A REMARK ON STOCHASTIC OSCILLATORY INTEGRALS WITH RESPECT TO A PINNED WIENER MEASURE

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1. Introduction

In [6], the asymptotic behavior of stochastic oscillatory integrals with quadratic phase function and analytic amplitude function were studied by using a complex change of variable formula established by Malliavin and Taniguchi [4]. In many situations, stochastic oscillatory integrals and their asymptotics are studied with respect to not only the Wiener measure but also the pinned measure (for example, see [1]). For the study of heat kernels, the estimate of asymptotics with respect to pinned measures is indispensable. In this paper, we investigate stochastic oscillatory integrals of the form

$$\int_X e^{\sqrt{-1} Q_A(x)} \psi(x) \delta_\eta(\mathbf{e}(x)) \mu(dx),$$

where (X, H, μ) is an abstract Wiener space, Q_A is a quadratic form on X , ψ is an analytic function on X , and $\delta_\eta(\mathbf{e}(x)) \mu(dx)$ is Watanabe's pullback of Dirac's δ -function δ_η by the non-degenerate Wiener functional $\mathbf{e}$. For details, see Section 2, Theorems 3.1 and 4.1 and Remark 4.5. The aim of the present paper is to extend the results in [6] to such pinned stochastic oscillatory integrals. For this purpose, we first extend the complex change of variable formula in [4] to pinned Wiener measures, and then we shall study the asymptotic behavior of stochastic oscillatory integrals relative to the pinned measure.

2. Preliminaries

In this section, we review several results in [4, 6]. Let (X, H, μ) be a real abstract Wiener space; X is a real separable Banach space, H is a real separable Hilbert space imbedded continuously and densely in X , and μ is a Gaussian measure on X such that

$$\int_X \exp[\sqrt{-1} \langle x, \ell \rangle] \mu(dx) = \exp[-\|\ell\|_H^2/2] \quad \text{for every } \ell \in X^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing of X and its dual X^* , $\|\cdot\|_H$ does the norm of H , and the identification that $H = H^*$ has been used so that X^* is included continuously in H .

We shall write $X \oplus \sqrt{-1} H$ for $X \times H$, and think of it as a complexification of X . Then the element $(x, h) \in X \times H$ is denoted by $x + \sqrt{-1} h$. The complexified space $H^{\mathbb{C}} := H \otimes \mathbb{C} = H \oplus \sqrt{-1} H$ is contained in $X \oplus \sqrt{-1} H$. In particular, for $x \in X$, $h \in H$, and $\zeta \in \mathbb{C}$, $x + \zeta h := (x + (\operatorname{Re} \zeta)h) + \sqrt{-1} (\operatorname{Im} \zeta)h$ belongs to $X \oplus \sqrt{-1} H$ and defines a complex line in $X \oplus \sqrt{-1} H$.

The Sobolev spaces over X are defined as usual; for a real separable Hilbert space E , we shall denote by $\mathbf{D}^{n,p}(X; E)$ the Sobolev space of E -valued n -times differentiable Wiener functionals with p th integrable derivatives of orders less than or equal to n in the sense of the Malliavin calculus. The j th derivative of $F \in \mathbf{D}^{n,p}(X; E)$, $1 \leq j \leq n$, is denoted by $\nabla^j F$ and is an $(H^{\otimes j} \otimes E)$ -valued functional, where $H^{\otimes j} \otimes E$ denotes the Hilbert space of Hilbert–Schmidt type multilinear operators $f : H^j \rightarrow E$ with norm $\|f\|_{H^{\otimes j} \otimes E} = \sum_{i_1, \dots, i_j=1}^{\infty} \|f(h_{i_1}, \dots, h_{i_j})\|_E^2$, $\{h_i\}_{i=1}^{\infty}$ being a CONS of H . For simplicity, we write $H^{\otimes j}$ for $H^{\otimes j} \otimes \mathbb{R}$. We set

$$\mathbf{D}^{n,\infty-}(X; E) = \bigcap_{1 < p < \infty} \mathbf{D}^{n,p}(X; E), \quad \mathbf{D}^{\infty,\infty-}(X; E) = \bigcap_{n \in \mathbb{N}} \mathbf{D}^{n,\infty-}(X; E).$$

The dual space of $\mathbf{D}^{n,p}(X; E)$ is denoted by $\mathbf{D}^{-n,q}(X; E)$, where $(1/p) + (1/q) = 1$, and we put $\mathbf{D}^{-\infty,q}(X; E) = \bigcup_{n \in \mathbb{N}} \mathbf{D}^{-n,q}(X; E)$. Let ∇^* be the adjoint operator of $\nabla : L^2(X; \mathbb{R}; \mu) \supset \mathbf{D}^{1,2}(X; \mathbb{R}) \ni F \mapsto \nabla F \in L^2(X; H; \mu)$. ∇^* extends to a continuous linear operator of $\mathbf{D}^{n,p}(X; H \otimes E)$ to $\mathbf{D}^{n-1,p}(X; E)$ (say ∇^* again). For details of Sobolev spaces, see [5, 8].

