EXISTENCE OF POSITIVE SOLUTIONS TO 
\((k, n - k)\) CONJUGATE BOUNDARY VALUE PROBLEMS†

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1. Introduction and main results

This paper is concerned with the existence of positive solutions to the following 
\((k, n - k)\) conjugate boundary value problem (BVP):

\[
\begin{cases}
(-1)^{n-k} y^{(n)} = a(t) f(y), & 0 < t < 1, \\
y^{(i)}(0) = 0, & 0 \leq i \leq k - 1, \\
y^{(j)}(1) = 0, & 0 \leq j \leq n - k - 1,
\end{cases}
\]  
(1.1)

where \(k\) is a positive integer less than \(n - 1\). The following hypotheses are adopted throughout:

(H1) \(a(t)\) is a non-negative measurable function defined on \([0, 1]\) and satisfies 

\[
0 < \int_0^1 h(s) a(s) \, ds < +\infty; \quad h(t) := \min\{t^{n-k}, (1-t)^{k}\};
\]  
(1.2)

(H2) \(f(u)\) is a non-negative continuous function defined on \(R_+ := [0, +\infty)\), and satisfies either

(i) \(f_0 = 0\) and \(f_\infty = +\infty\) (superlinear), or

(ii) \(f_0 = +\infty\) and \(f_\infty = 0\) (sublinear), where 

\[
f_0 := \lim_{u \downarrow 0} \frac{f(u)}{u}, \quad f_\infty := \lim_{u \uparrow \infty} \frac{f(u)}{u}.
\]

It is of interest to note that (H1) allows \(a(t) \equiv 0\) on some subintervals of \([0, 1]\) and to have singularity at \(t = 0\) and \(t = 1\). For example,

\[
a(t) = t^{-\alpha} (1-t)^{-\beta} (|2\pi t| + \cos 2\pi t),
\]

satisfies (1.2) provided \(\alpha \in (0, n + 1 - k), \beta \in (0, k + 1)\).

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The BVP (1.1) arises in many different areas of applied mathematics and physics. For examples, see [1, 2, 7, 10] when \( n = 2 \). However, higher-order boundary value problems for ordinary differential equations are not documented as well as for second-order problems. Possibly, the best known setting of a boundary value problem for a higher-order ordinary differential equation involves two-point problems for the fourth-order equation,

\[ y^{(4)} - p(t)g(y) = 0, \]

which in certain cases arises in describing deformations of an elastic beam with the boundary conditions often reflecting both ends simply supported, or one end simply supported and other end clamped by sliding clamps, while vanishing moments and shear force at rail ends are frequently included in boundary conditions; for references, see Gupta [8] and Yang [11]. One derivation of this fourth-order equation plus the two-point boundary conditions has resulted when the method of lines is used in the discretization over regions of certain partial differential equations describing the deflection of an elastic rail.

With relation to the results of this paper is the recent work by Ma and Wang [9] in which they established the existence of at least one positive solution of the above fourth-order equation satisfying (2, 2) conjugate conditions for the cases when \( g \) is superlinear or sublinear.

The main purpose of this paper is to extend and improve the existence results in [5, 6, 9]. All the results in [5, 6, 9] require that:

(A) \( a : [0, 1] \to [0, \infty) \) is continuous and does not vanish identically on any subinterval.

In addition, [6] deals only with the case of \( n = 2 \), [9] only the case where \( n = 4 \) and \( k = 2 \), and [5] deals only with the case where \( k = n - 1 \). The paper [4] and [10] deals with the problem (1.1) when \( f \) is singular at \( y = 0 \).

In Section 2, we will state and prove the following theorem, which is fundamental for our main result.

**Theorem 1.** There exist two positive numbers \( 0 < \lambda < 1 \) and \( B > 0 \) such that

\[ \lambda Bg(t)h(s) \leq G(t, s) \leq Bh(s) \quad \text{on} \quad [0, 1] \times [0, 1], \]

where \( G(t, s) \) is the Green function for \( (-1)^{n-k} y^{(n)} = 0 \) with boundary conditions

\( y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1; \quad y^{(j)}(1) = 0, \quad 0 \leq j \leq n - k - 1, \)

and

\[ g(t) := \min(t^k, (1-t)^{n-k}) = h(1-t). \]
In Section 3, we will establish the following existence result.

