SCHWARZ MAPS OF ${}_3F_2$ WITH FINITE IRREDUCIBLE MONODROMY GROUPS

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1. Preliminaries

It is known all the cases when the generalized hypergeometric function ${}_3F_2$ has finite irreducible primitive monodromy group [BH, No. 2–No. 12], but nothing is known about its Schwarz map. This paper determines the image of the Schwarz map and describes its inverse map for each case. The images turned out not to be as simple as several specialists expected.

1.1. Notation

The generalized hypergeometric function ${}_{n+1}F_n$ is defined by the following power series:

$${}_{n+1}F_n(a_0, a_1, \dots, a_n; b_1, \dots, b_n; z) = \sum_{m=0}^{\infty} \frac{\prod_{j=0}^n (a_j, m)}{\prod_{j=1}^n (b_j, m)} \frac{z^m}{(1, m)},$$

where $(a, m) = \Gamma(a+m)/\Gamma(a)$. This is a solution of a Fuchsian differential equation

$${}_{n+1}E_n = {}_{n+1}E_n(a_0, a_1, \dots, a_n; b_1, \dots, b_n)$$

of rank $n+1$ with singularities at $z = 0, 1, \infty$.

The ratio of $n+1$ linearly independent solutions of ${}_{n+1}E_n$ defines a multi-valued map $\mathbf{P}^1 - \{0, 1, \infty\} \rightarrow \mathbf{P}^n$ which we call a *Schwarz map* of ${}_{n+1}E_n$.

The characteristic exponents of ${}_{n+1}E_n$ at these singular points are

$$\begin{array}{ll} 0, 1 - b_1, 1 - b_2, \dots, 1 - b_n & \text{at } z = 0, \\ 0, 1, \dots, n - 1, \sum b_j - \sum a_j & \text{at } z = 1, \\ a_0, a_1, \dots, a_n & \text{at } z = \infty. \end{array}$$

Beukers and Heckman [BH] have determined when ${}_{n+1}E_n$ has finite irreducible monodromy group. They used $2n+2$ parameters α_j, β_j , $0 \leq j \leq n$, to define the

hypergeometric differential equation which we denote by

$${}_{n+1}D_n(\alpha_0, \alpha_1, \dots, \alpha_n; \beta_0, \beta_1, \dots, \beta_n).$$

We note [BH, p. 327]

$${}_{n+1}D_n(\alpha_0, \alpha_1, \dots, \alpha_n; \beta_0, \beta_1, \dots, \beta_n) = z^{1-\beta_0} {}_{n+1}E_n(a_0, a_1, \dots, a_n; b_1, \dots, b_n), \quad (1.1)$$

where

$$a_j = \alpha_j + 1 - \beta_0, \quad b_j = \beta_j + 1 - \beta_0.$$

This means that $z^{1-\beta_0}u(z)$ is a solution of ${}_{n+1}D_n(\alpha; \beta)$ if and only if $u(z)$ is a solution of ${}_{n+1}E_n(a; b)$.

In this paper, we deal with ${}_3E_2$ with finite irreducible primitive monodromy group, that is, No. 2–No. 12 in Table 8.3 of [BH].

Put

$$\begin{aligned} a'_j &= a_j + 1 - b_1, & b'_2 &= b_2 + 1 - b_1, & b'_1 &= 2 - b_1, \\ a''_j &= a_j + 1 - b_2, & b''_1 &= b_1 + 1 - b_2, & b''_2 &= 2 - b_2. \end{aligned}$$

Then we have

$$a_j, a'_j, a''_j, b_k, b'_k, b''_k \in \mathbf{Q} - \mathbf{Z}$$

in these cases. Therefore the differential equation ${}_3E_2$ has the following fundamental system of solutions:

$$\begin{aligned} v_0 &:= \frac{\Gamma(a_0)\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1)\Gamma(b_2)} {}_3F_2(a_0, a_1, a_2; b_1, b_2; z), \\ v_1 &:= \frac{\Gamma(a'_0)\Gamma(a'_1)\Gamma(a'_2)}{\Gamma(b'_1)\Gamma(b'_2)} z^{1-b_1} {}_3F_2(a'_0, a'_1, a'_2; b'_1, b'_2; z), \\ v_2 &:= \frac{\Gamma(a''_0)\Gamma(a''_1)\Gamma(a''_2)}{\Gamma(b''_1)\Gamma(b''_2)} z^{1-b_2} {}_3F_2(a''_0, a''_1, a''_2; b''_1, b''_2; z), \end{aligned} \quad (1.2)$$

which we fix throughout this paper. The Schwarz map

$$v : z \mapsto v(z) = [v_0(z) : v_1(z) : v_2(z)] \quad (1.3)$$

is defined on $\mathbf{P}^1$ and its image is then an irreducible plane curve C .

Important Remark. If the parameters a_j and b_j change by integers, then the monodromy group of ${}_3E_2(a_0, a_1, a_2; b_1, b_2)$ does not change, but its Schwarz map v does change. Therefore we need to specify these parameters. We will make a careful choice of a_j and b_j to ensure that the map v behaves mildly.

1.2. Main results

For the Schwarz map (1.3) of ${}_3E_2$ with carefully chosen parameters a_j and b_j , we collect our main results in Table 1.

TABLE 1.