We say that $F \in \mathbf{D}^{\infty,\infty-}(X; \mathbb{R})$ is *analytic* ($F \in C^\omega(X)$ in notation) if, for some $p \in (1, \infty)$,

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|\nabla^n F\|_{L^p(X, H^{\otimes n}; \mu)} < \infty \quad \text{for every } s \in (0, \infty).$$

As was seen in [6], for each $F \in C^\omega(X)$, choosing μ -versions of $\nabla^n F$'s appropriately, we have that

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|\nabla^n F(x)\|_{H^{\otimes n}} < \infty, \quad F(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \nabla^n F(x), h^{\otimes n} \rangle$$

for any $s \in (0, \infty)$, $x \in X$, $h \in H$, (2.1)

where we have also used $\langle \cdot, \cdot \rangle$ to denote inner products of $H^{\otimes n}$, $n \in \mathbb{N}$. Then the holomorphic prolongation of $F \in C^\omega(X)$ to $X \oplus \sqrt{-1} H$, which will be denoted by

the same letter F , is given by

$$F(x + \sqrt{-1}h) = \sum_{n=0}^{\infty} \frac{\sqrt{-1}^n}{n!} \langle \nabla^n F(x), h^{\otimes n} \rangle, \quad x \in X, h \in H.$$

From this definition and (2.1), we obtain that

$$F((x+h) + \sqrt{-1}k) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \nabla^n F(x), (h + \sqrt{-1}k)^{\otimes n} \rangle$$

for any $s \in (0, \infty)$, $x \in X$, $h, k \in H$. (2.2)

Let

$$H^{(n)} = \{g \in H^{\otimes n} : g \text{ is symmetric}\}.$$

The multiple Wiener integral associated with $f \in H^{(n)}$ is denoted by $I_n(f)$; $I_n(f) = (\nabla^*)^n f$. Then $I_n(f) \in C^\omega(X)$. See [6, Section 1.3]. A symmetric Hilbert–Schmidt operator A is an element of $H^{(2)}$. We shall write Q_A for $I_2(A)$. Moreover, we call $L_A := \nabla^* A \in \mathbf{D}^{\infty, \infty-}(X; H)$ a stochastic extension of A . Extending like $L_{A+\sqrt{-1}B} = L_A + \sqrt{-1}L_B$, we put

$$\varphi_A(x) = x + L_{((I-2\sqrt{-1}A)^{-1/2}-I)}(x).$$

3. Change of variable formula

In this section, we extend the change of variable formula [4, Theorem 7.8] to a pinned measure on X . To do this, consider $e_1, \dots, e_d \in H$ with $\langle e_i, e_j \rangle = \delta_{ij}$, and let $\pi : H \rightarrow H$ be the orthogonal projection onto the space spanned by them. Then $\mathbf{e} := (\langle *, e_1 \rangle, \dots, \langle *, e_d \rangle) : X \rightarrow \mathbb{R}^d$ determines a non-degenerate Wiener functional belonging to $\mathbf{D}^{\infty, \infty-}(X; \mathbb{R}^d)$. For $\eta \in \mathbb{R}^d$, $\delta_\eta(\mathbf{e}(x))\mu(dx)$ denotes the generalized Wiener functional obtained as Watanabe’s pullback of Dirac’s δ -function δ_η concentrating at η via $\mathbf{e}$ (see [8]). Our change of variable formula is as follows.

THEOREM 3.1. *Let e_i , $\mathbf{e}$ and π be as above. Consider a $B \in H^{(2)}$ and define $A = (I - \pi)B(I - \pi)$. Let $\psi \in C^\omega(X)$ satisfy*

$$M_0[\psi, r] := \sum_{n=0}^{\infty} \frac{r^n}{n!} \|\nabla^n F\|_{H^{\otimes n}}^2 \in L^1(X, \mathbb{R}; \mu) \quad \text{for any } r \in (0, \infty).$$

Then, it holds that

$$\begin{aligned} & \int_X e^{\sqrt{-1} Q_B(x)} \psi(x) \delta_\eta(\mathbf{e}(x)) \mu(dx) \\ &= \{\det_2(I - 2\sqrt{-1} A)\}^{-1/2} e^{\sqrt{-1} \langle \eta, B_1 \eta \rangle - \text{tr } B_1} \\ & \quad \times \int_X \psi(x + \varphi_A(x)) e^{2\sqrt{-1} \sum_{i=1}^d \eta_i \langle x, B_0 e_i \rangle} \delta_\eta(\mathbf{e}(x)) \mu(dx), \end{aligned} \quad (3.2)$$

where $B_1 = (\langle e_i, B e_j \rangle)_{1 \leq i, j \leq d} \in \mathbb{R}^d \otimes \mathbb{R}^d$, $B_0 = (I - 2\sqrt{-1} A)^{-1/2} (I - \pi) B$, and $\eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$.