**Theorem 2.** Let \((H_1)\) and \((H_2)\) hold. The problem (1.1) has a positive solution \(y(t)\).

Here we say that a function \(y(t)\) is a positive solution to the problem (1.1) if it satisfies the following conditions:

(i) \(y(t) \in C([0, 1]; R_+) \cap C^n((0, 1); R), y(t) > 0\) on \((0, 1);\)

(ii) \(y^{(n-1)}(t)\) is locally absolutely continuous in \((0, 1);\)

(iii) \((-1)^{n-k} y^{(n)} = a(t) f(y), \) for a.e. \(t \in [0, 1],\) and \(y^{(i)}(0) = y^{(i)}(1) = 0,\)

\[0 \leq i \leq k - 1, 0 \leq j \leq n - k - 1.\]

It is clear that \(y(t) \equiv 0\) is a trivial solution to the problem (1.1) when \(f(0) = 0.\)

Because of the physical background of the problem above, we are mainly interested in positive solutions.

Besides Theorem 1, the proof of Theorem 2 will be based on an application of the following fixed point theorem due to Krasnoselskii (which is quoted from [6]).

**Theorem 3.** Let \(E\) be a Banach space, and let \(K \subset E\) be a cone in \(E\). Assume \(\Omega_1, \Omega_2\) are open subset of \(E\) with \(0 \in \Omega_1, \Omega_1 \subset \Omega_2,\) and let

\[\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K\]

be a completely continuous operator such that either

(i) \(\|\Phi y\| \leq \|y\| \forall y \in K \cap \partial \Omega_1 \) and \(\|\Phi y\| \geq \|y\| \forall y \in K \cap \partial \Omega_2;\) or

(ii) \(\|\Phi y\| \geq \|y\| \forall y \in K \cap \partial \Omega_1 \) and \(\|\Phi y\| \leq \|y\| \forall y \in K \cap \partial \Omega_2.\)

Then \(\Phi\) has a fixed point in \(K \cap (\bar{\Omega}_2 \setminus \Omega_1).\)

2. Proof of Theorem 1

In this section, we explore some positive properties of the Green function, which will be used in the next section.

Let \(F(t) \in C[0, 1]\) and \(F(t) > 0\) on \((0, 1).\) Then, it is well-known from references [3] and [4] that the linear \((k, n - k)\) conjugate boundary value problem

\[
\begin{cases}
(-1)^{n-k} y^{(n)} = F(t), & 0 \leq t \leq 1, \\
y^{(i)}(0) = 0, & 0 \leq i \leq k - 1, \\
y^{(j)}(0) = 0, & 0 \leq j \leq n - k - 1,
\end{cases}
\]

has a unique positive solution \(y(t),\) which can be represented as

\[y(t) = \int_0^1 G(t, s) F(s) \, ds, \quad 0 \leq t \leq 1,
\]

(2.1)
where the Green’s function $G(t, s)$ possesses the following properties:

$$
\begin{align*}
G(t, s) &> 0, & \text{on } (0, 1) \times (0, 1), \\
\frac{\partial^i}{\partial t^i} G(0, s) &= 0, & 0 < s < 1, \quad 0 \leq i \leq k - 1, \\
\frac{\partial^k}{\partial t^k} G(0, s) &> 0, & 0 < s < 1, \\
\frac{\partial^j}{\partial t^j} G(1, s) &= 0, & 0 < s < 1, \quad 0 \leq j \leq n - k - 1, \\
(-1)^{n-k} \frac{\partial^{n-k}}{\partial t^{n-k}} G(1, s) &> 0, & 0 < s < 1.
\end{align*}
$$

(2.3)

Since $y \in C^n[0, 1]$, we have the Taylor expansion

$$
y(t) = a_0 + a_1 t + \cdots + \frac{a_{n-1}}{(n-1)!} t^{n-1} \\
+ \frac{(-1)^{n-k}}{(n-1)!} \int_0^t (t-s)^{n-1} F(s) \, ds, \quad 0 \leq t \leq 1,
$$

