POLYNOMIAL HAMILTONIAN STRUCTURE FOR THE A_4 -SYSTEM

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0. Introduction

Consider a linear ordinary differential equation of the form

$$\frac{d^2 y}{dx^2} + p_1(x, t) \frac{dy}{dx} + p_2(x, t) y = 0, \quad (0.1)$$

defined on the Riemann sphere $\mathbf{P}^1$, with coefficients

$$\begin{aligned} p_1(x, t) &= -A_g(x, t) - \sum_{k=1}^g \frac{1}{x - \lambda_k}, \\ p_2(x, t) &= -(2\alpha + 1)x^g - 2 \sum_{j=1}^g H_j x^{g-j} + \sum_{k=1}^g \frac{\mu_k}{x - \lambda_k}, \end{aligned} \quad (0.2)$$

where g is a positive integer and

$$A_g(x, t) = 2x^{g+1} + \sum_{j=1}^g j t_j x^{j-1}. \quad (0.3)$$

Since $A_g(x, t)$ is a polynomial representing the versal deformation of the singularity of A_g -type, we call (0.1) and (0.2) the linear differential equations of A_g -type.

In this paper, we will study the holonomic deformation of linear differential equations of A_g -type when $g = 4$. Here the holonomic deformation means that there exists a fundamental system of solutions of equations of A_g -type whose monodromy representation and Stokes multipliers at irregular singular points are invariant under the variation of deformation parameters $t = (t_1, \dots, t_g)$. In the case of $g = 1$, by considering the holonomic deformation of linear differential equations of A_1 -type, Okamoto [4] obtained the following Hamiltonian system:

$$\frac{d\lambda}{dt} = \mu - \lambda^2 - \frac{1}{2}t, \quad \frac{d\mu}{dt} = 2\lambda\mu + \alpha + \frac{1}{2},$$

$$L_j = \overline{H}_j \quad (j = 1, \dots, 4). \quad (1.3)$$

Furthermore, throughout this paper we suppose that $\sigma_0 = -1$ and $\sigma_l = 0$ if $l \geq 5$ or $l < 0$. We can prove the following.

PROPOSITION 1.1. *The transformation of the Hamiltonian system $(\overline{H})$ defined by (1.1)–(1.3),*

$$(\lambda, \mu, \overline{H}, t) \rightarrow (\sigma, \rho, L, t)$$

is canonical, where $\lambda = (\lambda_1, \dots, \lambda_4)$, $\mu = (\mu_1, \dots, \mu_4)$, $\overline{H} = (\overline{H}_1, \dots, \overline{H}_4)$ and $t = (t_1, \dots, t_4)$.

Proof. Noting

$$\begin{aligned} \frac{\partial \sigma_j}{\partial \lambda_i} &= (-1)^{j-1} e_{j-1}^{(i)} = N^{ji}, \\ d\sigma_j &= \sum_{k=1}^4 \frac{\partial \sigma_j}{\partial \lambda_k} d\lambda_k = \sum_{k=1}^4 N^{jk} d\lambda_k, \end{aligned}$$

we obtain

$$\begin{aligned} \sum_{j=1}^4 \mu_j d\lambda_j &= \sum_{j=1}^4 \left(\sum_{k=1}^4 N^{kj} \rho_k \right) d\lambda_j \\ &= \sum_{k=1}^4 \sum_{j=1}^4 N^{jk} \rho_j d\lambda_k \\ &= \sum_{j=1}^4 \rho_j \left(\sum_{k=1}^4 N^{jk} d\lambda_k \right) \\ &= \sum_{j=1}^4 \rho_j d\sigma_j. \end{aligned}$$

Hence we have

$$\sum_{k=1}^4 d\rho_k \wedge d\sigma_k - \sum_{j=1}^4 dL_j \wedge dt_j = \sum_{k=1}^4 d\mu_k \wedge d\lambda_k - \sum_{j=1}^4 d\overline{H}_j \wedge dt_j.$$

This proves the Proposition. $\square$

By means of (0.5), (0.6) and (1.1)–(1.3), we compute the explicit form of L_j as a function of the variables σ , ρ and t .

First we rewrite (0.5) in the following form:

$$H_j = \frac{1}{2}(\mu_1, \dots, \mu_4) \begin{pmatrix} N_1 N^{j1} & 0 & 0 & 0 \\ 0 & N_2 N^{j2} & 0 & 0 \\ 0 & 0 & N_3 N^{j3} & 0 \\ 0 & 0 & 0 & N_4 N^{j4} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} \\ - \frac{1}{2}(\mu_1, \mu_2, \mu_3, \mu_4) \begin{pmatrix} U_{j1} \\ U_{j2} \\ U_{j3} \\ U_{j4} \end{pmatrix} - (\alpha + \frac{1}{2}) \sum_{k=1}^4 N_k N^{jk} \lambda_k^4. \quad (1.4)$$

Substituting (1.2) into (1.4), and by some computation, we have

$$H_j = (\rho_1, \rho_2, \rho_3, \rho_4) X^{(j)} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix} + (\rho_1, \rho_2, \rho_3, \rho_4) Y^{(j)} + Z^{(j)}, \quad (1.5)$$

where a 4×4 symmetric matrix $X^{(j)} = (X_{nm}^{(j)})$ is given by

$$X_{nm}^{(j)} = \frac{1}{2} \sum_{k=1}^4 N^{nk} N^{jk} N^{mk} N_k,$$

a 4×1 matrix $Y^{(j)} = (Y_i^{(j)})$ is given by

$$Y_i^{(j)} = -\frac{1}{2} \sum_{k=1}^4 N^{ik} U_{jk},$$

and

$$Z^{(j)} = -(\alpha + \frac{1}{2}) \sum_{k=1}^4 N_k N^{jk} \lambda_k^4.$$

Now we compute $X_{nm}^{(j)}$, $Y_i^{(j)}$ and $Z^{(j)}$ respectively. In order to do that, we first prove the following lemmas.