Remark 3.3. (i) As will be seen in Lemma 3.4, the functional $\psi(* + \varphi_A(*))$ belongs to the class $\mathbf{D}^{\infty, 2-}(X; \mathbb{C}) := \bigcap_{p \in (1, 2)} \mathbf{D}^{\infty, p}(X; \mathbb{C})$. Since $\delta_\eta(\mathbf{e}(x)) \in \mathbf{D}^{-\infty, q}(X; \mathbb{R})$ for any $q \in (1, \infty)$ [5], the right-hand side of (3.2) is well defined.

(ii) The identity (3.2) with $\psi \equiv 1$ was also discussed in [1, 7].

The proof of the theorem will be divided into several steps, each being a lemma. In every step, it is always assumed that $\psi \in C^\omega(X)$ satisfies that $M_0[\psi, r] \in L^1(X; \mathbb{R}; \mu)$, $r \in (0, \infty)$. We shall use the expression that $A = \sum_{n=1}^\infty a_n h_n \otimes h_n$ for some $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ with $\sum_{n=1}^\infty a_n^2 < \infty$ and $\text{CONS } \{h_n\}_{n=1}^\infty$ of H . The real numbers s_n and t_n are defined by $(1 - 2\sqrt{-1} a_n)^{-1/2} - 1 = s_n + \sqrt{-1} t_n$, where the branch of $\zeta^{-1/2}$ so that $1^{-1/2} = 1$ was considered. Without loss of generality, we may and will assume that $a_n = s_n = t_n = 0$ and $h_n = e_n$, $1 \leq n \leq d$.

LEMMA 3.4. *It holds that*

$$\psi(* + \varphi_A(*)) \in \mathbf{D}^{\infty, 2-}(X; \mathbb{C}).$$

Proof. By (2.2),

$$\psi(* + \varphi_A(*)) = \sum_{n=0}^\infty \frac{1}{n!} \langle \nabla^n \psi, \varphi_A^{\otimes n} \rangle.$$

Let $S = \sum_{n=1}^\infty s_n h_n \otimes h_n$ and $T = \sum_{n=1}^\infty t_n h_n \otimes h_n$. Then $\nabla \varphi_A = S + \sqrt{-1} T$ and $\nabla^2 \varphi_A = 0$. Hence

$$\begin{aligned} & \|\nabla^k \{\psi(* + \varphi_A(*))\}\|_{H^{\otimes k} \otimes \mathbb{C}} \\ & \leq \sum_{m=0}^k \frac{k!}{m!(k-m)!} \sum_{n=m}^\infty \frac{1}{(n-m)!} \|\nabla^{n+k-m} \psi\|_{H^{\otimes n+k-m}} \|\nabla \varphi_A\|_{H^{\otimes 2} \otimes \mathbb{C}}^m \|\varphi_A\|_{H^{\otimes \mathbb{C}}}^{n-m} \end{aligned}$$

$$\begin{aligned}
&\leq (1 + \|S\|_\infty + \|T\|_\infty)^k \sum_{m=0}^k \left\{ \left(\sum_{n \geq m} \frac{r^{n-m}}{(n-m)!} \|\nabla^{n+k-m} \psi\|_{H^{\otimes n+k-m}} \right)^{1/2} \right. \\
&\quad \times \left. \left(\sum_{n \geq m} \frac{r^{-(n-m)}}{(n-m)!} \|\varphi_A\|_{H^{\otimes \mathbb{C}}}^{n-m} \right)^{1/2} \right\} \\
&\leq CM_0[\psi, r+1]^{1/2} e^{\|\varphi_A\|_H^2/(2r)},
\end{aligned}$$

where C is a constant depending only on r, k, S , and T , and $\|\alpha + \sqrt{-1}\beta\|_{H^{\otimes n} \otimes \mathbb{C}}^2 = \|\alpha\|_H^2 + \|\beta\|_H^2$, $\alpha, \beta \in H^{\otimes n}$. Since $e^{\varepsilon\|\varphi_A\|_H^2} \in L^1(X, \mathbb{R}; \mu)$ for some $\varepsilon > 0$, the desired assertion holds. $\square$

LEMMA 3.5. *It holds that, for every $h \in H$ and $\eta \in \mathbb{R}^d$,*

$$\begin{aligned}
&\int_X e^{\sqrt{-1}Q_A(x)} \psi(x) e^{\sqrt{-1}\langle x, h \rangle} \delta_\eta(\mathbf{e}(x)) \mu(dx) \\
&= \{\det_2(I - 2\sqrt{-1}A)\}^{-1/2} \\
&\quad \times \int_X \psi(x + \varphi_A(x)) e^{\sqrt{-1}\langle x, (I - 2\sqrt{-1}A)^{-1/2}h \rangle} \delta_\eta(\mathbf{e}(x)) \mu(dx).
\end{aligned}$$

Proof. By [4, Theorem 7.8] and [6, Lemma 2.11], for any $\xi, \eta \in \mathbb{R}^d$, it holds that

$$\begin{aligned}
&\int_X e^{\sqrt{-1}Q_A(x)} \psi(x) e^{\sqrt{-1}\langle x, h \rangle} e^{\sqrt{-1}\langle \xi, \mathbf{e}(x) - \eta \rangle} \mu(dx) \\
&= \{\det_2(I - 2\sqrt{-1}A)\}^{-1/2} \\
&\quad \times \int_X \psi(x + \varphi_A(x)) e^{\sqrt{-1}\langle x + \varphi_A(x), h \rangle} e^{\sqrt{-1}\langle \xi, \mathbf{e}(x + \varphi_A(x)) - \eta \rangle} \mu(dx).
\end{aligned}$$