No.	a_0	a_1	a_2	b_1	b_2	g'	ST-No.	I_d	f_C	v^{-1}
2	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{2}{3}$	$\frac{1}{3}$	168	24	I_4, I_6, I_{14}	$I_6^2 - I_4^3$	$I_{14}/I_4^2 I_6$
3	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{5}{4}$	$\frac{3}{4}$	168	24	I_4, I_6, I_{14}	I_6	I_{14}^2/I_4^7
4	$-\frac{1}{14}$	$\frac{1}{14}$	$\frac{9}{14}$	$\frac{6}{7}$	$\frac{2}{7}$	168	24	I_4, I_6, I_{14}	I_6	—
5	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{4}{3}$	$\frac{2}{3}$	60	23	I_2, I_6, I_{10}	I_2	I_{10}^3/I_6^5
6	$\frac{5}{10}$	$\frac{7}{10}$	$\frac{13}{10}$	$\frac{8}{5}$	$\frac{7}{5}$	60	23	I_2, I_6, I_{10}	I_6	I_2^5/I_{10}
7	$-\frac{1}{30}$	$\frac{5}{30}$	$\frac{11}{30}$	$\frac{4}{5}$	$\frac{1}{5}$	360	27	I_6, I_{12}, I_{30}	I_{12}	I_{30}/I_6^5
8	$-\frac{1}{30}$	$\frac{5}{30}$	$\frac{11}{30}$	$\frac{3}{4}$	$\frac{1}{4}$	360	27	I_6, I_{12}, I_{30}	I_{12}	I_{30}/I_6^5
9	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{5}{4}$	$\frac{3}{4}$	216	25	I_6, I_9, I_{12}	$I_9^2 - I_6 I_{12}$	I_6^2/I_{12}
10	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{7}{6}$	$\frac{3}{6}$	216	25	I_6, I_9, I_{12}	I_{12}	I_9^2/I_6^3
11	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{7}{9}$	$\frac{5}{4}$	$\frac{3}{4}$	216	25	I_6, I_9, I_{12}	*	I_6^2/I_{12}
12	$-\frac{1}{12}$	$\frac{2}{12}$	$\frac{5}{12}$	$\frac{3}{4}$	$\frac{1}{4}$	36	2	I_3, I'_3, I_6	I'_3	I_3^4/I_6^2

* = $I_9^2 I_{12} - I_6(I_{12} + c I_6^2)^2$ for some constant c .

We denote by g' the order of the projective monodromy group of ${}_3E_2$. The reflection subgroup H_r of the monodromy group H is a finite unitary reflection group (see Section 3). Its Shephard–Todd number [ST, p. 301], which we denote by ‘ST-No.’, is already given in [BH]. The ring $\mathbb{C}[v_0, v_1, v_2]^{H_r}$ of invariants under the action of H_r is generated by three basic invariants I_d ($\deg I_d = d$) [ST] which are used to express the defining function f_C of C and the inverse map v^{-1} of v .

Except for the case of No. 4, v has single valued inverse

$$v^{-1} : C \longrightarrow \mathbf{P}^1,$$

which is equal to the quotient map of C by the action of the (projective) monodromy group, whence the degree of v^{-1} is g' ; that is, v^{-1} is a $g' : 1$ mapping of C to $\mathbf{P}^1$. In

the case of No. 4, v^{-1} is 2-valued. In fact, the differential equation No. 4 is derived from that of No. 3 under a quadratic transformation.

The inverse map v^{-1} (except for No. 4) ramifies at $z = 0, 1, \infty$. Let p, q, r be the ramification indices at these points. Then the projective monodromy group of ${}_3E_2$ is a quotient group Δ/N of the Schwarz triangle group $\Delta = \Delta(p, q, r)$ under a normal subgroup N with index g' . There exists a Gauss hypergeometric differential equation ${}_2E_1$ whose projective monodromy group is the above Δ and whose Schwarz map $u : \mathbf{P}^1 \rightarrow M$ ($= \mathbf{P}^1$ or $\mathbf{C}$ or $\mathbf{H}$) has the single valued inverse

$$u^{-1} : M \rightarrow \mathbf{P}^1.$$

Then there exists a map

$$\varpi : M \rightarrow C$$

satisfying

$$u^{-1} = v^{-1} \circ \varpi.$$

This implies that the curve C is the quotient of M under the normal subgroup N of Δ . We explain this fact for the cases of No. 5 and No. 12 explicitly.

In the case of No. 5, $\Delta = \Delta(3, 2, 5)$, $M = \mathbf{P}^1$, and u is a Schwarz map of ${}_2E_1(-1/60, 11/60; 2/3)$. The curve C is a smooth quadric curve and ϖ is a usual Veronese map $\mathbf{P}^1 \xrightarrow{\sim} C \subset \mathbf{P}^2$.

In the case of No. 12, $\Delta = \Delta(4, 2, 4)$, $M = \mathbf{C}$, and u is a Schwarz map of ${}_2E_1(0, 1/4; 3/4)$. The curve C is a smooth cubic curve. The Schwarz map u and its single valued inverse map u^{-1} are given by

$$u(z) = z^{1/4} {}_2F_1(1/4, 1/2; 5/4; z) \quad \text{and} \quad u^{-1} = (\wp(u|\Omega))^{-2},$$

where $\wp(u|\Omega)$ is the Weierstrass $\wp$ function with period lattice Ω [Iwn, p. 69]. Then the map $\varpi : \mathbf{C} \rightarrow C$ is given by

$$\varpi(u) = [c_0 \wp'(u|3\Omega) : c_1 \wp(u|3\Omega) : 1]$$

for some $c_0, c_1 \in \mathbf{C}$.

As for the remaining case, M is the upper half plane $\mathbf{H}$.

The proof of the statement in this subsection is given in Section 5.

2. Invariants of finite unitary reflection groups

In this section, we recall some results from [ST].

Let G be a finite group generated by unitary reflections acting irreducibly on an $(n + 1)$ -dimensional complex vector space V . We describe the action of G as

$$xg \quad \text{for } x \in V, \ g \in G.$$

We say that G is *imprimitive* if V has a direct sum decomposition $V_1 + V_2 + \cdots + V_k$ with $\dim V_j \geq 1$ and $k \geq 2$ such that any element of G induces a permutation of $\{V_1, V_2, \dots, V_k\}$.