LEMMA 1.1. *If we define $Q_j(x) = -\frac{1}{2} \sum_{n=0}^{j-1} \sigma_n x^{j-1-n}$ for $j = 1, 2, 3, 4$, then we have*

$$N^{jk} = 2Q_j(\lambda_k). \quad (1.6)$$

where

$$\begin{aligned}
 R_1 &= p_2^2 - q_2 p_3^2 - 2q_1 p_2 p_3 + 2p_1 p_3, \\
 R_2 &= -2q_1 p_2^2 + (q_1 q_2 + \tfrac{1}{2} t_3 q_1 + q_3 - \tfrac{1}{2} t_2) p_3^2 \\
 &\quad + 2p_1 p_2 + (2q_1^2 - t_3) p_2 p_3 - 2q_1 p_1 p_3, \\
 R_3 &= p_1^2 + (q_1^2 - \tfrac{1}{2} t_3) p_2^2 + (q_2^2 - q_1 q_3 - \tfrac{1}{2} t_3 q_1^2 + \tfrac{1}{2} t_2 q_1 - \tfrac{1}{2} t_3 q_2 + \tfrac{1}{4} t_3^2) p_3^2 \\
 &\quad - 2q_1 p_1 p_2 + (2q_1 q_2 + t_3 q_1 + 2q_3 - t_2) p_2 p_3 - 2q_2 p_1 p_3,
 \end{aligned}$$

and $D = (D_{jk})_{j,k=1,2,3}$ is the symmetric matrix defined by

$$\begin{aligned}
 D_{11} &= q_1^2 + q_2 + \tfrac{1}{2} t_3, \\
 D_{22} &= q_2^2 - t_2 q_1 - \tfrac{1}{2} t_3 q_2 + \tfrac{1}{2} t_1 - \tfrac{1}{4} t_3^2, \\
 D_{33} &= q_3^2 + \tfrac{1}{2} t_2 t_3 q_1 - \tfrac{1}{2} t_1 q_2 - \tfrac{1}{4} t_2^2 + \tfrac{1}{8} t_3^3, \\
 D_{12} &= D_{21} = q_1 q_2 - \tfrac{1}{2} t_3 q_1 + q_3 + \tfrac{1}{2} t_2, \\
 D_{13} &= D_{31} = q_1 q_3 - \tfrac{1}{2} t_2 q_1 - \tfrac{1}{2} t_3 q_2 + \tfrac{1}{2} t_1, \\
 D_{23} &= D_{32} = q_2 q_3 + (\tfrac{1}{4} t_3^2 - \tfrac{1}{2} t_1) q_1 - \tfrac{1}{2} t_2 q_2 - \tfrac{1}{2} t_2 t_3.
 \end{aligned}$$

Further, by using the Laplace transformation, we obtained a particular solution of the A_3 -system expressed by the following integral:

$$\int_{\eta_1} \exp \left[- \int_0^{\xi} A_3(x, t) dx - \tfrac{1}{2} t_2 t_3 \right] d\xi. \quad (0.8)$$

In this paper, we generalize these results to the holonomic deformation of A_4 -type, that is, we will determine a polynomial Hamiltonian structure for the A_4 -system and then compute particular solutions of the A_4 -system having integral representations such as (0.8).

1. Polynomial Hamiltonian structure

In this section, we determine a polynomial Hamiltonian structure for the A_4 -system $(\bar{H})$. In order to do that, we consider new variables $\sigma = (\sigma_1, \dots, \sigma_4)$, $\rho = (\rho_1, \dots, \rho_4)$ and $L = (L_1, \dots, L_4)$ defined by

$$\sigma_j = (-1)^{j-1} e_j \quad (j = 1, \dots, 4), \quad (1.1)$$

$$\mu_j = \sum_{k=1}^4 N^{kj} \rho_k \quad (j = 1, \dots, 4), \quad (1.2)$$

with $\Lambda'(x)$ being the derivative of $\Lambda(x) = \prod_{k=1}^4 (x - \lambda_k)$. Throughout this paper, we suppose that H_j are the rational functions given by (0.5).

Using these functions $H_j = H_j(t, \lambda, \mu)$, we proved the following theorem in [3].

THEOREM. ([3]) *The holonomic deformation of linear ordinary differential equations of A_4 -type is governed by the completely integrable Hamiltonian system*

$$\frac{\partial \lambda_k}{\partial t_j} = \frac{\partial \bar{H}_j}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial t_j} = -\frac{\partial \bar{H}_j}{\partial \lambda_k} \quad (k, j = 1, \dots, 4) \quad (\bar{H})$$

with the following Hamiltonian functions $\bar{H}_j$ ($j = 1, \dots, 4$):

$$\begin{pmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{H}_3 \\ \bar{H}_4 \end{pmatrix} = W(t) \left[\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix} + \begin{pmatrix} t_1 t_4 + \frac{3}{2} t_2 t_3 \\ t_4 + \frac{1}{2} t_2^2 + \frac{3}{4} t_1 t_3 \\ \frac{3}{2} t_3 + \frac{1}{2} t_1 t_2 \\ \frac{3}{2} t_2 + \frac{1}{8} t_1^2 \end{pmatrix} \right], \quad (0.6)$$

where

$$W(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2t_4 & 0 & 1 & 0 \\ -\frac{3}{2}t_3 & -2t_4 & 0 & 1 \end{pmatrix}$$

and H_j ($j = 1, \dots, 4$) are given by (0.5).

Since the completely integrable Hamiltonian system $(\bar{H})$ determines the holonomic deformation of A_4 -type, we call $(\bar{H})$ the A_4 -system.