It is easily seen that

$$\langle x + \varphi_A(x), h \rangle = \langle x, (I - 2\sqrt{-1}A)^{-1/2}h \rangle \quad \text{and} \quad \mathbf{e}(x + \varphi_A(x)) = \mathbf{e}(x) \quad \mu\text{-a.e.}$$

(For the first, see [6, (2.14)].) We then have that

$$\begin{aligned}
&\int_X e^{\sqrt{-1}Q_A(x)} \psi(x) e^{\sqrt{-1}\langle x, h \rangle} e^{\sqrt{-1}\langle \xi, \mathbf{e}(x) - \eta \rangle} \mu(dx) \\
&= \{\det_2(I - 2\sqrt{-1}A)\}^{-1/2} \\
&\quad \times \int_X \psi(x + \varphi_A(x)) e^{\sqrt{-1}\langle x, (I - 2\sqrt{-1}A)^{-1/2}h \rangle} e^{\sqrt{-1}\langle \xi, \mathbf{e}(x) - \eta \rangle} \mu(dx).
\end{aligned}$$

Integrating both sides with respect to $(2\pi)^{-d} e^{-t|\xi|^2/2} d\xi$ on $\mathbb{R}^d$, we obtain that

$$\begin{aligned} & \int_X e^{\sqrt{-1} Q_A(x)} \psi(x) e^{\sqrt{-1} \langle x, h \rangle} \frac{1}{\sqrt{2\pi t}} e^{-|\mathbf{e}(x) - \eta|^2/(2t)} \mu(dx) \\ &= \{\det_2(I - 2\sqrt{-1} A)\}^{-1/2} \\ & \quad \times \int_X \psi(x + \varphi_A(x)) e^{\sqrt{-1} \langle x, (I - 2\sqrt{-1} A)^{-1/2} h \rangle} \frac{1}{\sqrt{2\pi t}} e^{-|\mathbf{e}(x) - \eta|^2/(2t)} \mu(dx). \end{aligned} \quad (3.6)$$

Note that

$$e^{\sqrt{-1} Q_A}, \psi, e^{\sqrt{-1} \langle *, h \rangle}, e^{\sqrt{-1} \langle *, (I - 2\sqrt{-1} A)^{-1/2} h \rangle} \in \mathbf{D}^{\infty, \infty-}(X; \mathbb{C}).$$

Due to Lemma 3.4, letting $t \rightarrow 0$ in (3.6), we obtain the desired identity. $\square$

LEMMA 3.7. *The identity (3.2) holds.*

Proof. For a Hilbert–Schmidt operator $U : H \rightarrow H$, let $L_U : X \rightarrow H$ be the stochastic extension. It then holds that $Q_B = \nabla^* L_B$. Hence

$$Q_B = Q_A + \nabla^* L_{(I-\pi)B\pi} + \nabla^* L_{\pi B(I-\pi)} + Q_{\pi B\pi}. \quad (3.8)$$

By a straightforward computation, we obtain

$$\begin{aligned} L_{(I-\pi)B\pi} &= \sum_{m=1}^d \langle *, h_m \rangle (I - \pi)[Bh_m], \quad L_{\pi B(I-\pi)} = \sum_{n=d+1}^{\infty} \langle *, h_n \rangle \pi[Bh_n], \\ \nabla^* L_{(I-\pi)B\pi} &= \sum_{m=1}^d \langle *, h_m \rangle \langle *, (I - \pi)[Bh_m] \rangle = \sum_{m=1}^d \sum_{n=d+1}^{\infty} \langle *, h_m \rangle \langle *, h_n \rangle \langle h_n, Bh_m \rangle \\ \nabla^* L_{\pi B(I-\pi)} &= \sum_{n=d+1}^{\infty} \langle *, h_n \rangle \langle *, \pi[Bh_n] \rangle = \sum_{m=1}^d \sum_{n=d+1}^{\infty} \langle *, h_m \rangle \langle *, h_n \rangle \langle h_n, Bh_m \rangle \\ Q_{\pi B\pi} &= \sum_{n,m=1}^d \langle *, h_n \rangle \langle *, h_m \rangle \langle h_n, Bh_m \rangle - \sum_{n=1}^d \langle h_n, Bh_n \rangle. \end{aligned}$$

Since $h_n = e_n$, $1 \leq n \leq d$, combined with (3.8), these imply that

$$Q_B = Q_A + 2 \sum_{i=1}^d \eta_i \langle *, (I - \pi)Be_i \rangle + \langle \eta, B_1 \eta \rangle - \text{tr } B_1, \quad \delta_\eta(\mathbf{e}(x)) \mu(dx)\text{-a.e.}$$

Now the assertion is an immediate consequence of Lemma 3.5. $\square$

4. The asymptotic behavior

In this section, we consider e_i , $\mathbf{e}$, and $B, A \in H^{(2)}$ similar to those in Section 3. As stated before Lemma 3.4, we fix a CONS $\mathcal{H} = \{h_n\}_{n=1}^\infty$ of H consisting of eigenfunctions of A , denote by a_n the eigenvalue corresponding to h_n , and assume that $a_n = 0$ and $h_n = e_n$ for $1 \leq n \leq d$.