We choose a basis $\{v_0, v_1, \dots, v_n\}$ of V . Then G can be considered as a subgroup of $GL(n + 1, \mathbb{C})$. The action of G on V is extended naturally to the symmetric tensor product $S(V)$ which is isomorphic to the polynomial ring $\mathbb{C}[v_0, v_1, \dots, v_n]$. A polynomial $P(v)$ is called an *invariant* of G if

$$P(v)g = P(vg) = P(v) \quad \text{for any } g \in G$$

and is called an *anti-invariant* if

$$P(v)g = (\det g)^{-1} P(v) \quad \text{for any } g \in G.$$

We know [ST] that there are $n + 1$ algebraically independent invariant forms ('form' means homogeneous polynomial) $I_{d_0}, I_{d_1}, \dots, I_{d_n}$ of degree $d_0, d_1, \dots, d_n$ with

$$\prod_{j=0}^n d_j = g \quad (= \text{the order of } G)$$

such that any invariant form is expressible as a polynomial in $I_{d_0}, I_{d_1}, \dots, I_{d_n}$. Any anti-invariant form $P(v)$ is expressible as a product

$$P(v) = I(v)J(v)$$

of some invariant form $I(v)$ and the Jacobian

$$J(v) = \partial(I_{d_0}, I_{d_1}, \dots, I_{d_n}) / \partial(v_0, v_1, \dots, v_n)$$

(Bourbaki [Brb]). In particular, for any anti-invariant form $P(v)$, we have

$$\deg(P(v)) \geq \sum d_j - n - 1. \quad (2.1)$$

3. Monodromy group of ${}_3E_2$

Put $X = \mathbf{C} - \{0, 1\}$ and fix $z_0 \in X$. Let L_0 (respectively L_1) be a closed curve once surrounding 0 (respectively 1) positively starting and ending at z_0 . We denote by $V(z_0)$ the set of germs of holomorphic solutions of ${}_3E_2$ at the point z_0 . Then for any $L \in \pi_1(X, z_0)$ and $f \in V(z_0)$, the analytic continuation fL_* of f along L again belongs to $V(z_0)$. We write

$$f(LL')_* = (fL_*)L'_* = fL_*L'_*$$

if L' is continued after L . Recall the fundamental system $v = {}^t(v_0, v_1, v_2)$ of solutions of ${}_3E_2$ given by (1.2). We denote by vL_* the analytic continuation ${}^t(v_0L_*, v_1L_*, v_2L_*)$ of v along L . Then there exists a non-singular matrix $\rho_v(L)$ satisfying

$$vL_* = \rho_v(L)v.$$

It is easily seen that ρ_v defines a group homomorphism (called a monodromy representation)

$$\pi_1(X, z_0) \xrightarrow{\rho_v} GL(3, \mathbf{C}).$$

We call

$$H = H_v(a_0, a_1, a_2; b_1, b_2; z_0) = \rho_v(\pi_1(X, z_0)) \quad (3.1)$$

the *monodromy group* of ${}_3E_2$ with respect to the fundamental system v of solutions at z_0 . We call

$$H/Z(H)$$

the *projective monodromy group*, where $Z(H)$ is the center of H .

We have [Kt]

$$\rho_v(L_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e(1-b_1) & 0 \\ 0 & 0 & e(1-b_2) \end{pmatrix}, \quad (3.2)$$

$$\rho_v(L_1) = I - 2\sqrt{-1}e \left(\frac{b_1 + b_2 - a_0 - a_1 - a_2}{2} \right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (c_0, c_1, c_2), \quad (3.3)$$

$$c_0 = \frac{\sin \pi a_0 \sin \pi a_1 \sin \pi a_3}{\sin \pi b_1 \sin \pi b_2},$$

$$c_1 = \frac{\sin \pi a'_0 \sin \pi a'_1 \sin \pi a'_3}{\sin \pi b'_1 \sin \pi b'_2},$$

$$c_2 = \frac{\sin \pi a''_0 \sin \pi a''_1 \sin \pi a''_3}{\sin \pi b''_1 \sin \pi b''_2}$$

and

$$e(x) = \exp(2\pi\sqrt{-1}x).$$

Let H_r be the smallest normal subgroup of H containing $\rho_v(L_1)$; that is,

$$H_r = (H_v)_r = \langle \rho_v(L_0^k L_1 L_0^{-k}) k \in \mathbf{Z} \rangle. \quad (3.4)$$

If

$$b_1 + b_2 - a_0 - a_1 - a_2 \in \mathbf{Q} - \mathbf{Z},$$

then $\rho_v(L_1)$ is a reflection and H_r is generated by reflections. So we call H_r the *reflection subgroup* of H [BH]. By definition, we have

$$H = H_r \cdot \langle \rho_v(L_0) \rangle. \quad (3.5)$$

4. Lemmas

The defining function $f_C(v)$ of $C = v(\mathbf{P}^1) \subset \mathbf{P}^2$ is an irreducible homogeneous polynomial in $v = (v_0, v_1, v_2)$. Since the analytic continuation $f_C(v)L_*$ of $f_C(v)$ along $L \in \pi_1(X, z_0)$ is again a defining function of C , we obtain a one-dimensional representation ρ_C of $\pi_1(X, z_0)$ such that

$$f_C(v)L_* = \rho_C(L)f_C(v), \quad L \in \pi_1(X, z_0).$$

By definition, we have $\rho_C(L) = 1$ if $\rho_v(L) = I$.

LEMMA 4.1. *Assume that $\rho_v(L) \in H_r$ and that $\det(\rho_v(L)) = 1$. Then we have $\rho_C(L) = 1$.*

Proof. Since $\rho_v(L_1)$ is a reflection, it is diagonalizable and has eigenvalues 1, 1, x for some primitive n th root x of 1. The fact that $\rho_v(L) \in H_r$ implies that L is a product of m matrices of the form $\rho_v(L_0^k L_1 L_0^{-k})$ (see (3.4)). The fact that $\det(\rho_v(L)) = 1$ implies that this number m is a multiple of n . Since $(\rho_C(L_1))^n = 1$, we have $\rho_C(L) = 1$. $\square$

Let $(\alpha; \beta) = (\alpha_0, \alpha_1, \alpha_2; \beta_0, \beta_1, \beta_2)$ and $(a; b) = (a_0, a_1, a_2; b_1, b_2)$ be related by $a_j = \alpha_j + 1 - \beta_0$, $b_j = \beta_j + 1 - \beta_0$. Then equation (1.1) has the form

$${}_3D_2(\alpha; \beta) = z^{1-\beta_0} {}_3E_2(a; b).$$

We fix the fundamental system

$$\tilde{v} := z^{1-\beta_0} v = z^{1-\beta_0} (v_0, v_1, v_2) \quad (4.1)$$

of solutions of ${}_3D_2(\alpha; \beta)$ at z_0 . Similar to the definitions (3.1) and (3.4) of $H_v(a; b)$ and $(H_v)_r(a; b)$, we put

$$H_{\tilde{v}}(\alpha; \beta; z_0) = \rho_{\tilde{v}}(\pi_1(X, z_0)), \quad (4.2)$$

$$(H_{\tilde{v}})_r(\alpha; \beta; z_0) = \langle \rho_{\tilde{v}}(L_0^k L_1 L_0^{-k}) k \in \mathbf{Z} \rangle. \quad (4.3)$$

