 v_{\zeta_0}^{\bullet} \\
& = -\frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^*} + 1)(-\alpha \tilde{a} P_{\pm})) \frac{P_{\pm} \alpha \xi}{2\zeta_0 \xi} \right] q \alpha \zeta_0 v_{\zeta_0}^{\bullet} \\
& \quad - \frac{1}{4} \alpha \zeta_0 \mathcal{F}^{-1} \left[\mathcal{F}((\overline{v_{\zeta_0}^*} + 1) \tilde{q}_2 P_{\mp}) \frac{2P_{\mp} P_{+}}{-2\zeta_0 \xi} \right] q \alpha \zeta_0 v_{\zeta_0}^{\bullet} \\
& = \frac{1}{4} P_{\pm} \alpha \zeta_0 Y_{\zeta_0} q \alpha \zeta_0 P_{\pm} v_{\zeta_0}^{\bullet} \quad (\text{by } (\alpha \zeta_0)^2 = 0 \text{ and (3.31)}). \tag{3.33}
\end{aligned}$$

Similarly, we have

$$P_{\pm} N_3(\lambda) P_{\pm} \sim \frac{1}{4} P_{\pm} \alpha \zeta_0 q X_{\zeta_0} \alpha \zeta_0 P_{\pm} \overline{v_{\zeta_0}^*}. \tag{3.34}$$

For $N_4(\lambda)$, noting $\alpha k q \alpha \zeta_0 = -\alpha \zeta_0 q \alpha k$ (by $k \zeta_0 = 0$), we obtain, by a similar computation as above,

$$\begin{aligned}
N_4(\lambda) & \sim \frac{1}{4} \alpha \zeta_0 N_{\zeta_0} (-\tilde{v}_{\zeta_0}^* + 1) \tilde{Q}_2 \alpha k q \alpha \zeta_0 N_{\zeta_0} (\tilde{Q}_1 (\tilde{v}_{\zeta_0} + 1)) \alpha \zeta_0 \\
& = \frac{1}{4} \alpha \zeta_0 N_{\zeta_0} (-\overline{v_{\zeta_0}^*} + 1) \tilde{Q}_2 \alpha k q \alpha \zeta_0 N_{\zeta_0} ((-\alpha \tilde{a})(v_{\zeta_0}^{\bullet} + 1)) \alpha \zeta_0 \\
& = -\frac{1}{4} \alpha \zeta_0 N_{\zeta_0} (-\overline{v_{\zeta_0}^*} + 1)(-\alpha \tilde{a}) \alpha \zeta_0 q \alpha k (-v_{\zeta_0}^{\bullet}) \\
& = \frac{1}{4} \overline{v_{\zeta_0}^*} \alpha k q \alpha \zeta_0 v_{\zeta_0}^{\bullet},
\end{aligned}$$

and so

$$P_{\pm} N_4(\lambda) P_{\pm} \sim \frac{1}{4} P_{\pm} \alpha k q \alpha \zeta_0 P_{\pm} \overline{v_{\zeta_0}^*} v_{\zeta_0}^{\bullet}. \tag{3.35}$$

Hence (3.19)–(3.21), (3.29), (3.30) and (3.33)–(3.35) yield

$$\begin{aligned}
P_{\pm} \lambda^{-1} K(\lambda) P_{\pm} & \sim \frac{1}{4} P_{\pm} [\alpha k q \alpha \zeta_0 (1 + v_{\zeta_0}^{\bullet})(1 + \overline{v_{\zeta_0}^*}) \\
& \quad + (1 + v_{\zeta_0}^{\bullet}) \alpha \zeta_0 Y_{\zeta_0} q \alpha \zeta_0 + (1 + \overline{v_{\zeta_0}^*}) \alpha \zeta_0 q X_{\zeta_0} \alpha \zeta_0] P_{\pm}. \tag{3.36}
\end{aligned}$$

To obtain Proposition 3.3 we note that $(1 + v_{\zeta_0}^{\bullet})(1 + \overline{v_{\zeta_0}^*}) = e^{\varphi_{\zeta_0}} e^{\overline{\varphi_{\zeta_0}}} = 1$ and

$$(1 + v_{\zeta_0}^{\bullet}) \alpha \zeta_0 Y_{\zeta_0} q \alpha \zeta_0 + (1 + \overline{v_{\zeta_0}^*}) \alpha \zeta_0 q X_{\zeta_0} \alpha \zeta_0 = 0. \tag{3.37}$$

In fact, by the definitions of X_{ζ_0} and Y_{ζ_0} and (3.14), the left-hand side of (3.37) equals, with the convention $\varphi = \varphi_{\zeta_0}$,

$$e^{\varphi} \alpha \zeta_0 \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^{-\varphi} \alpha \tilde{a}) \alpha \xi}{2\zeta_0 \xi} \right] q \alpha \zeta_0 + e^{-\varphi} \alpha \zeta_0 q \mathcal{F}^{-1} \left[\frac{\alpha \xi \mathcal{F}(\alpha \tilde{a} e^{\varphi})}{2\zeta_0 \xi} \right] \alpha \zeta_0$$

$$\begin{aligned}
&= q' \left(e^\varphi \alpha \zeta_0 \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^{-\varphi} \alpha \tilde{a}) \alpha \xi}{2 \zeta_0 \xi} \right] - e^{-\varphi} \alpha \zeta_0 \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^\varphi \alpha \tilde{a}) \alpha \xi}{2 \zeta_0 \xi} \right] \right) \alpha \zeta_0 \\
&= q' \left(e^\varphi \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^{-\varphi} 2 \zeta_0 \tilde{a}) \alpha \xi}{2 \zeta_0 \xi} \right] - e^{-\varphi} \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^\varphi 2 \zeta_0 \tilde{a}) \alpha \xi}{2 \zeta_0 \xi} \right] \right) \alpha \zeta_0 \\
&\quad - q' \left(e^\varphi \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^{-\varphi} \alpha \tilde{a}) \alpha \zeta_0 \alpha \xi}{2 \zeta_0 \xi} \right] - e^{-\varphi} \mathcal{F}^{-1} \left[\frac{\mathcal{F}(e^\varphi \alpha \tilde{a}) \alpha \zeta_0 \alpha \xi}{2 \zeta_0 \xi} \right] \right) \alpha \zeta_0.
\end{aligned}$$

The first term in the right-hand side vanishes since $\mathcal{F}(e^{\pm\varphi} \zeta_0 \tilde{a}) / \zeta_0 \xi = \pm \mathcal{F}(e^{\pm\varphi} - 1)$ and the second term vanishes since $\alpha \zeta_0 \alpha \xi \alpha \zeta_0 = 2 \zeta_0 \xi \alpha \zeta_0$. $\square$

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