To state our asymptotic behavior, we prepare some notation. Put

$$\begin{aligned} \mathcal{A} &= (\mathbb{N} \cup \{0\})^d, \quad \mathcal{A}_2 = \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathcal{A} : \text{every } \alpha_j \text{ is even}\}, \\ |\alpha| &= \sum_{i=1}^d \alpha_i, \quad \alpha! = \prod_{i=1}^d (\alpha_i!), \\ e^{\otimes \alpha} &= e_1^{\otimes \alpha_1} \otimes \dots \otimes e_d^{\otimes \alpha_d} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_d) \in \mathcal{A}, \\ \beta/2 &= (\beta_1/2, \dots, \beta_d/2) \quad \text{for } \beta = (\beta_1, \dots, \beta_d) \in \mathcal{A}_2. \end{aligned}$$

Denoting by $H_n(\xi)$ the n th Hermite polynomial,

$$H_n(\xi) = e^{\xi^2/2} \frac{d^n}{dx^n} (e^{-\xi^2/2}),$$

we set

$$H_\alpha(0) = \prod_{i=1}^d H_{\alpha_i}(0) \quad \text{for } \alpha \in \mathcal{A}.$$

Then, we have that

$$H_\alpha(0) = \begin{cases} (-1)^{|\alpha|} \frac{\alpha!}{2^{|\alpha|/2} (\alpha/2)!}, & \text{if } \alpha \in \mathcal{A}_2, \\ 0, & \text{if } \alpha \notin \mathcal{A}_2. \end{cases}$$

We are now ready to state our asymptotic behavior.

THEOREM 4.1. *Consider $g \in H^{(n)}$ satisfying that, for every $1 \leq k \leq [n/2]$,*

$$\begin{aligned} \sum_{j_{2k+1}, \dots, j_n=1}^\infty \left\{ \sum_{i_1, \dots, i_k=1}^\infty |g(h_{i_1} \otimes h_{i_1} \otimes h_{i_2} \otimes h_{i_2} \otimes \right. \\ \left. \dots \otimes h_{i_k} \otimes h_{i_k} \otimes h_{j_{2k+1}} \otimes \dots \otimes h_{j_n})| \right\}^2 < \infty. \end{aligned} \quad (4.2)$$

In the case where n is even and $k = n/2$, this is understood as follows:

$$\sum_{i_1, \dots, i_{n/2}=1}^\infty |g(h_{i_1} \otimes h_{i_1} \otimes h_{i_2} \otimes h_{i_2} \otimes \dots \otimes h_{i_{n/2}} \otimes h_{i_{n/2}})| < \infty. \quad (4.3)$$

Assume that A is non-degenerate on $H_0 = \{h \in H : \langle h, e_n \rangle = 0, 1 \leq n \leq d\}$, i.e. $a_n \neq 0$ for every $n > d$. Then, for any $h_0, h, \ell \in H_0$, it holds that, as $|\lambda| \rightarrow \infty$,

$$\begin{aligned} & \int_X e^{\sqrt{-1}\lambda Q_A(x-h_0)} e^{\sqrt{-1}\langle x, \ell \rangle} I_n(g)(x+h) \delta_0(\mathbf{e}(x)) \mu(dx) \\ & \sim \{\det_2(I - 2\sqrt{-1}\lambda A)\}^{-1/2} \frac{e^{-\sqrt{-1}\lambda \operatorname{tr} B_1}}{\sqrt{2\pi}^d} \exp[-\frac{1}{2}\|h_0\|_H^2 + \sqrt{-1}\langle h_0, \ell \rangle] \\ & \quad \times \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]-k} \sum_{r=0}^{[n/2]-k-j} \sum_{\substack{\alpha \in \mathcal{A}_2, \\ |\alpha|=2r}} \left\{ \left(-\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^j \frac{n! H_\alpha(0)}{k!(n-2k-2j-2r)! \alpha!} \right. \\ & \quad \left. \times \langle T_j(g, \mathcal{H}, k), (h_0 + h)^{\otimes n-2k-2j-2r} \otimes e^{\otimes \alpha} \rangle \right\}, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} & \langle T_j(g, \mathcal{H}, k), \eta_1 \otimes \cdots \otimes \eta_{n-2k-2j} \rangle \\ & = \sum_{i_1, \dots, i_j=1}^d \sum_{i_{j+1}, \dots, i_{j+k}=1}^{\infty} g(h_{i_1} \otimes h_{i_1} \otimes h_{i_2} \otimes h_{i_2} \otimes \\ & \quad \cdots \otimes h_{i_k} \otimes h_{i_k} \otimes \eta_1 \otimes \cdots \otimes \eta_{n-2k}) \end{aligned}$$

(for $j = 0$, the first sums over $\{1, \dots, d\}$ are omitted).