LEMMA 4.2. *Let $\tilde{v}$ and v be as above. Then we have $(H_{\tilde{v}})_r(\alpha; \beta) = (H_v)_r(a; b)$ as subsets of $GL(3, \mathbf{C})$. If $e(1 - \beta_0)I \in H_r(a; b)$, then $H_{\tilde{v}}(\alpha; \beta) = H_v(a; b)$ in $GL(3, \mathbf{C})$.*

Proof. From (4.1), we obtain

$$(1) \quad \rho_{\tilde{v}}(L_0) = e(1 - \beta_0) \rho_v(L_0)$$

$$(2) \quad \rho_{\tilde{v}}(L_1) = \rho_v(L_1).$$

Hence we have

$$\rho_{\tilde{v}}(L_0^k L_1 L_0^{-k}) = \rho_v(L_0^k L_1 L_0^{-k}), \quad k \in \mathbf{Z}.$$

Then (3.4) and (4.3) imply the first part of Lemma 4.2. Since

$$(H_v)_r(a; b) = \langle \rho_v(L_0), \rho_v(L_1) \rangle \quad \text{and} \quad (H_{\tilde{v}})_r(\alpha; \beta) = \langle \rho_{\tilde{v}}(L_0), \rho_{\tilde{v}}(L_1) \rangle,$$

the second part also follows from (1) and (2). $\square$

LEMMA 4.3. *Let $P(v)$ be an invariant form of the reflection subgroup H_r , and let L be an element of $\pi_1(X, z_0)$. We assume that*

$$P(v) = \sum_{k=0}^m P^{(k)}(v) \quad \text{with} \quad P^{(k)}(v)L_* = \lambda_k P^{(k)}(v),$$

where λ_k $0 \leq k \leq m$ are mutually distinct constant numbers. Then each $P^{(k)}(v)$ is itself an invariant form of H_r .

Proof. Because $\rho_v(L^{j+k}L_1L^{-j-k}) \in H_r$, we have $P(v)(L^{j+k}L_1L^{-j-k})_* = P(v)$, equivalently

$$(P(v)L_*^j)(L^kL_1L^{-k})_* = P(v)L_*^j.$$

This means that $P_j(v) := P(v)L_*^j$ is an invariant form of H_r . Because

$$P_j(v) = \sum_{k=0}^m \lambda_k^j P^{(k)}(v),$$

and because the Vandermonde determinant $\det(\lambda_k^j)_{0 \leq j, k \leq m}$ is not zero, we know that each $P^{(k)}(v)$ is a linear combination of $P_j(v)$, $0 \leq j \leq m$. This completes the proof. $\square$

5. Proof of main results 1.2

No. 2 $(a_0, a_1, a_2; b_1, b_2) = (-1/14, 3/14, 5/14; 2/3, 1/3)$

In this case, the characteristic exponents are

$$\begin{aligned} 0, 1/3, 2/3 & \quad \text{at } z = 0, \\ 0, 1, 1/2 & \quad \text{at } z = 1, \\ -1/14, 3/14, 5/14 & \quad \text{at } z = \infty. \end{aligned}$$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(3, 2, 7)$.

Since $H (= H_r)$ is of ST-No. 24 [BH], we know that the order g' of the projective monodromy group $H/Z(H)$ is 168 and that there exist three generators of invariants $I_4(v)$, $I_6(v)$, $I_{14}(v)$ of degree 4, 6, 14.

The single valued functions $I_d(v(z))$, $d = 4, 6, 14$, are holomorphic at $z = 0$ and $z = 1$. At $z = \infty$, $I_d(v(z))$ has pole of order at most $d/14$. This implies that $I_d(v(z)) = c_d$ is constant for $d = 4, 6$ and $I_{14}(v(z))$ is a polynomial of degree at most one.

Since $\det(\rho_v(L_0)) = 1$ and $\rho_v(L_0) \in H_r (= H)$, we have $\rho_C(L_0) = 1$ by Lemma 4.1. Since $\rho_v(L_1)^2 = I$, we have $\rho_C(L_1) = \pm 1$. We note that if $\rho_C(L_1) = 1$ then f_C is an invariant form and if $\rho_C(L_1) = -1$ then f_C is an anti-invariant form, hence of degree ≥ 21 by (2.1).

We next prove that c_4 and c_6 are not zero. By the definition of characteristic exponents at $z = \infty$, there exists a fundamental system of solutions of the following form:

$$w_j(z) = z^{-a_j} h_j(1/z), \quad j = 0, 1, 2,$$

where $h_j(x)$ are holomorphic at $x = 0$ and $h_j(0) \neq 0$. As a polynomial in w_0, w_1 and w_2 ,

$$\begin{aligned} I_4 &\text{ is a linear combination of } w_0^3 w_1, w_0 w_2^3, w_1^3 w_2, \\ I_6 &\text{ is a linear combination of } w_0^5 w_2, (w_0 w_1 w_2)^2, w_0 w_1^5, w_1 w_2^5. \end{aligned}$$

If $c_4 = 0$ then I_4 cannot contain the monomial $w_0^3 w_1$. Hence I_4 is reducible. Because f_C is an irreducible component of I_4 , we have $\deg(f_C) < 4 < 21$ and hence f_C is an invariant. This is a contradiction. This concludes that c_4 is not zero.

If $c_6 = 0$ then I_6 cannot contain the monomial $w_0^5 w_2$. Hence I_6 is reducible. Then, by the same argument as above, we know that f_C is an invariant with $\deg(f_C) < 6$ whence a constant multiple of I_4 . But this is impossible because $c_4 \neq 0$. This proves that c_6 is not zero.