On the other hand, for the holonomic deformation of A_i -type ($i = 1, 2, 3$) the corresponding theorem has already been proved in [4], [1] and [2] (for $i = 1, 2, 3$ respectively) and, moreover, polynomial Hamiltonian structure and particular solutions having integral representation were also obtained there. For example, for A_3 -type, we obtained the following polynomial Hamiltonian system $(L)_3$ which is equivalent to the A_3 -system:

$$\frac{\partial q_k}{\partial t_j} = \frac{\partial L_j}{\partial p_k}, \quad \frac{\partial p_k}{\partial t_j} = -\frac{\partial L_j}{\partial q_k} \quad (k, j = 1, 2, 3) \quad (L)_3$$

with the following Hamiltonian functions L_j ($j = 1, 2, 3$):

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} - D \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} - (\alpha + \frac{1}{2}) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \quad (0.7)$$

By computing the two sides of (1.6) in the cases of $j = 1, 2, 3, 4$ respectively, we can prove the lemma. Here we omit its proof.

LEMMA 1.2. *If we define Δ_j ($j = 0, 1, \dots$) by the following local expansion around $x = \infty$:*

$$\frac{1}{\Lambda(x)} = \Delta_0 x^{-4} + \Delta_1 x^{-5} + \dots,$$

and especially we define $\Delta_l = 0$ if $l < 0$, then we have

$$\sum_{k=1}^4 \frac{\lambda_k^l}{\Lambda'(\lambda_k)} = \Delta_{l-3}.$$

This lemma can be easily verified. In fact, considering the function $f(x) = x^l/\Lambda(x)$, we obtain

$$\begin{aligned} \sum_{k=1}^4 \frac{\lambda_k^l}{\Lambda'(\lambda_k)} &= \sum_{k=1}^4 \operatorname{Res}_{x=\lambda_k} f(x) = -\operatorname{Res}_{x=\infty} f(x) \\ &= -\operatorname{Res}_{x=\infty} \frac{x^l}{\Lambda(x)} = -\operatorname{Res}_{x=\infty} [x^l (\Delta_0 x^{-4} + \Delta_1 x^{-5} + \dots)] \\ &= \Delta_{l-3}. \end{aligned}$$

Using Lemma 1.2, we can verify that for non-negative integers i, j , we have

$$\sum_{i+j=l} \sigma_i \Delta_j = \begin{cases} -1 & \text{when } l = 0 \\ 0 & \text{when } l \geq 1. \end{cases} \quad (1.7)$$

Now we compute $X_{nm}^{(j)}$. Noting Lemma 1.1, we have

$$\begin{aligned} X_{nm}^{(j)} &= \frac{1}{2} \sum_{k=1}^4 \frac{N^{nk} N^{jk} N^{mk}}{\Lambda'(\lambda_k)} \\ &= -\frac{1}{2} \sum_{p=0}^{n-1} \sum_{q=0}^{j-1} \sum_{r=0}^{m-1} \sigma_p \sigma_q \sigma_r \sum_{k=1}^4 \frac{\lambda_k^{m+n+j-p-q-r-3}}{\Lambda'(\lambda_k)}. \end{aligned}$$

Using Lemma 1.2, we obtain

$$\begin{aligned} X_{nm}^{(j)} &= -\frac{1}{2} \sum_{p=0}^{n-1} \sum_{q=0}^{j-1} \sum_{r=0}^{m-1} \sigma_p \sigma_q \sigma_r \Delta_{m+n+j-p-q-r-6} \\ &= -\frac{1}{2} \sum_{p=0}^{n-1} \sum_{q=0}^{j-1} \sigma_p \sigma_q [\sigma_0 \Delta_{m+n+j-p-q-6} + \dots + \sigma_{m-1} \Delta_{n+j-p-q-5}]. \end{aligned}$$

By some computation and using (1.7), we can prove

$$X_{nm}^{(j)} = \frac{1}{2} \left(\sum_{l=m+j-5}^{j-1} \sigma_l \sigma_{m+n+j-l-6} - \sum_{l=0}^{n+j-6} \sigma_l \sigma_{m+n+j-l-6} \right). \quad (1.8)$$

Next we compute $Y_i^{(j)}$.

$$\begin{aligned} Y_i^{(j)} &= -\frac{1}{2} \sum_{k=1}^4 N^{ik} U_{jk} \\ &= -\frac{1}{2} \sum_{k=1}^4 N^{ik} \left\{ N_k N^{jk} \left[2\lambda_k^5 + \sum_{m=1}^4 m t_m \lambda_k^{m-1} \right] - \sum_{l=1, \neq k}^4 \frac{N_k N^{jk} + N_l N^{jl}}{\lambda_l - \lambda_k} \right\} \\ &= \Omega_1 + \Omega_2 + \Omega_3, \end{aligned}$$

where

$$\begin{aligned} \Omega_1 &= -\sum_{k=1}^4 N_k N^{ik} N^{jk} \lambda_k^5, \\ \Omega_2 &= -\frac{1}{2} \sum_{k=1}^4 N_k N^{ik} N^{jk} \sum_{m=1}^4 m t_m \lambda_k^{m-1}, \\ \Omega_3 &= -\frac{1}{2} \sum_{k=1}^4 \sum_{l=1, \neq k}^4 \frac{N_k N^{jk} + N_l N^{jl}}{\lambda_k - \lambda_l} N^{ik}. \end{aligned}$$