Before getting into the proof, we make several remarks.

Remark 4.5. (i) The elements h, ℓ in the theorem need not be in H_0 . Indeed, mention that

$$\langle x, \ell \rangle = \langle x, (I - \pi)\ell \rangle, \quad \delta_0(\mathbf{e}(x))\mu(dx)\text{-a.e.}$$

$$I_n(g)(x+h) = \sum_{k=0}^n \frac{n!}{k!(n-k)!} I_{n-k}(g((\pi h)^{\otimes k}))(x + (I - \pi)h).$$

(ii) Let X_0 be the completion of H_0 in X . When we are dealing with $\delta_0(\mathbf{e}(x))\mu(dx)$, we are working on the hypersurface X_0 (not on the full space X). The assumption that A is non-degenerate means that the restriction of Q_B to X_0 admits only one stationary point at the origin. Hence the condition $h_0 \in H_0$ in the theorem is natural when $\delta_0(\mathbf{e}(x))\mu(dx)$ is considered; namely, this condition means that we are restricting ourselves to the shifts translating the stationary point from the origin to $h_0 \in H_0$ on X_0 .

If $h_0 \notin H_0$, the restriction of $Q_B(x - h_0)$ to X_0 may have no stationary point on X_0 , and the asymptotic behavior like (4.4) may not occur. We shall see such phenomenon in Example 4.10.

(iii) The asymptotic behavior for $\delta_\eta(\mathbf{e}(x))\mu(dx)$ follows from that for $\delta_0(\mathbf{e}(x))\mu(dx)$. In fact, when $\delta_\eta(\mathbf{e}(x))\mu(dx)$ is considered, the hypersurface on which we are is $\{x + \sum_{i=1}^d \eta_i e_i : x \in X_0\}$. By the same reasoning as above, suitable stationary points to appear in Q_B are of the form $\sum_{i=1}^d \eta_i e_i + h_0$ with $h_0 \in H_0$, and hence the stochastic oscillatory integral of interest is

$$\int_X \exp \left[\sqrt{-1} Q_B \left(x - \left\{ \sum_{i=1}^d \eta_i e_i + h_0 \right\} \right) \right] e^{\sqrt{-1} \langle x, \ell \rangle} I_n(g)(x + h) \delta_\eta(\mathbf{e}(x)) \mu(dx)$$

which, by the Cameron–Martin formula, coincides with

$$\int_X e^{\sqrt{-1} Q_B(x-h_0)} e^{\sqrt{-1} \langle x, \ell \rangle} I_n(g) \left(x + \sum_{i=1}^d \eta_i e_i + h \right) \delta_0(\mathbf{e}(x)) \mu(dx) e^{-|\eta|^2/2}.$$

We now turn to the proof of Theorem 4.1. We prepare a lemma.

LEMMA 4.6. Consider $g \in H^{(n)}$ satisfying (4.2) and $U \in H^{(2)} \oplus \sqrt{-1} H^{(2)}$. Suppose that U has $\mathcal{H} = \{h_i\}_{i=1}^\infty$ as its eigenfunctions, and let $\{u_i\}_{i=1}^\infty$ be the corresponding eigenvalues. Set

$$\begin{aligned} & J_{k,j,r,\alpha}(g, U, h) \\ &= \sum_{\substack{i_1, \dots, i_{j+k}=1, \\ s_1, \dots, s_{n-2j-2k-2r}=1}}^\infty \left\{ \prod_{p=1}^j (1 + u_{i_p})^2 \prod_{q=1}^{n-2k-2j-2r} (1 + u_{s_q}) \langle h, h_{s_q} \rangle \right\} (1 + u)^\alpha \\ & \times g(h_{i_1} \otimes h_{i_1} \otimes \cdots \otimes h_{i_{k+j}} \otimes h_{i_{k+j}} \otimes h_{s_1} \otimes \cdots \otimes h_{s_{n-2k-2j-2r}} \otimes e^{\otimes \alpha}), \end{aligned}$$

where $(1 + u)^\alpha = \prod_{i=1}^d (1 + u_i)^{\alpha_i}$. Then, for each $h \in H_0 \oplus \sqrt{-1} H_0$, it holds that

$$\begin{aligned} & \int_X e^{\sqrt{-1} \langle x, h \rangle} I_n(g)(x + Ux) \delta_0(\mathbf{e}(x)) \mu(dx) \\ &= \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]-k} \sum_{r=0}^{[n/2]-k-j} \sum_{\substack{\alpha \in \mathcal{A}_2, \\ |\alpha|=2r}} \left\{ \left(-\frac{1}{2} \right)^k \left(\frac{1}{2} \right)^j \frac{n! H_\alpha(0)}{k!(n-2k-2j-2r)! \alpha!} \right. \\ & \quad \left. \times (-1)^{(n/2)-k-j-r} \frac{e^{-\langle h, h \rangle/2}}{\sqrt{2\pi}^d} J_{k,j,r,\alpha}(g, U, h) \right\}. \end{aligned} \quad (4.7)$$