Now we have

$$(I_6(v(z))/c_6)^2 - (I_4(v(z))/c_4)^3 = 0,$$

whence $\deg(f_C) \leq 12 < 21$ which again implies that f_C is an invariant form. If $\deg(f_C) < 12$, then f_C is a constant multiple of one of I_4, I_6, I_4^2 and $I_4 I_6$. This cannot happen because $c_4 c_6 \neq 0$. Therefore $\deg(f_C) = 12$ and

$$f_C(v) = (I_6(v)/c_6)^2 - (I_4(v)/c_4)^3.$$

By adding a constant multiple of $I_4^2(v)I_6(v)$ to $I_{14}(v)$ if necessary, we may assume that

$$I_{14}(v(z)) = c_{14}z$$

for some constant c_{14} . If $c_{14} = 0$, then $I_{14}(v)$ is a multiple of f_C and we obtain an invariant form I_{14}/f_C of degree two. This is a contradiction. Hence we have $c_{14} \neq 0$.

Now we know that the single valued inverse v^{-1} of $v = v(z)$ is given by

$$z = c_4^2 c_6 I_{14}(v)/c_{14} I_4(v)^2 I_6(v).$$

Hence v^{-1} is a $12 \cdot 14 (= g') : 1$ mapping of C to $\mathbf{P}^1$.

Remark. The basic invariants I_4, I_6 and I_{14} in Table 1 stand for $I_4/c_4, I_6/c_6$ and I_{14}/c_{14} above. Similar replacements of the basic invariants by constant multiplications are necessary in the following cases in order to obtain Table 1.

No. 3 $(a_0, a_1, a_2; b_1, b_2) = (3/14, 5/14, 13/14; 5/4, 3/4)$

In this case, the characteristic exponents are

$$\begin{array}{ll} 0, -1/4, 1/4 & \text{at } z = 0, \\ 0, 1, 1/2 & \text{at } z = 1, \\ 3/14, 5/14, 13/14 & \text{at } z = \infty. \end{array}$$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(4, 2, 7)$.

Since $H (= H_r)$ is of ST-No. 24 [BH], we know that the order g' of the projective monodromy group $H/Z(H)$ is 168 and that there exist three generators of invariants $I_4(v)$, $I_6(v)$, $I_{14}(v)$ of degree 4, 6, 14.

The single valued functions $I_d(v(z))$, $d = 4, 6, 14$, are holomorphic at $z = 1$ and $z = \infty$. At $z = \infty$, $I_d(v(z))$ has zero of order at least $3d/14$. At $z = 0$, $I_d(v(z))$ has pole of order at most $d/4$. Consequently, we have

$$I_4(v(z)) = c_4/z, \quad I_6(v(z)) = 0, \quad I_{14}(v(z)) = c_{14}/z^3$$

for some constant numbers c_4 and c_{14} .

As in the previous example, we have $\rho_C(L_0) = 1$ and $\rho_C(L_1) = \pm 1$. The defining function f_C of C is an invariant form or an anti-invariant form of H according to whether $\rho_C(L_1) = 1$ or $\rho_C(L_1) = -1$. If $\deg f_C < 21$, then f_C is an invariant form.

Assume that $c_4 = 0$. Then f_C is a common factor of I_4 and I_6 . But since there is no non-constant invariant form of degree less than four, f_C must be a constant multiple of I_4 . Then I_6 is divisible by I_4 and we obtain an invariant form I_6/I_4 of degree two. This is a contradiction. This proves that c_4 is not zero. This again implies that $f_C = I_6$.

Assume that $c_{14} = 0$. Then I_{14} is divisible by I_6 and I_{14}/I_6 must be a constant multiple of I_4^2 . This is a contradiction. This proves that c_{14} is not zero. Therefore, we know that the single valued inverse v^{-1} of $v = v(z)$ is given by

$$z = c_4^7 I_{14}(v)^2 / c_{14}^2 I_4(v)^7.$$

Hence v^{-1} is a $6 \cdot 28 (= g') : 1$ mapping of C to $\mathbf{P}^1$.

Remark. The differential equation ${}_3E_2(3/14, 5/14, 13/14; 5/4, 3/4)$ has a solution

$$z^{-3/14} {}_3F_2(-1/28, 3/14, 13/28; 6/7, 2/7; 1/z).$$

Hence, for the parameters

$$(a_0, a_1, a_2; b_1, b_2) = (-1/28, 3/14, 13/28; 6/7, 2/7),$$

f_C is an invariant form of H_r of degree six, and the Schwarz map $v(z)$ has a single valued inverse v^{-1} of the form

$$z = I_4(v)^7 / I_{14}(v)^2,$$

where $I_d(v)$ is an invariant form of $H_r(-1/28, 3/14, 13/28; 6/7, 2/7)$ of degree d .

We will refer to this parameter set in the following example.

No. 4 $(a_0, a_1, a_2; b_1, b_2) = (-1/14, 1/14, 9/14; 6/7, 2/7)$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(7, 2, 7)$.

In Table 8.3 of [BH], the following parameters are given for No. 4:

$$(\alpha_0, \alpha_1, \alpha_2; \beta_0, \beta_1, \beta_2) = (3/14, 5/14, 13/14; 2/7, 1/7, 4/7).$$

Since

$${}_3D_2(3/14, 5/14, 13/14; 2/7, 1/7, 4/7) = z^{5/7} {}_3E_2(13/14, 15/14, 23/14; 6/7, 9/7)$$

and since [BH]

$$H(3/14, 5/14, 13/14; 2/7, 1/7, 4/7) = H_r(3/14, 5/14, 13/14; 2/7, 1/7, 4/7),$$

$I(z^{5/7}v(z))$ is a single valued function of z for any invariant form $I(v)$ of $H_r = H_r(-1/14, 1/14, 9/14; 6/7, 2/7)$.