(2.4)

holds. It follows from (2.3) that $a_0 = a_1 = \cdots = a_{k-1} = 0$, $a_k > 0$, and

$$
\frac{a_k}{(k+1-j)!} + \cdots + \frac{a_{n-1}}{(n-j)!} + \frac{(-1)^{n-k}}{(n-j)!} \int_0^1 (1-s)^{n-j} F(s) \, ds = 0,
$$

for $1 \leq j \leq n - k$, and hence, by Cramer’s rule, $a_k, \ldots, a_{n-1}$ are all linear combinations of

$$
\int_0^1 (1-s)^k F(s) \, ds, \ldots, \int_0^1 (1-s)^{n-1} F(s) \, ds.
$$

From (2.2)–(2.4) and the arbitrariness of $F(s)$, we conclude that

$$
G(t, s) = \begin{cases} 
  t^k (1-s)^k P(t, s), & 0 \leq t \leq s \leq 1, \\
  t^k (1-s)^k P(t, s) + \frac{(-1)^{n-k}}{(n-1)!} (t-s)^{n-1}, & 0 \leq s \leq t \leq 1,
\end{cases}
$$

(2.5)

where $P(t, s)$ is a polynomial of order $(n - k - 1)$ of variables $t$ and $s$,

$$
P(0, s) > 0 \text{ on } (0, 1) \quad \text{and} \quad P(t, 1) > 0 \text{ on } (0, 1).
$$

Similarly, we have

$$
y(t) = b_0 + b_1 (t - 1) + \cdots + \frac{b_{n-1}}{(n-1)!} (1-t)^{n-1} \\
+ \frac{(-1)^k}{(n-1)!} \int_t^1 (s-t)^{n-1} F(s) \, ds, \quad 0 \leq t \leq 1.
$$

(2.6)
It follows again from (2.3) that \( b_0 = b_1 = \cdots = b_{n-k-1} = 0, b_{n-k} > 0, \) and
\[
\frac{b_{n-k}}{(n-k+1-i)!} + \cdots + \frac{b_{n-1}}{(n-i)!} + \frac{(-1)^k}{(n-i)!} \int_0^1 s^{n-i} F(s) \, ds = 0,
\]
for \( 1 \leq i \leq k, \) and hence, by Cramer's rule, \( b_{n-k}, \ldots, b_{n-1} \) are all linear combinations of
\[
\int_0^1 s^{n-k} F(s) \, ds, \ldots, \int_0^1 s^{n-1} F(s) \, ds.
\]
From (2.2), (2.3), (2.6) and the arbitrariness of \( F(s) \), we conclude that
\[
G(t, s) = \begin{cases} 
(1-t)^{n-k} s^{n-k} Q(t, s) + \frac{(-1)^k}{(n-1)!} (s-t)^n, & 0 \leq t \leq s \leq 1, \\
(1-t)^{n-k} s^{n-k} Q(t, s), & 0 \leq s \leq t \leq 1,
\end{cases} \tag{2.7}
\]
where \( Q(t, s) \) is a polynomial of order \( k-1 \) of variables \( t \) and \( s \), satisfying
\[Q(1, s) > 0 \text{ on } (0, 1) \quad \text{and} \quad Q(t, 0) > 0 \text{ on } (0, 1) .\]

From (2.5) and (2.7) we conclude that there exists a number \( B > 0 \), such that if \( 0 \leq t \leq s \leq 1, \) then
\[G(t, s) = t^k (1-s)^k P(t, s) \leq B(1-s)^k ,\]
and
\[
G(t, s) = (1-t)^{n-k} s^{n-k} Q(t, s) + \frac{(-1)^k}{(n-1)!} (s-t)^n
\leq (1-t)^{n-k} s^{n-k} Q(t, s) + \frac{1}{(n-1)!} s^{n-1}
\leq Bs^{n-k}.
\]
Therefore, we have
\[
G(t, s) \leq B \min\{s^{n-k}, (1-s)^k\} = B h(s), \quad 0 \leq t \leq s \leq 1. \tag{2.8}
\]
In the same way, we have
\[
G(t, s) \leq B \min\{s^{n-k}, (1-s)^k\} = B h(s), \quad 0 \leq s \leq t \leq 1. \tag{2.9}
\]
From (2.5) and (2.7)-(2.9), we conclude that there exists positive number \( 0 < \lambda < 1 \) such that
\[
\lambda B g(t) h(s) \leq G(t, s) \leq B h(s) \text{ on } [0, 1] \times [0, 1].
\]
Remark 1. For example, when \( k = n - 1 \), we find that the Green’s function \( G(t, s) \) is explicitly given by