Proof. Let $f \in H^{(p)}$. For a multi-index $\alpha \in \mathcal{A}$, denote by ∂^α the associated partial differentiation on $\mathbb{R}^d$. Since $I_p(\bullet) = (\nabla^*)^p$ and f is symmetric, we have that

$$\begin{aligned}
& \int_X e^{\sqrt{-1}\langle x, h \rangle} I_p(f)(x) \delta_0(\mathbf{e}(x)) \mu(dx) \\
&= \sum_{r=0}^p \frac{p!}{r!(p-r)!} \int_X \langle \nabla^{p-r} (e^{\sqrt{-1}\langle x, h \rangle}) \otimes \nabla^r (\delta_0(\mathbf{e}(x))), f \rangle \mu(dx) \\
&= \sum_{r=0}^p \sum_{\substack{\alpha \in \mathcal{A} \\ |\alpha|=r}} \frac{p!}{(p-r)! \alpha!} \sqrt{-1}^{p-r} \langle h^{\otimes p-r} \otimes e^{\otimes \alpha}, f \rangle \\
&\quad \times \int_X e^{\sqrt{-1}\langle x, h \rangle} (\partial^\alpha \delta_0)(\mathbf{e}(x)) \mu(dx),
\end{aligned}$$

where the integrals $\int_X (\cdots) \mu(dx)$ should be understood as pairings of generalized Wiener functionals and smooth functionals. Since $\langle \nabla^{|\alpha|} (\delta_0(\mathbf{e}(x))), e^{\otimes \alpha} \rangle = \partial^\alpha \delta_0(\mathbf{e}(x))$ and $h \in H_0 \oplus \sqrt{-1} H_0$, by a straightforward computation, we observe that

$$\begin{aligned}
\int_X e^{\sqrt{-1}\langle x, h \rangle} (\partial^\alpha \delta_0)(\mathbf{e}(x)) \mu(dx) &= \int_X (\nabla^*)^{|\alpha|} (e^{\otimes \alpha}) \delta_0(\mathbf{e}(x)) \mu(dx) e^{-\langle h, h \rangle / 2} \\
&= \frac{H_\alpha(0)}{\sqrt{2\pi}^d} e^{-\langle h, h \rangle / 2}.
\end{aligned}$$

We therefore obtain that

$$\begin{aligned}
& \int_X e^{\sqrt{-1}\langle x, h \rangle} I_p(f)(x) \delta_0(\mathbf{e}(x)) \mu(dx) \\
&= \sum_{r=0}^{\lfloor p/2 \rfloor} \sum_{\substack{\alpha \in \mathcal{A}_2 \\ |\alpha|=2r}} \frac{p!}{(p-2r)! \alpha!} (-1)^{(p/2)-r} \frac{H_\alpha(0)}{\sqrt{2\pi}^d} e^{-\langle h, h \rangle / 2} \langle h^{\otimes p-2r} \otimes e^{\otimes \alpha}, f \rangle.
\end{aligned} \tag{4.8}$$

Now, by [6, Lemma 2.16], repeating the argument in [6, Lemma 2.18], we obtain the desired conclusion. $\square$

Proof of Theorem 4.1. By virtue of Theorem 3.1, Lemma 4.6, and the Cameron–

Martin formula, we obtain that, for $h_0, h, \ell \in H_0$,

$$\begin{aligned}
& \int_X e^{\sqrt{-1} Q_B(x-h_0)} e^{\sqrt{-1} \langle x, \ell \rangle} I_n(g)(x+h) \delta_0(\mathbf{e}(x)) \mu(dx) \\
&= \{\det_2(I - 2\sqrt{-1} A)\}^{-1/2} \frac{e^{-\sqrt{-1} \operatorname{tr} B_1}}{\sqrt{2\pi}^d} \\
&\quad \times \exp[\sqrt{-1} \langle A(h_0+h), (h_0+h) \rangle - (\|h\|_H^2/2) - \sqrt{-1} \langle h, \ell \rangle] \\
&\quad \times \exp[-(1/2) \langle (I - 2\sqrt{-1} A)^{-1} \{h' - 2A(h_0+h)\}, \{h' - 2A(h_0+h)\} \rangle] \\
&\quad \times \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]-k} \sum_{r=0}^{[n/2]-k-j} \sum_{\substack{\alpha \in \mathcal{A}_2, \\ |\alpha|=2r}} \left\{ \left(-\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^j \frac{n! H_\alpha(0)}{k!(n-2k-2j-2r)! \alpha!} \right. \\
&\quad \left. \times (-1)^{(n/2)-k-j-r} J_{k,j,r,\alpha}(g, U_A, (I - 2\sqrt{-1} A)^{-1/2} \{h' - 2A(h_0+h)\}) \right\},
\end{aligned} \tag{4.9}$$

where $h' = \ell - \sqrt{-1} h$ and $U_A = (I - 2\sqrt{-1} A)^{-1/2} - I$.