The reflection subgroup H_r is of ST-No. 24. Let $I_6(v)$ be an invariant form of degree six. Since $I_6(z^{5/7}v(z)) (= z^{30/7}I_6(v(z)))$ is single valued and since the characteristic exponents are 0, 1/7, 5/7 at $z = 0$, $I_6(v(z))$ has the form

$$I_6(v(z)) = z^{5/7}(c_0 + c_1z + \cdots).$$

Since the characteristic exponents are $-1/14, 1/14, 9/14$ at $z = \infty$, $I_6(v(z))$ cannot have pole of order greater than 6/14 at $z = \infty$. This implies that

$$I_6(v(z)) = 0.$$

By the same reasoning used in the previous example, we know that

$$f_C(v) = I_6(v).$$

By direct calculations, we know that the image C of the Schwarz map $v(z)$ is smooth and projectively equivalent to the image of the Schwarz map of ${}_3E_2(-1/28, 3/14, 13/28; 6/7, 2/7)$ (see the Remark of the previous example). But this fact also follows from the following functional identity:

$$\begin{aligned} (1-z)^{1/14} {}_3F_2\left(-1/28, 3/14, 13/28; 6/7, 2/7; \frac{-4z}{(1-z)^2}\right) \\ = {}_3F_2(-1/14, 1/14, 9/14; 6/7, 2/7; z) \end{aligned}$$

which is a special case of

$$\begin{aligned} (1-z)^{-a} {}_3F_2\left(\frac{a}{2}, \frac{1+a}{2}, 1+a-b-c; 1+a-b, 1+a-c; \frac{-4z}{(1-z)^2}\right) \\ = {}_3F_2(a, b, c; 1+a-b, 1+a-c; z) \end{aligned}$$

(see [Bly, p. 97]). In particular, the Schwarz map $v(z)$ has two-valued inverse.

No. 5 $(a_0, a_1, a_2; b_1, b_2) = (3/10, 1/2, 7/10; 4/3, 2/3)$

In this case, the characteristic exponents are

$$\begin{aligned} 0, -1/3, 1/3 & \text{ at } z = 0, \\ 0, 1, 1/2 & \text{ at } z = 1, \\ 3/10, 1/2, 7/10 & \text{ at } z = \infty. \end{aligned}$$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(3, 2, 5)$. In fact we have $H/Z(H) \simeq \Delta$ which is proved below.

In Table 8.3 of [BH], the parameters $0, 1/5, 4/5; 1/2, 1/6, 5/6$ are given. For these parameters, as stated in [BH], $H_{\bar{v}} = (H_{\bar{v}})_r$ which is of ST-No. 23 and hence contains $-I$. Then, by Lemma 4.2, we have

$$\begin{aligned} H_v(3/10, 1/2, 7/10; 4/3, 2/3) &= H_v(13/10, 1/2, 7/10; 2/3, 4/3) \\ &= H_{\bar{v}}(0, 1/5, 4/5; 1/2, 1/6, 5/6) \\ &= (H_{\bar{v}})_r(0, 1/5, 4/5; 1/2, 1/6, 5/6) \\ &= (H_v)_r(13/10, 1/2, 7/10; 2/3, 4/3) \\ &= (H_v)_r(3/10, 1/2, 7/10; 4/3, 2/3) \end{aligned}$$

as subsets of $GL(3, \mathbb{C})$. Therefore $H = H(3/10, 1/2, 7/10; 4/3, 2/3)$ coincides with its reflection subgroup H_r of type ST-No. 23. This proves that

$$H/Z(H) \simeq \Delta(3, 2, 5).$$

Let I_2, I_6, I_{10} be generators of invariant forms of H . The single valued function $I_d(v(z))$ has pole at $z = 0$ of order at most $d/3$, holomorphic at $z = 1$ and has zero at $z = \infty$ of order at least $3d/10$. Hence we have

$$I_2(v(z)) = 0, \quad I_6(v(z)) = c_6 z^{-2}, \quad I_{10}(v(z)) = c_{10} z^{-3}$$

for some constant numbers c_6 and c_{10} . We know that $c_6, c_{10} \neq 0$ by the same argument used in the previous examples. Thus we conclude that

$$f_C = I_2$$

and that the single valued inverse v^{-1} of $v(z)$ is given by

$$z = c_6^5 I_{10}(v)^3 / c_{10}^3 I_6(v)^5.$$

The degree of v^{-1} is thus 60 which is equal to g' .

The projective monodromy group of the Gauss hypergeometric differential equation

$${}_2E_1(-1/60, 11/60; 2/3) \tag{5.1}$$

is also $\Delta(3, 2, 5)$. The ratio of linearly independent solutions

$$u_0 = F(-1/60, 11/60; 2/3; z), \quad u_1 = z^{1/3} F(19/60, 31/60; 4/3; z)$$

of (5.1) defines a Schwarz map u of (5.1) which has a single valued inverse

$$u^{-1} : \mathbf{P}^1 \longrightarrow \mathbf{P}^1.$$

Then there exists a Veronese map

$$\varpi : \mathbf{P}^1 \longrightarrow C \subset \mathbf{P}^2$$

of degree two, such that

$$u^{-1} = v^{-1} \circ \varpi.$$

In fact, we have the following functional equalities:

$$u_0^2 = z^{1/3} f_1, \quad u_0 u_1 = z^{1/3} f_0, \quad u_1^2 = z^{1/3} f_2,$$

where

$$\begin{aligned} f_0 &= {}_3F_2(3/10, 7/10, 1/2; 4/3, 2/3; z), \\ f_1 &= z^{-1/3} {}_3F_2(-1/30, 11/30, 1/6; 1/3, 2/3; z), \\ f_2 &= z^{1/3} {}_3F_2(19/30, 31/30, 5/6; 5/3, 4/3; z), \end{aligned}$$

are solutions of ${}_3E_2(3/10, 7/10, 1/2; 4/3, 2/3)$. These functional equalities are special cases of the following equalities due to Clausen and Orr (see [Bly, p. 86 and p. 100]):

$$\begin{aligned} (F(a, b; a + b + 1/2; z))^2 &= {}_3F_2(2a, 2b, a + b; 2a + 2b, a + b + 1/2; z), \\ F(a, b; a + b + 1/2; z)F(1/2 - a, 1/2 - b; 3/2 - a - b; z) \\ &= {}_3F_2(a - b + 1/2, b - a + 1/2, 1/2; a + b + 1/2, 3/2 - a - b; z). \end{aligned}$$

No. 6 $(a_0, a_1, a_2; b_1, b_2) = (13/10, 1/2, 7/10; 8/5, 7/5)$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(5, 2, 5)$.