Using Lemma 1.1, we have

$$\begin{aligned} \Omega_1 &= -\sum_{k=1}^4 \frac{1}{\Lambda'(\lambda_k)} \sum_{m=0}^{j-1} (-\sigma_m \lambda_k^{j-1-m}) \sum_{n=0}^{i-1} (-\sigma_n \lambda_k^{i-1-n}) \lambda_k^5 \\ &= -\sum_{m=0}^{j-1} \sum_{n=0}^{i-1} \sigma_m \sigma_n \sum_{k=1}^4 \frac{1}{\Lambda'(\lambda_k)} \lambda_k^{i+j+3-m-n} \\ &= -\sum_{m=0}^{j-1} \sum_{n=0}^{i-1} \sigma_m \sigma_n \Delta_{i+j-m-n} \\ &= -\sum_{m=0}^{j-1} \sigma_m [\sigma_0 \Delta_{i+j-m} + \sigma_1 \Delta_{i+j-m-1} + \cdots + \sigma_{i-1} \Delta_{j+1-m}]. \end{aligned}$$

Furthermore, by using (1.7), we obtain

$$\Omega_1 = -\sigma_{i+j} - \sigma_i \sigma_j.$$

In the same way, we have

$$\begin{aligned}\Omega_2 &= \frac{1}{2} \sum_{l=1}^{5-i} l t_l \sigma_{i+j+l-6} - \frac{1}{2} \sum_{l=6-j}^4 l t_l \sigma_{i+j+l-6}, \\ \Omega_3 &= \frac{1}{2} (5-i) \sigma_{i+j-6}.\end{aligned}$$

Hence we have

$$Y_i^{(j)} = -\sigma_{i+j} - \sigma_i \sigma_j + \frac{1}{2} \sum_{l=1}^{5-i} l t_l \sigma_{i+j+l-6} - \frac{1}{2} \sum_{l=6-j}^4 l t_l \sigma_{i+j+l-6} + \frac{1}{2} (5-i) \sigma_{i+j-6}. \quad (1.9)$$

Finally, we have

$$\begin{aligned}Z^{(j)} &= -(\alpha + \frac{1}{2}) \sum_{k=1}^4 N_k N^{jk} \lambda_k^4 \\ &= -(\alpha + \frac{1}{2}) \sum_{k=1}^4 \frac{1}{\Lambda'(\lambda_k)} [\lambda_k^{j+3} - \sigma_1 \lambda_k^{j+2} - \dots - \sigma_{j-1} \lambda_k^4].\end{aligned}$$

Using Lemma 1.2 and (1.7), we have

$$\begin{aligned}Z^{(j)} &= -(\alpha + \frac{1}{2}) [\Delta_j - \sigma_1 \Delta_{j-1} - \dots - \sigma_{j-1} \Delta_1 - \sigma_j + \sigma_j] \\ &= -(\alpha + \frac{1}{2}) \sigma_j.\end{aligned} \quad (1.10)$$

Noting (0.6), (1.3) and (1.5), we obtain the following explicit form of the Hamiltonian functions L_j ($j = 1, \dots, 4$) as a polynomial of $t = (t_1, \dots, t_4)$, $\sigma = (\sigma_1, \dots, \sigma_4)$ and $\rho = (\rho_1, \dots, \rho_4)$:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{pmatrix} = W(t) \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix} + W(t) \begin{pmatrix} t_1 t_4 + \frac{3}{2} t_2 t_3 \\ t_4 + \frac{1}{2} t_2^2 + \frac{3}{4} t_1 t_3 \\ \frac{3}{2} t_3 + \frac{1}{2} t_1 t_2 \\ \frac{3}{2} t_2 + \frac{1}{8} t_1^2 \end{pmatrix}, \quad (1.11)$$

where H_j ($j = 1, \dots, 4$) are given by (1.5) with the coefficient matrices $X^{(j)}$, $Y^{(j)}$ and $Z^{(j)}$ determined by (1.8), (1.9) and (1.10) respectively.

By Proposition 1.1 and the theorem referred to in the Introduction, we have the following.

PROPOSITION 1.2. *The holonomic deformation of linear differential equations of A_4 -type is governed by the completely integrable Hamiltonian system*

$$\frac{\partial \sigma_k}{\partial t_j} = \frac{\partial L_j}{\partial \rho_k}, \quad \frac{\partial \rho_k}{\partial t_j} = -\frac{\partial L_j}{\partial \sigma_k} \quad (k, j = 1, \dots, 4), \quad (L)$$

with the polynomial Hamiltonian functions L_j ($j = 1, \dots, 4$) given by (1.11).

Now the Hamiltonian functions L_j are polynomials in the canonical variables. But they are not symmetric about the variables ρ_k ($k = 1, \dots, 4$), that is, at the point $(\rho_1, \dots, \rho_4) = (0, \dots, 0)$,

$$\frac{\partial L_j}{\partial \rho_k} \neq \frac{\partial L_k}{\partial \rho_j}, \quad (j, k = 1, \dots, 4; j \neq k).$$

In order to obtain a particular solution of the Hamiltonian system (L), we want L_j ($j = 1, \dots, 4$) to be symmetric about the variables ρ_k ($k = 1, \dots, 4$). Hence we will rewrite the Hamiltonian system (L) and obtain a polynomial Hamiltonian, symmetric with respect to the independent variables ρ_k ($k = 1, \dots, 4$).

For the polynomial Hamiltonian system (L), we make the following transformations:

$$\begin{aligned} L_1 &= R_1, \\ L_2 &= R_2 - \frac{1}{2}p_4, \\ L_3 &= R_3 - s_4 R_1 - p_3 + \frac{1}{2}q_1 p_4, \\ L_4 &= R_4 - s_3 R_1 - \frac{3}{2}s_4 R_2 - \frac{3}{2}p_2 + q_1 p_3 + s_4 p_4 + \frac{1}{2}q_2 p_4. \end{aligned} \quad (1.12)$$

$$\begin{aligned} \rho_1 &= p_1 - s_4 p_3 - \frac{1}{2}s_3 p_4, \\ \rho_2 &= p_2 - \frac{1}{2}s_4 p_4, \\ \rho_3 &= p_3, \\ \rho_4 &= p_4. \end{aligned} \quad (1.13)$$

$$\begin{aligned} \sigma_1 &= q_1, \\ \sigma_2 &= q_2 - \frac{3}{2}s_4, \\ \sigma_3 &= q_3 + s_4 q_1 - s_3, \\ \sigma_4 &= q_4 + \frac{1}{2}s_4 q_2 + \frac{1}{2}s_3 q_1 - \frac{1}{2}s_2 - \frac{1}{4}s_4^2. \end{aligned} \quad (1.14)$$

$$\begin{aligned} t_1 &= s_1 + s_3 s_4, \\ t_2 &= s_2 + \frac{3}{4}s_4^2, \\ t_3 &= s_3, \\ t_4 &= s_4. \end{aligned} \quad (1.15)$$