Remembering that $a_n = 0$, $1 \leq n \leq d$, and applying the dominated convergence theorem, we see that

$$\begin{aligned}
& J_{k,j,r,\alpha}(g, U_{\lambda A}, (I - 2\sqrt{-1} \lambda A)^{-1/2} \{h' - 2\lambda A(h_0+h)\}) \\
& \longrightarrow (-1)^{-(n/2)+k+j+r} \langle T_j(g, \mathcal{H}, k), (h_0+h)^{\otimes n-2k-2j-2r} \otimes e^{\otimes \alpha} \rangle \quad \text{as } |\lambda| \rightarrow \infty
\end{aligned}$$

(cf. the proof of [6, Theorem 2.2]). Combining this with (4.9) and [6, (2.22)], we arrive at the desired assertion. $\square$

Example 4.10. Let $h_0 \in H$ satisfy that $(I - \pi)Bh_0 = A\tilde{h}$ for some $\tilde{h} \in H_0$. Then

$$Q_B(x - h_0) = Q_B(x - \tilde{h}) + \langle Bh_0, h_0 \rangle - \langle B\tilde{h}, \tilde{h} \rangle, \quad \delta_0(\mathbf{e}(x)) \mu(dx)\text{-a.e.}$$

Hence an asymptotic behavior similar to (4.4) occurs.

Let $h_0 \in H$ and suppose a condition that $(I - \pi)Bh_0 \neq A\tilde{h}$ for any $\tilde{h} \in H_0$, which may be thought of as a condition that the restriction of $Q_B(* - h_0)$ has no stationary point on X_0 . It then holds that

$$\sum_{n=d+1}^{\infty} \left(\frac{\langle Bh_0, h_n \rangle}{a_n} \right)^2 = \infty. \tag{4.11}$$

Since

$$Q_B(x - h_0) = Q_B(x) - 2\langle x, (I - \pi)Bh_0 \rangle + \langle Bh_0, h_0 \rangle, \quad \delta_0(\mathbf{e}(x)) \mu(dx)\text{-a.e.},$$

applying Theorem 3.1, we can show that

$$\begin{aligned} & \int_X e^{\sqrt{-1}\lambda Q_B(x-h_0)} \delta_0(\mathbf{e}(x)) \mu(dx) \\ &= \{\det_2(I - 2\sqrt{-1}\lambda A)\}^{-1/2} \frac{e^{-\sqrt{-1} \operatorname{tr} B_1}}{\sqrt{2\pi}^d} e^{\sqrt{-1}\lambda \langle Bh_0, h_0 \rangle} \\ & \quad \times \exp[-\frac{1}{2} \langle (1 - 2\sqrt{-1}\lambda A)^{-1} (I - \pi)\lambda Bh_0, (I - \pi)\lambda Bh_0 \rangle]. \end{aligned} \quad (4.12)$$

By Fatou's lemma and (4.11), we have that

$$\begin{aligned} & \operatorname{Re} \langle (1 - 2\sqrt{-1}\lambda A)^{-1} (I - \pi)\lambda Bh_0, (I - \pi)\lambda Bh_0 \rangle \\ &= \sum_{n=d+1}^{\infty} \frac{\lambda^2 \langle (I - \pi)Bh_0, h_n \rangle^2}{1 + 4\lambda^2 a_n^2} \rightarrow \infty \quad \text{as } |\lambda| \rightarrow \infty. \end{aligned}$$

Plugging this into (4.12), we obtain a faster decay than (4.4).

REFERENCES

- [1] N. Ikeda and S. Manabe. Asymptotic formulae for stochastic oscillatory integrals. *Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics* (Pitman Research Notes in Mathematics, 284). Eds. K. D. Elworthy and N. Ikeda. Longman, 1993, pp. 136–155.
- [2] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North-Holland and Kodansha, Amsterdam and Tokyo, 1981 (2nd edn, 1989).
- [3] Y. Z. Hu and P. A. Meyer. Sur les intégrales multiples de Stratonovich. *Séminaire de Probabilités XXII* (Lecture Notes in Mathematics, 1321). Springer, Berlin, Heidelberg, New York, 1988, pp. 72–81.
- [4] P. Malliavin and S. Taniguchi. Analytic functions, Cauchy formula and stationary phase on a real abstract Wiener space. *J. Funct. Anal.* **143** (1997), 470–528.
- [5] H. Sugita. Sobolev spaces of Wiener functionals and Malliavin's calculus. *J. Math. Kyoto Univ.* **25** (1985), 31–48.
- [6] H. Sugita and S. Taniguchi. Oscillatory integrals with quadratic phase function on a real abstract Wiener space. *J. Funct. Anal.* **155** (1998), 229–262.
- [7] S. Takanobu and S. Watanabe. Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener integrations. *Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics* (Pitman Research Notes in Mathematics, 284). Eds. K. D. Elworthy and N. Ikeda. Longman, 1993, pp. 194–241.
- [8] S. Watanabe. Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels. *Ann. Prob.* **15** (1987), 1–39.

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