From the same reasoning as used in the previous example, we can draw the following conclusion.

We have $H = H_r$ which is of ST-No. 23. We can choose generators I_2, I_6 and I_{10} of invariants of H such that

$$I_2(v(z)) = 1/z, \quad I_6(v(z)) = 0, \quad I_{10}(v(z)) = 1/z^6.$$

Consequently we have

$$f_C = I_6$$

and the single valued inverse v^{-1} of $v(z)$ is obtained by

$$z = I_2(v)^5 / I_{10}(v),$$

which is a $60(= g') : 1$ mapping of C to $\mathbf{P}^1$.

No. 7 $(a_0, a_1, a_2; b_1, b_2) = (-1/30, 1/6, 11/30; 4/5, 1/5)$

No. 8 $(a_0, a_1, a_2; b_1, b_2) = (-1/30, 1/6, 11/30; 3/4, 1/4)$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(5, 2, 5)$ in case of No. 7 and $\Delta(4, 2, 5)$ in case of No. 8.

In these two cases, $H_r = H$ and are of ST-No. 27 [BH]. In both cases, we know from direct calculations that there exist invariant forms I_6 , I_{12} and I_{30} such that

$$I_6(v(z)) = 1, \quad I_{12}(v(z)) = 1, \quad I_{30}(v(z)) = z.$$

Therefore, we have

$$f_C = I_{12} - I_6^2.$$

The single valued inverse v^{-1} of $v(z)$ is given by

$$z = I_{30}(v)/I_6(v)^5,$$

which is a 360 ($= g'$) : 1 mapping of C to $\mathbf{P}^1$.

In the following three examples (No. 9, 10, 11), $H_r = H$ and are of ST-No. 25 [BH].

$$\text{No. 9 } (a_0, a_1, a_2; b_1, b_2) = (1/6, 2/3, 5/6; 5/4, 3/4)$$

$$\text{No. 10 } (a_0, a_1, a_2; b_1, b_2) = (1/9, 4/9, 7/9; 7/6, 1/2)$$

$$\text{No. 11 } (a_0, a_1, a_2; b_1, b_2) = (1/9, 4/9, 7/9; 5/4, 3/4)$$

The projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle groups $\Delta(4, 3, 6)$, $\Delta(6, 3, 3)$, $\Delta(4, 3, 3)$, in cases No. 9, 10, 11 respectively.

By direct calculations, we know that there exist generators I_6 , I_9 and I_{12} of invariants of H such that

$$I_6(v(z)) = 1/z, \quad I_9(v(z)) = 1/z^2, \quad I_{12}(v(z)) = 1/z^3$$

for No. 9,

$$I_6(v(z)) = 1/z, \quad I_9(v(z)) = 1/z, \quad I_{12}(v(z)) = 0$$

for No. 10, and

$$I_6(v(z)) = 1/z, \quad I_9(v(z)) = (1 + cz)/z^2, \quad I_{12}(v(z)) = 1/z^3$$

for No. 11, where c is a constant. Hence we have

$$f_C = I_9^2 - I_6 I_{12} \quad \text{for No. 9,}$$

$$f_C = I_{12} \quad \text{for No. 10,}$$

$$f_C = I_9^2 I_{12} - I_6 (I_{12} + c I_6^2)^2 \quad \text{for No. 11.}$$

The single valued inverses v^{-1} of $v(z)$ are given by

$$\begin{aligned} z &= I_6(v)^2/I_{12}(v) && \text{for No. 9, 11,} \\ z &= I_9(v)^2/I_6(v)^3 && \text{for No. 10.} \end{aligned}$$

In any case, v^{-1} is a 216 ($= g'$) : 1 mapping.

No. 12 $(a_0, a_1, a_2; b_1, b_2) = (-1/12, 2/12, 5/12; 3/4, 1/4)$

In this case, the characteristic exponents are

$$\begin{aligned} 0, 1/4, 3/4 &&& \text{at } z = 0, \\ 0, 1, 1/2 &&& \text{at } z = 1, \\ -1/12, 2/12, 5/12 &&& \text{at } z = \infty, \end{aligned}$$

and the projective monodromy group $H/Z(H)$ is a quotient group of the Schwarz triangle group $\Delta(4, 2, 4)$.

As stated in [BH], the reflection subgroup H_r of the monodromy group H is imprimitive and of type $G(3, 3, 3)$ (see [ST] for this notation). In fact, if we choose the fundamental system of solutions

$$\varphi = {}^t(\varphi_0, \varphi_1, \varphi_2) := \begin{pmatrix} 1 & -(\sqrt{3}+1)/2 & (\sqrt{3}-1)/2 \\ -1 & -(\sqrt{3}+1)/2 & (\sqrt{3}-1)/2 \\ 0 & 1 & 1 \end{pmatrix} v,$$

the reflection subgroup $H_r = \langle \rho_v(L_0^k L_1 L_0^{-k}), 0 \leq k \leq 3 \rangle$ of H is conjugate to the group generated by the following four matrices:

$$\begin{aligned} \rho_\varphi(L_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \rho_\varphi(L_0 L_1 L_0^{-1}) &= \begin{pmatrix} 0 & \omega^2 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \rho_\varphi(L_0^2 L_1 L_0^{-2}) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \rho_\varphi(L_0^3 L_1 L_0^{-3}) &= \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\omega = e(1/3)$ is a primitive cubic root of 1.