PROPOSITION 1.3. *The changes*

$$(\sigma, \rho, L, t) \rightarrow (q, p, R, s)$$

defined by (1.12)–(1.15) is a canonical transformation of the system (L) , where $L = (L_1, \dots, L_4)$, $q = (q_1, \dots, q_4)$, $p = (p_1, \dots, p_4)$, $R = (R_1, \dots, R_4)$ and $s = (s_1, \dots, s_4)$.

Proof. First we compute

$$\begin{aligned} \sum_{j=1}^4 L_j dt_j &= \sum_{j=1}^4 R_j ds_j - \frac{1}{2} p_4 ds_2 + \left(\frac{1}{2} q_1 p_4 - p_3\right) ds_3 \\ &\quad + \left(-\frac{3}{2} p_2 + q_1 p_3 + \frac{1}{4} s_4 p_4 + \frac{1}{2} q_2 p_4\right) ds_4. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{k=1}^4 \rho_k d\sigma_k &= \sum_{k=1}^4 p_k dq_k - \frac{1}{2} p_4 ds_2 + \left(\frac{1}{2} q_1 p_4 - p_3\right) ds_3 \\ &\quad + \left(-\frac{3}{2} p_2 + q_1 p_3 + \frac{1}{4} s_4 p_4 + \frac{1}{2} q_2 p_4\right) ds_4. \end{aligned}$$

Hence we have

$$\sum_{j=1}^4 L_j dt_j - \sum_{k=1}^4 \rho_k d\sigma_k = \sum_{j=1}^4 R_j ds_j - \sum_{k=1}^4 p_k dq_k;$$

that is,

$$\sum_{k=1}^4 dp_k \wedge dq_k - \sum_{j=1}^4 dR_j \wedge ds_j = \sum_{k=1}^4 d\rho_k \wedge d\sigma_k - \sum_{j=1}^4 dL_j \wedge dt_j.$$

This proves the Proposition. $\square$

Furthermore, we can deduce from (1.12)–(1.15) and (1.11) the explicit form of the Hamiltonian functions R_j .

LEMMA 1.3. *The Hamiltonian functions R_j can be written as follows:*

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = A \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} + B \begin{pmatrix} p_1 p_2 \\ p_1 p_3 \\ p_1 p_4 \\ p_2 p_3 \\ p_2 p_4 \\ p_3 p_4 \end{pmatrix} + C \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

$$+(\alpha + \frac{1}{2}) \begin{pmatrix} q_1 \\ q_2 + \frac{3}{2}s_4 \\ q_3 + s_3 \\ q_4 + \frac{1}{2}s_2 + \frac{1}{2}s_4^2 \end{pmatrix} + W(s) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}, \quad (1.16)$$

where a 4×4 matrix $A = (A_{ij})$ is given by

$$\begin{aligned} A_{11} &= 0, & A_{12} &= 0, & A_{13} &= -\frac{1}{2}q_1, & A_{14} &= -\frac{1}{2}q_3, \\ A_{21} &= 0, & A_{22} &= \frac{1}{2}, & A_{23} &= \frac{1}{2}q_1^2 - \frac{1}{2}q_2 - \frac{1}{4}s_4, \\ A_{24} &= \frac{1}{2}q_1q_3 + \frac{1}{2}q_4 + \frac{1}{4}s_4q_2 + \frac{1}{4}s_3q_1, \\ A_{31} &= 0, & A_{32} &= -q_1, & A_{33} &= q_1q_2 + \frac{1}{2}q_3 + \frac{1}{2}s_4q_1 - \frac{1}{2}s_3, \\ A_{34} &= \frac{1}{2}q_2q_3 - \frac{1}{2}q_1q_4 - \frac{1}{4}s_4q_3 - \frac{1}{4}s_4q_1q_2 \\ &\quad + \frac{1}{4}(s_2 - \frac{1}{2}s_4^2)q_1 - \frac{1}{4}s_3q_1^2 + \frac{1}{4}s_3s_4, \\ A_{41} &= \frac{1}{2}, & A_{42} &= \frac{1}{2}q_1^2 - \frac{1}{4}s_4, \\ A_{43} &= \frac{1}{2}q_2^2 + \frac{1}{2}q_4 - \frac{1}{4}s_4q_1^2 + \frac{1}{2}s_3q_1 + \frac{1}{8}s_4^2 - \frac{1}{4}s_2, \\ A_{44} &= \frac{1}{2}q_3^2 - \frac{1}{2}q_2q_4 + \frac{1}{4}s_4q_1q_3 - \frac{1}{4}s_3q_3 - \frac{1}{4}s_4q_2^2 - \frac{1}{4}s_3q_1q_2 \\ &\quad + \frac{1}{8}s_4q_1^2 + \frac{1}{4}(s_2 - s_4^2)q_2 - \frac{1}{4}s_3s_4q_1 + \frac{1}{8}s_3^2 - \frac{1}{16}s_4^3, \end{aligned}$$