\[
G(t, s) = \begin{cases} 
\frac{t^{n-1} \left(1 - s\right)^{n-1} - (t - s)^{n-1}}{(n - 1)!}, & 0 \leq s \leq t \leq 1, \\
\frac{t^{n-1} \left(1 - s\right)^{n-1}}{(n - 1)!}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

A direct calculation shows that \( G(t, s) \) satisfies Theorem 1. However, the explicit formula of Green’s function cannot be found in [5].

Remark 2. Let \( y(t) \) be the unique solution to the boundary value problem (2.1). Then

\[
y(t) \leq B \int_0^1 h(s) F(s) \, ds, \quad 0 \leq t \leq 1,
\]

i.e.

\[
\|y\| := \max\{|y(t)|; 0 \leq t \leq 1\} \leq B \int_0^1 h(s) F(s) \, ds,
\]

and hence,

\[
y(t) \geq \lambda B g(t) \int_0^1 h(s) F(s) \, ds \geq \lambda \|y\| g(t), \quad 0 \leq t \leq 1.
\]

3. Proof of Theorem 2

Superlinear case

Suppose then that \( f_0 = 0 \) and \( f_\infty = +\infty \). We now consider the problem (1.1) and wish to prove the existence of a positive solution. Clearly, \( y(t) \) is a solution of (1.1) if and only if \( y(t) \) satisfies the integral equation

\[
y(t) = \int_0^1 G(t, s) a(s) f(y(s)) \, ds =: (\Phi y)(t).
\]

Let \( K \) be a cone in \( E \) given by

\[
K := \{ y \in E; \ y(t) \geq \lambda g(t) \|y\| \text{ on } [0, 1]\}
\]

while \( E \) is the Banach space of continuous functions defined on \([0, 1]\) with the norm

\[
\|y\| := \max\{|y(t)|; 0 \leq t \leq 1\}.
\]
Let us define an operator $\Phi : K \to K$ by the right-hand side of (3.1). From Theorem 1 and the definition of $\Phi$, we have for any $y \in K$,

$$0 \leq (\Phi y)(t) \leq B \int_0^1 h(s)a(s)f(y(s))\, ds, \quad 0 \leq t \leq 1,$$

$$(\Phi y)(t) \geq \lambda B g(t) \int_0^1 h(s)a(s)f(y(s))\, ds \geq \lambda g(t)\|\Phi y\|, \quad 0 \leq t \leq 1,$$

$$(\Phi y)^{(i)}(0) = (\Phi y)^{(j)}(1) = 0, \quad 0 \leq i \leq k - 1, \quad 0 \leq j \leq n - k - 1.$$ 

This shows that $\Phi(K) \subset K$ and each fixed point of $\Phi$ is a solution to (1.1). Moreover, we have the following lemma which will be proved at the end of this section.

**Lemma 1.** $\Phi : K \to K$ is completely continuous.

Now suppose that $f_0 = 0$ and $f_{+\infty} = +\infty$. Since $f_0 = 0$, we can choose a $\rho_1 > 0$ such that

$$f(u) \leq \varepsilon u \quad \text{whenever } 0 \leq u \leq \rho_1,$$

where $\varepsilon > 0$ satisfies the condition

$$0 < \varepsilon B \int_0^1 h(s)a(s)\, ds < 1/2. \quad (3.3)$$

Thus, if $y \in K$ with $\|y\| = \rho_1$, then from (3.2) and (3.3)

$$0 \leq (\Phi y)(t) \leq B \int_0^1 h(s)a(s)f(y(s))\, ds$$

$$\leq \varepsilon B\|y\| \int_0^1 h(s)a(s)\, ds$$

$$< \|y\|,$$

i.e.