As stated in [ST], H_r has the basic invariants

$$\sum_{k=0}^2 \varphi_k^3, \quad \prod_{k=0}^2 \varphi_k, \quad \sum_{0 \leq j < k \leq 2} (\varphi_j \varphi_k)^3.$$

Then, by Lemma 4.3, we can find another basic set of invariants of H_r of the following form:

$$\begin{aligned} I_3 &= 2v_0^2v_1 - (2 - \sqrt{3})v_2^3 - \sqrt{3}v_1^2v_2, \\ I'_3 &= 2v_0^2v_2 - (2 + \sqrt{3})v_1^3 + \sqrt{3}v_1v_2^2, \\ I_6 &= 4v_0^6 + 12v_0^4v_1v_2 + 3(11 + 8\sqrt{3})v_0^2v_1^4 + 90v_0^2v_1^2v_2^2 + 3(11 - 8\sqrt{3})v_0^2v_2^4 \\ &\quad + 3(15 + 8\sqrt{3})v_1^5v_2 - 6v_1^3v_2^3 + 3(15 - 8\sqrt{3})v_1v_2^5. \end{aligned} \quad (5.2)$$

By a similar argument as used in the previous cases, we have (by constant multiplications)

$$I_3(v(z)) = z^{1/4}, \quad I'_3(v(z)) = 0, \quad I_6(v(z)) = 1.$$

Hence we know that

$$f_C = I'_3$$

and that the single valued inverse v^{-1} of v is given by

$$z = I_3(v)^4/I_6(v)^2, \quad (5.3)$$

which is a $36 : 1$ mapping of the smooth cubic curve C to $\mathbf{P}^1$, whence we have $g' = 36$.

Because the order of $G(3, 3, 3)$ is 54 and $H = H_r \cdot \langle \rho_v(L_0) \rangle$ with $H_r \cap \langle \rho_v(L_0) \rangle = \{I\}$, the order of H is $4 \cdot 54$. Since $(L_0L_1)^{-1}$ is a loop surrounding the point $z = \infty$, we have

$$\rho_v((L_0L_1)^4) = e(1/3)I. \quad (5.4)$$

By direct calculations, we have

$$\rho_v((L_1L_0^2)^3) = \rho_v((L_0L_1L_0)^3) = -I. \quad (5.5)$$

From (5.4) and (5.5), we can re-prove that $g' = 36$.

The projective monodromy group of the Gauss hypergeometric differential equation

$${}_2F_1(0, 1/4; 3/4), \quad (5.6)$$

is isomorphic to $\Delta(4, 2, 4)$. If we define the Schwarz map u of (5.6) by

$$u = z^{1/4} {}_2F_1(1/4, 1/2; 5/4; z),$$

then its single valued inverse

$$u^{-1} : \mathbf{C} \longrightarrow \mathbf{P}^1$$

is given by

$$z = (\wp(u|\Omega))^{-2}.$$

Here $\wp(u|\Omega)$ is the Weierstrass $\wp$ function with period lattice Ω generated by two constant numbers

$$\omega_1 := (u(L_1 L_0^2)_* - u)(z) \quad \text{and} \quad \omega_2 := (u(L_0 L_1 L_0)_* - u)(z). \quad (5.7)$$

It is well known (see [Iwn, p. 69]) that Ω is a square lattice, that is, $\omega_1 \in \mathbf{R}$ and $\omega_2 = \sqrt{-1}\omega_1$. In this case, $\wp(u|\Omega)$ and z have the following form at $u = 0$:

$$\wp(u|\Omega) = u^{-2}p(u^4), \quad z = u^4\tilde{p}(u^4), \quad (5.8)$$

where $p(x)$ and $\tilde{p}(x)$ are holomorphic at $x = 0$ and $p(0) = \tilde{p}(0) = 1$.

Since $\mathbf{C}$ is simply connected, there exists a map

$$\varpi : \mathbf{C} \longrightarrow C$$

such that

$$(\wp(u|\Omega))^{-2} = v^{-1} \circ \varpi(u). \quad (5.9)$$

We will show that ϖ is given by

$$(v_0/v_2) \circ \varpi(u) = c_0 \wp'(u|3\Omega), \quad (v_1/v_2) \circ \varpi(u) = c_1 \wp(u|3\Omega),$$

for some $c_0, c_1 \in \mathbf{C}$.

First we note that $f_{jk}(u) := (v_j/v_k) \circ \varpi(u)$ is a single valued meromorphic function of u . Since Ω is generated by (5.7) and since $f_{jk}(u+3\omega_1) = (v_j/v_k)(L_1 L_0^2)_*^3$ and $f_{jk}(u+3\omega_2) = (v_j/v_k)(L_0 L_1 L_0)_*^3$, we know, from (5.5), that f_{jk} is an elliptic function with period 3Ω . We denote by D a fundamental region of 3Ω containing the point $u = 0$.

We will find poles of f_{12} in D . Assume $f_{12}(u) = \infty$. Then $\varpi(u) = [x : 1 : 0]$, with $x \in \mathbf{P}^1$. From the equality $I'_3(x, 1, 0) = 0$, we know that $x = \infty$. This means that $\varpi(u) = [1 : 0 : 0]$. Then, by (5.2) and (5.3), we have $z = 0$. There are nine points u in D satisfying $(\wp(u|\Omega))^{-2} = 0$. On the other hand, since v^{-1} is $36 : 1$ and has the ramification index 4 at $z = 0$, there are $36/4$ points in C over $z = 0$.

Hence, $\varpi^{-1}([1 : 0 : 0]) = \{0\} \bmod 3\Omega$. This means that $f_{12}(u)$ has a pole only at $u = 0 \bmod 3\Omega$. By (5.8), we know that at $u = 0$, $f_{12}(u)$ has the form

$$f_{12} = z^{-1/2} f(z) = u^{-2} \tilde{f}(u^4),$$

where $f(x)$ and $\tilde{f}(x)$ are holomorphic at $x = 0$ and $f(0) \neq 0$, $\tilde{f}(0) \neq 0$. This proves that

$$f_{12}(u) = c_1 \wp(u|3\Omega)$$

for some constant number c_1 .

By the equation $I'_3 = 0$ of C (see (5.2)), we know

$$f_{02}(u) = c_0 \wp'(u|3\Omega)$$

for some constant number c_0 .

The equality (5.9) then has the form

$$\wp(u|\Omega) = (I_6/I_3^2)(c_0 \wp'(u|3\Omega), c_1 \wp(u|3\Omega), 1). \quad (5.10)$$

The right-hand side of (5.10) is a rational function of $\wp(u|3\Omega)$ and $\wp'(u|3\Omega)^2$ (see (5.2)), whence reduces to a rational function of $\wp(u|3\Omega)$, which is nothing but the formula expressing $\wp(u|\Omega) = \wp(3u|3\Omega)/9$ as a rational function of $\wp(u|3\Omega)$.

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