a 4×6 matrix $B = (B_{ij})$ is given by

$$\begin{aligned} B_{11} &= 0, & B_{12} &= 0, & B_{13} &= 0, & B_{14} &= 1, & B_{15} &= -q_1, & B_{16} &= -q_2, \\ B_{21} &= 0, & B_{22} &= 1, & B_{23} &= -q_1, & B_{24} &= -2q_1, \\ B_{25} &= q_1^2 - \frac{1}{2}s_4, & B_{26} &= q_1q_2 + \frac{1}{2}s_4q_1 - \frac{1}{2}s_3, \\ B_{31} &= 1, & B_{32} &= -q_1, & B_{33} &= -q_2, & B_{34} &= q_1^2 - q_2 - \frac{1}{2}s_4, \\ B_{35} &= q_1q_2 - \frac{1}{2}s_3 + \frac{1}{2}s_4q_1, & B_{36} &= q_2^2 + q_4 + s_3q_1 - \frac{1}{2}s_4q_1^2 + \frac{1}{4}s_4^2 - \frac{1}{2}s_2, \\ B_{41} &= -q_1, & B_{42} &= -q_2, & B_{43} &= -q_3, & B_{44} &= q_1q_2 + \frac{1}{2}s_4q_1 - \frac{1}{2}s_3, \\ B_{45} &= q_1q_3 + q_4 + \frac{1}{2}s_4q_2 + \frac{1}{2}s_3q_1 - \frac{1}{2}s_2, \\ B_{46} &= q_2q_3 - q_1q_4 - \frac{1}{2}s_4q_3 + \frac{1}{2}(s_2 - \frac{1}{2}s_4^2)q_1 - \frac{1}{2}s_3q_1^2 - \frac{1}{2}s_4q_1q_2 + \frac{1}{2}s_3s_4, \end{aligned}$$

a 4×4 symmetric matrix $C = (C_{ij})$ is given by

$$\begin{aligned} C_{11} &= -q_1^2 - q_2 - \frac{1}{2}s_4, & C_{22} &= -q_2^2 - q_4 + \frac{1}{2}s_4q_2 + s_3q_1 + \frac{1}{4}s_4^2 - \frac{1}{2}s_2, \\ C_{33} &= -q_3^2 + \frac{1}{2}s_3q_3 - \frac{1}{4}s_4^2q_2 - s_3s_4q_1 + \frac{1}{2}s_1q_1 + s_2q_2 + \frac{1}{2}s_3^2 + \frac{1}{2}s_2s_4 - \frac{1}{8}s_4^3, \end{aligned}$$

$$\begin{aligned}
C_{44} &= -q_4^2 + \frac{1}{2}s_1q_3 - \frac{1}{2}(s_2s_4 + \frac{1}{2}s_3^2)q_2 + \frac{1}{2}(\frac{3}{4}s_3s_4^2 - s_2s_3)q_1 \\
&\quad + \frac{1}{4}(s_2^2 - \frac{3}{2}s_3^2s_4 + \frac{1}{4}s_4^4), \\
C_{12} &= C_{21} = -q_1q_2 - q_3 + \frac{1}{2}s_4q_1 - \frac{1}{2}s_3, \\
C_{13} &= C_{31} = -q_1q_3 - q_4 + \frac{1}{2}s_4q_2 + \frac{1}{2}s_3q_1 - \frac{1}{2}s_2, \\
C_{14} &= C_{41} = -q_1q_4 + \frac{1}{2}s_4q_3 + \frac{1}{2}s_3q_2 + \frac{1}{2}s_2q_1 - \frac{1}{2}s_1, \\
C_{23} &= C_{32} = -q_2q_3 + \frac{1}{2}s_4q_3 + s_3q_2 + (s_2 - \frac{1}{4}s_4^2)q_1 + \frac{1}{2}s_3s_4 - \frac{1}{2}s_1, \\
C_{24} &= C_{42} = -q_2q_4 + \frac{1}{2}s_3q_3 + \frac{1}{2}(s_2 - \frac{1}{2}s_4^2)q_2 + \frac{1}{2}(s_1 - s_3s_4)q_1 \\
&\quad + \frac{1}{2}(s_2s_4 + \frac{1}{2}s_3^2), \\
C_{34} &= C_{43} = -q_3q_4 + \frac{1}{2}s_2q_3 + \frac{1}{2}(s_1 - s_3s_4)q_2 - \frac{1}{2}(s_3^2 - \frac{1}{4}s_4^3 + s_2s_4)q_1 \\
&\quad + \frac{1}{2}(s_2s_3 - \frac{3}{4}s_3s_4^2),
\end{aligned}$$

a 4×1 matrix $E = (E_i)$ is given by

$$\begin{aligned}
E_1 &= s_1s_4 + \frac{3}{2}s_2s_3 + \frac{17}{8}s_3s_4^2, \\
E_2 &= s_4 + \frac{1}{2}s_2^2 + \frac{3}{4}s_2s_4^2 + \frac{9}{32}s_4^4 + \frac{3}{4}s_1s_3 + \frac{3}{4}s_3^2s_4, \\
E_3 &= \frac{3}{2}s_3 + \frac{1}{2}s_1s_2 - s_2s_3s_4 - \frac{5}{8}s_1s_4^2 - \frac{7}{4}s_3s_4^3, \\
E_4 &= \frac{3}{2}s_2 + \frac{5}{8}s_4^2 + \frac{1}{8}s_1^2 - \frac{5}{8}s_1s_3s_4 - \frac{21}{16}s_3^2s_4^2 - \frac{3}{4}s_2s_3^2 - \frac{1}{4}s_2^2s_4 - \frac{3}{8}s_2s_4^3 - \frac{9}{64}s_4^5.
\end{aligned} \tag{1.17}$$

By Propositions 1.2 and 1.3, we have the following.

THEOREM 1.1. *The holonomic deformation of linear ordinary differential equations of A_4 -type is governed by the completely integrable Hamiltonian system*

$$\frac{\partial q_k}{\partial s_j} = \frac{\partial R_j}{\partial p_k}, \quad \frac{\partial p_k}{\partial s_j} = -\frac{\partial R_j}{\partial q_k} \quad (k, j = 1, \dots, 4) \tag{R}$$

with the polynomial Hamiltonian functions R_j defined by (1.16).