$$\|\Phi y\| < \|y\| \quad \forall y \in K \cap \partial \Omega_1,$$

where $\Omega_1 := \{y \in E; \|y\| < \rho_1\}$. Furthermore, since $f_\infty = +\infty$, there exists a $\rho_2 > \rho_1$ such that

$$f(u) \geq Mu \quad \text{whenever } u > \lambda \delta^n \rho_2, \quad (3.4)$$

where the constant $\delta \in (0, \frac{1}{4})$ is chosen so that

$$\alpha := \int_{\delta}^{1-\delta} h(s)a(s)\, ds > 0. \quad (3.5)$$
In the sequel, \( \delta \) always satisfies (3.5). Moreover, \( M > 0 \) is chosen so that
\[
\alpha \lambda^2 Bg(1/2)\delta^n M > 1. \tag{3.6}
\]
Let \( \Omega_2 := \{ y \in E : \| y \| = \rho_2 \} \). Since \( y \in K \) with \( \| y \| = \rho_2 \) implies that
\[
y(t) \geq \lambda \delta^n \rho_2 \quad \text{on } [\delta, 1-\delta], \tag{3.7}
\]
it follows from (3.4)–(3.7) that
\[
(\Phi y)(\frac{1}{2}) = \int_0^1 G(\frac{1}{2}, s) a(s)f(y(s)) \, ds \\
\geq \lambda Bg(\frac{1}{2}) \int_{\delta}^{1-\delta} h(s)a(s)f(y(s)) \, ds \\
\geq \alpha \lambda^2 Bg(\frac{1}{2})\delta^n M \rho_2 > \rho_2 = \| y \|.
\]
This shows that
\[
\| \Phi y \| > \| y \| \quad \forall y \in K \cap \partial \Omega_2.
\]
Therefore, from the first part of Theorem 3, we conclude that \( \Phi \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \). Let \( y(t) \) be the fixed point. Then \( 0 < \rho_1 \leq \| y \| \leq \rho_2 \). This shows that the fixed point \( y(t) \) is a positive solution to the problem (1.1).

\textbf{Sublinear case}

Suppose next that \( f_0 = +\infty \) and \( f_\infty = 0 \).

Since \( f_0 = +\infty \), we may choose \( \rho_1 > 0 \) so that
\[
f(u) \geq Mu \quad \text{whenever } 0 \leq u \leq \rho_1,
\]
where the constant \( M \) satisfies (3.6). Since \( y \in K \) with \( \| y \| = \rho_1 \) implies that
\[
y(t) \geq \lambda \delta^n \rho_1 \quad \text{on } [\delta, 1-\delta], \tag{3.8}
\]
we have
\[
(\Phi y)(\frac{1}{2}) = \int_0^1 G(\frac{1}{2}, s) a(s)f(y(s)) \, ds \\
\geq \lambda Bg(\frac{1}{2}) \int_{\delta}^{1-\delta} h(s)a(s)f(y(s)) \, ds \\
\geq \alpha \lambda^2 Bg(\frac{1}{2})\delta^n \rho_1 > \rho_1 = \| y \|.
\]
This shows that

$$\|\Phi y\| > \|y\| \quad \forall y \in K \cap \partial \Omega_1,$$

where $\Omega_1 := \{y \in E; \|y\| < \rho_1\}$. Furthermore, since $f_\infty = 0$, there exists an $N > \rho_1$ such that

$$f(u) \leq \varepsilon u \quad \text{whenever } u \geq N,$$

where the constant $\varepsilon$ satisfies (3.3). Let

$$\rho_2 > 2N + 2B \max\{f(u); 0 \leq u \leq N\} \int_0^1 h(s)a(s)\,ds.$$

Then for $y \in K$ with $\|y\| = \rho_2$, we have

$$(\Phi y)(t) \leq \int_0^1 B h(s)a(s)f(y(s))\,ds$$

$$< N + B \max\{f(u); 0 \leq u \leq N\} \int_0^1 h(s)a(s)\,ds$$

$$+ \varepsilon B \|y\| \int_0^1 h(s)a(s)\,ds$$

$$\leq \frac{1}{2} \rho_2 + \frac{1}{2} \|y\|,$$

i.e.

$$\|\Phi y\| < \|y\| \quad \forall y \in K \cap \partial \Omega_2,$$

where $\Omega_2 := \{y \in E; \|y\| < \rho_2\}$. Therefore, by the second part of Theorem 3, it follows that $\Phi$ has a fixed point $y(t)$ in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is a positive solution to the problem (1.1).