2. Particular solutions

In this section, we study particular solutions of the A_4 -system in the case of $\alpha = -\frac{1}{2}$.

Firstly we consider transformations of the Hamiltonians:

$$R_j = K_j + f_j(s), \tag{2.1}$$

where $f_j(s)$ are arbitrary functions of s . Then, the Hamiltonian system (R) is transformed to the following equivalent Hamiltonian system:

$$\frac{\partial q_k}{\partial s_j} = \frac{\partial K_j}{\partial p_k}, \quad \frac{\partial p_k}{\partial s_j} = -\frac{\partial K_j}{\partial q_k} \quad (k, j = 1, \dots, 4) \quad (K)$$

with the Hamiltonian functions K_j ($j = 1, \dots, 4$) defined by (2.1). As a special case of such transformations (2.1), we consider the following transformations:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} + W(s) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}, \quad (2.2)$$

where E_j ($j = 1, \dots, 4$) are determined by (1.17). Since the Hamiltonian system (R) is equivalent to the system (K) with the Hamiltonian functions given by (2.2), in what follows we will consider the Hamiltonian system (K) . Noting (1.16) and (2.2), we know that K_j ($j = 1, \dots, 4$) can be written in the form

$$\begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = A \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} + B \begin{pmatrix} p_1 p_2 \\ p_1 p_3 \\ p_1 p_4 \\ p_2 p_3 \\ p_2 p_4 \\ p_3 p_4 \end{pmatrix} + C \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \\ + (\alpha + \frac{1}{2}) \begin{pmatrix} q_1 \\ q_2 + \frac{3}{2}s_4 \\ q_3 + s_3 \\ q_4 + \frac{1}{2}s_2 + \frac{1}{2}s_4^2 \end{pmatrix}. \quad (2.3)$$

In the case of $\alpha = -\frac{1}{2}$, we obtain from (2.3),

$$\frac{\partial}{\partial q_k} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \frac{\partial}{\partial q_k} A \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \\ p_4^2 \end{pmatrix} + \frac{\partial}{\partial q_k} B \begin{pmatrix} p_1 p_2 \\ p_1 p_3 \\ p_1 p_4 \\ p_2 p_3 \\ p_2 p_4 \\ p_3 p_4 \end{pmatrix} + \frac{\partial}{\partial q_k} C \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}.$$

Hence the system of equations

$$\frac{\partial p_i}{\partial s_j} = -\frac{\partial K_j}{\partial q_i}$$

admits a particular solution of the form

$$(p_1, \dots, p_4) = (0, \dots, 0).$$

Then we deduce from (2.3) and Theorem 1.1 that particular solutions $(q_1, \dots, q_4; p_1, \dots, p_4) = (q_1, \dots, q_4; 0, \dots, 0)$ of the system (K) satisfy equations of the form

$$\frac{\partial q_i}{\partial s_j} = C_{ij} \quad (i, j = 1, \dots, 4), \quad (2.4)$$

where C_{ij} is the (i, j) element of the matrix C . Noting that $C_{ij} = C_{ji}$, we can easily obtain the following.

PROPOSITION 2.1. *For a solution $q = (q_1, \dots, q_4)$ of (2.4), the one-form*

$$\omega = \sum_{j=1}^4 q_j ds_j$$

is closed.

Now we introduce a new variable u defined by

$$\omega = d \log u, \quad (2.5)$$

or, equivalently,

$$q_j = \frac{\partial}{\partial s_j} \log u.$$

By substituting this into (2.4), we can verify the following proposition.

PROPOSITION 2.2. *The function u satisfies the following completely integrable linear system:*

$$\begin{aligned} \frac{\partial^2 u}{\partial s_1^2} &= -\frac{\partial u}{\partial s_2} - \frac{1}{2}s_4 u, \\ \frac{\partial^2 u}{\partial s_1 \partial s_2} &= -\frac{\partial u}{\partial s_3} + \frac{1}{2}s_4 \frac{\partial u}{\partial s_1} - \frac{1}{2}s_3 u, \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial s_1 \partial s_3} &= -\frac{\partial u}{\partial s_4} + \frac{1}{2}s_4 \frac{\partial u}{\partial s_2} + \frac{1}{2}s_3 \frac{\partial u}{\partial s_1} - \frac{1}{2}s_2 u, \\
\frac{\partial^2 u}{\partial s_1 \partial s_4} &= \frac{1}{2}s_4 \frac{\partial u}{\partial s_3} + \frac{1}{2}s_3 \frac{\partial u}{\partial s_2} + \frac{1}{2}s_2 \frac{\partial u}{\partial s_1} - \frac{1}{2}s_1 u, \\
\frac{\partial^2 u}{\partial s_2^2} &= -\frac{\partial u}{\partial s_4} + \frac{1}{2}s_4 \frac{\partial u}{\partial s_2} + s_3 \frac{\partial u}{\partial s_1} + \frac{1}{2}(\frac{1}{2}s_4^2 - s_2)u, \\
\frac{\partial^2 u}{\partial s_2 \partial s_3} &= \frac{1}{2}s_4 \frac{\partial u}{\partial s_3} + s_3 \frac{\partial u}{\partial s_2} + (s_2 - \frac{1}{4}s_4^2) \frac{\partial u}{\partial s_1} + \frac{1}{2}(s_3 s_4 - s_1)u, \\
\frac{\partial^2 u}{\partial s_2 \partial s_4} &= \frac{1}{2}s_3 \frac{\partial u}{\partial s_3} + \frac{1}{2}(s_2 - \frac{1}{2}s_4^2) \frac{\partial u}{\partial s_2} - \frac{1}{2}(s_3 s_4 - s_1) \frac{\partial u}{\partial s_1} + \frac{1}{2}(s_2 s_4 + \frac{1}{2}s_3^2)u, \\
\frac{\partial^2 u}{\partial s_3^2} &= \frac{1}{2}s_3 \frac{\partial u}{\partial s_3} + (s_2 - \frac{1}{4}s_4^2) \frac{\partial u}{\partial s_2} - (s_3 s_4 - \frac{1}{2}s_1) \frac{\partial u}{\partial s_1} + \frac{1}{2}(s_2 s_4 + s_3^2 - \frac{1}{4}s_4^3)u, \\
\frac{\partial^2 u}{\partial s_3 \partial s_4} &= \frac{1}{2}s_2 \frac{\partial u}{\partial s_3} - \frac{1}{2}(s_3 s_4 - s_1) \frac{\partial u}{\partial s_2} - \frac{1}{2}(s_2 s_4 + s_3^2 - \frac{1}{4}s_4^3) \frac{\partial u}{\partial s_1} \\
&\quad + \frac{1}{2}(s_2 s_3 - \frac{3}{4}s_3 s_4^2)u, \\
\frac{\partial^2 u}{\partial s_4^2} &= \frac{1}{2}s_1 \frac{\partial u}{\partial s_3} - \frac{1}{2}(s_2 s_4 + \frac{1}{2}s_3^2) \frac{\partial u}{\partial s_2} - \frac{1}{2}(s_2 s_3 - \frac{3}{4}s_3 s_4^2) \frac{\partial u}{\partial s_1} \\
&\quad + \frac{1}{4}(s_2^2 - \frac{3}{2}s_3^2 s_4 + \frac{1}{4}s_4^4)u.
\end{aligned} \tag{2.6}$$

We show the following.

PROPOSITION 2.3. *The system (2.6) has solutions of the form*

$$u_l(s) = \int_{r_l} \exp \left[- \int_0^\xi A_4(x, t) dx - \frac{1}{2}(s_2 s_4 + \frac{1}{2}s_3^2 + \frac{1}{6}s_4^3) \right] d\xi \quad (l = 1, \dots, 5), \tag{2.7}$$

where r_l are paths in the complex plane described in Figure 1, i.e. each r_l is in the angular domain of the angle $\pi/3$, and $t = (t_1, \dots, t_4)$ are determined by (1.15).

Proof. Supposing that u is written in the form

$$u(s) = \int_r e^{-s_1 \xi} \phi(\xi, s_2, s_3, s_4) d\xi,$$

FIGURE 1.

we obtain the following system of four independent equations:

$$\begin{aligned}
 \frac{\partial \phi}{\partial s_2} &= -(\xi^2 + \tfrac{1}{2}s_4)\phi, \\
 \frac{\partial \phi}{\partial s_3} &= -(\xi^3 + s_4\xi + \tfrac{1}{2}s_3)\phi, \\
 \frac{\partial \phi}{\partial s_4} &= -(\xi^4 + \tfrac{3}{2}s_4\xi^2 + s_3\xi + \tfrac{1}{2}s_2 + \tfrac{1}{4}s_4^2)\phi, \\
 \frac{\partial \phi}{\partial \xi} &= -[2\xi^5 + 4s_4\xi^3 + 3s_3\xi^2 + (2s_2 + \tfrac{3}{2}s_4^2)\xi + s_3s_4]\phi.
 \end{aligned} \tag{2.8}$$

By some computation, we can verify that

$$\phi(\xi, s_2, s_3, s_4) = \exp \left[- \int_0^\xi A_4(x, t) dx - \tfrac{1}{2}(s_2s_4 + \tfrac{1}{2}s_3^2 + \tfrac{1}{6}s_4^3) + s_1\xi \right]$$

solves (2.8). This proves Proposition 2.3. $\square$

Let J be the linear space of solutions of (2.6). It is known [5] that the linear map Φ from J to C^5 defined by

$$\Phi : J \ni u(t) \longrightarrow \begin{pmatrix} u(0) \\ u_1(0) \\ \vdots \\ u_4(0) \end{pmatrix} \in C^5 \quad (2.9)$$

is an isomorphism, where $u_i(0) = (\partial u / \partial s_i)|_{s=0}$ for $i = 1, \dots, 4$. Here we evaluate values of the functions at $s = (s_1, \dots, s_4) = (0, \dots, 0)$, for the right side of (2.6) is polynomial in $s = (s_1, \dots, s_4)$. Hence J is of dimension 5. Finally, we prove the following.

PROPOSITION 2.4. *The solutions (2.7) of (2.6) form a basis of the linear space J .*

Proof. By means of (2.9), it suffices to show that for u_l ($l = 1, \dots, 5$), $\Phi(u_l)$ are linear independent vectors in C^5 . In order to compute the vector $\Phi(u_l)$, we put $\theta = e^{\pi i/3}$ and

$$c_l = \int_0^\infty \xi^{l-1} e^{-\xi^6/3} d\xi, \quad (l = 1, \dots, 5).$$

Then we deduce from (2.7) that the matrix

$$(\Phi(u_1), \dots, \Phi(u_5))$$

is given by

$$\begin{pmatrix} (\theta - 1)c_1 & 0 & \dots & 0 \\ 0 & (1 - \theta^2)c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1 - \theta^5)c_5 \end{pmatrix} \begin{pmatrix} 1 & \theta & \dots & \theta^4 \\ 1 & \theta^2 & \dots & \theta^8 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta^5 & \dots & \theta^{20} \end{pmatrix}.$$

Since $(1 - \theta^l)c_l \neq 0$, $\Phi(u_l)$ ($l = 1, \dots, 5$) are linearly independent. It follows that u_l ($l = 1, \dots, 5$) form a basis of J . $\square$

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