This completes the proof of Theorem 2.

Proof of Lemma 1. We can obtain the continuity of $\Phi$ from the continuity of $f$. In fact, suppose that $y_m, y \in K$ and $\|y_m - y\| \to 0$ as $m \to \infty$. By Theorem 1, we get

$$\|\Phi y_m - \Phi y\| \leq \max_{0 \leq t \leq 1} |f(y_m(t)) - f(y(t))|B \int_0^1 h(s)a(s)\,ds.$$

This implies that $\|\Phi y_m - \Phi y\| \to 0$ as $m \to \infty$.

Now let $D$ be a bounded subset of $K$ and $M > 0$ is the constant such that $\|y\| \leq M$ for $y \in D$. Then we have

$$\|\Phi y\| \leq B \max\{f(y); 0 \leq y \leq M\} \int_0^1 h(s)a(s)\,ds,$$

(3.9)
which implies the boundedness of $\Phi(D)$.

From (2.5) and (2.7), it is easy to obtain that

\[
(\Phi y)'(t) = \int_0^t s^{n-k} \frac{d}{dt} [(1-t)^{n-k} Q(t, s)] a(s) f(y(s)) \, ds \\
+ \int_t^1 (1-s)^k \frac{d}{dt} [t^k P(t, s)] a(s) f(y(s)) \, ds, \quad 0 < t < 1. \quad (3.10)
\]

At last, we claim that $\Phi(D)$ is equicontinuous on $[0, 1]$. We will prove the claim. For any $y \in D$ we have

\[
(\Phi y)(t) \leq \max \{f(y); 0 \leq y \leq M\} \int_0^1 G(t, s) a(s) \, ds \\
=: F(t), \quad \text{for } 0 \leq t \leq 1.
\]

In view of $F(0) = F(1) = 0$ and the continuity of $F(t)$ on $t = 0$ and $t = 1$, for any $\varepsilon > 0$, there is a $\delta_1 \in (0, 1/4)$ such that

\[
(\Phi y)(t) \leq F(t) \leq \frac{\varepsilon}{2}, \quad \text{for } t \in [0, 2\delta_1] \cup [1 - 2\delta_1, 1]. \quad (3.11)
\]

Thus we derive from (3.10) and Theorem 1 that

\[
|{(\Phi y)'(t)}| \leq C_1 \max \{f(y); 0 \leq y \leq M\} \int_0^{1-\delta_1} s^{n-k} a(s) \, ds \\
+ C_1 \max \{f(y); 0 \leq y \leq M\} \int_{\delta_1}^1 (1-s)^k a(s) \, ds \\
\leq \frac{C_1}{\delta_1^k} \max \{f(y); 0 \leq y \leq M\} \int_{\delta_1}^1 s^{n-k} (1-s)^k a(s) \, ds \\
+ \frac{C_1}{\delta_1^{n-k}} \max \{f(y); 0 \leq y \leq M\} \int_{\delta_1}^1 s^{n-k} (1-s)^k a(s) \, ds \\
\leq \left( \frac{C_1}{\delta_1^k} + \frac{C_1}{\delta_1^{n-k}} \right) \max \{f(y); 0 \leq y \leq M\} \int_0^1 h(s) a(s) \, ds \\
=: L, \quad \text{for } t \in [\delta_1, 1 - \delta_1].
\]

Put $\delta_2 = \varepsilon/2L$, then for $t_1, t_2 \in [\delta_1, 1 - \delta_1]$, $|t_1 - t_2| < \delta_2$

\[
|{(\Phi y)(t_1) - (\Phi y)(t_2)}| \leq L|t_1 - t_2| < \varepsilon/2.
\]

Set $\delta_0 = \min\{\delta_1, \delta_2\}$. Then for $t_1, t_2 \in [0, 1]$, $|t_1 - t_2| < \delta_0$, one has

\[
|{(\Phi y)(t_1) - (\Phi y)(t_2)}| < \varepsilon.
\]
That is to say that $\Phi(D)$ is equicontinuous on $[0, 1]$.
This completes the proof of Lemma 1.

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REFERENCES


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