GROUP COMPLETION FOR MULTI H-MAPS

Tetsu NISHIMOTO

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1. Introduction

Let B_i be weak homotopy associative weak H-spaces and let Z be a weak homotopy associative, weak homotopy commutative H-space. We call a map $\psi : \prod B_i \rightarrow Z$ a weak multi H-map if the following diagram is weak homotopy commutative:

$$\begin{array}{ccc}
 \prod_{j=1}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j & \xrightarrow{1_{\prod B_j} \times \mu_{B_i} \times 1_{\prod B_j}} & \prod_{j=1}^n B_j \\
 \downarrow \Pi \Delta_{B_j} \times 1_{B_i^2} \times \Pi \Delta_{B_j} & & \downarrow \psi \\
 \prod_{j=1}^n B_j^2 & & Z \\
 \downarrow T & & \uparrow \mu_Z \\
 \left(\prod_{j=1}^n B_j \right)^2 & \xrightarrow{\psi^2} & Z \times Z.
 \end{array} \tag{1.1}$$

We call an H-map $\varphi : B \rightarrow \overline{B}$ a group completion if B is a homotopy associative H-space and $\overline{B}$ is a group-like H-space; and for any group-like weak H-space Z and for any weak H-map $\psi : B \rightarrow Z$, there is a weak H-map $\overline{\psi} : \overline{B} \rightarrow Z$, unique up to weak homotopy, such that $\psi \simeq_w \overline{\psi} \circ \varphi$. If $\varphi : B_i \rightarrow \overline{B}_i$ are group completions and Z is a weak homotopy commutative, group-like weak H-space, for any weak multi H-map $\psi : \prod B_i \rightarrow Z$, there exists a weak multi H-map $\overline{\psi} : \prod \overline{B}_i \rightarrow Z$ such that the following diagram is weak homotopy commutative:

$$\begin{array}{ccc}
 \prod B_i & & \\
 \Pi \varphi_i \downarrow & \searrow \psi & \\
 \prod \overline{B}_i & \xrightarrow{\overline{\psi}} & Z.
 \end{array} \tag{1.2}$$

The purpose of this paper is to introduce the notion of a weak multi H-map and to discuss the properties; and then we prove the Kahn–Priddy theorem as an application.

Section 2 is devoted to the group completion for weak multi H-maps.

Section 3 is devoted to giving another proof of the theorem of Kahn–Priddy as an application of the first section. The theorem insists that, for $q > 0$, the composition map

$$t_* \circ j_* : \{S^q, B\mathbb{Z}/p\} \longrightarrow \{S^q, B\mathbb{Z}/p_+\} \longrightarrow \{S^q, S^0\} \quad (1.3)$$

is a split epimorphism of the p -primary components, where $j : B\mathbb{Z}/p \rightarrow B\mathbb{Z}/p \vee S^0 \underset{S}{\simeq} B\mathbb{Z}/p_+$ is the inclusion map and t is the transfer.

2. Group completion for multi H-maps

Definition 2.1. We call a space B a weak H-space, if there is a map $\mu_B : B \times B \rightarrow B$ such that $\mu_B \circ (* \times 1_B) \underset{w}{\simeq} 1_B \underset{w}{\simeq} \mu_B \circ (1_B \times *)$. We say that a weak H-space is weak homotopy associative or weak homotopy commutative, if $\mu_B \circ (\mu_B \times 1) \underset{w}{\simeq} \mu_B \circ (1 \times \mu_B)$ or $\mu_B \circ T \underset{w}{\simeq} \mu_B$ respectively, where $T(b, b') = (b', b)$.

From now on, we use the terms WHA and WHC instead of weak homotopy associative and weak homotopy commutative respectively.

Definition 2.2. Let B be a homotopy associative H-space, let $\bar{B}$ be a group-like H-space and let $\varphi : B \rightarrow \bar{B}$ be an H-map. We call φ a group completion if, for any group-like weak H-space Z and for any weak H-map $\psi : B \rightarrow Z$, there is a weak H-map $\bar{\psi} : \bar{B} \rightarrow Z$, unique up to weak homotopy, such that $\psi \underset{w}{\simeq} \bar{\psi} \circ \varphi$.

Recall from [7], that if A is a Γ -space such that $A(\mathbf{1})$ has the homotopy type of a CW-complex and $\pi_0(A(\mathbf{1}))$ contains a free abelian monoid, then $A(\mathbf{1}) \rightarrow \Omega BA(\mathbf{1})$ is a group completion.

Definition 2.3. Let B_i be homotopy associative H-spaces and let Z be a homotopy associative homotopy commutative H-space. Let $\mu_{B_i} : B_i \times B_i \rightarrow B_i$ and $\mu_Z : Z \times Z \rightarrow Z$ be multiplications respectively. We say that a map $\psi : B_1 \times \cdots \times B_n \rightarrow Z$

is a multi H-map if, for any i , the following diagram is homotopy commutative:

$$\begin{array}{ccc}
 \prod_{j=1}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j & \xrightarrow{1_{\prod B_j} \times \mu_{B_i} \times 1_{\prod B_j}} & \prod_{j=1}^n B_j \\
 \downarrow \Pi \Delta_{B_j} \times 1_{B_i^2} \times \Pi \Delta_{B_j} & & \downarrow \psi \\
 \prod_{j=1}^n B_j^2 & & Z \\
 \downarrow T & & \uparrow \mu_Z \\
 \left(\prod_{j=1}^n B_j \right)^2 & \xrightarrow{\psi^2} & Z \times Z,
 \end{array} \quad (2.1)$$

where $\Delta_{B_j} : B_j \rightarrow B_j \times B_j$ is the diagonal map and T is the shuffling map.

If B_i are WHA weak H-spaces, Z is a WHA, WHC weak H-space and the diagram (2.1) is WHC, then we call ψ a weak multi H-map.

THEOREM 2.4. *Let B_i be a homotopy associative H-space and let $\bar{B}_i$ be a group-like H-space. Let $\varphi_i : B_i \rightarrow \bar{B}_i$ be a group completion. Then, for any WHC, group-like weak H-space Z and any weak multi H-map $\psi : B_1 \times \cdots \times B_n \rightarrow Z$, there is a weak multi H-map*

$$\bar{\psi} : \bar{B}_1 \times \cdots \times \bar{B}_n \longrightarrow Z, \quad (2.2)$$

uniquely up to weak homotopy, such that the following diagram is WHC:

$$\begin{array}{ccc}
 \prod B_i & & \\
 \Pi \varphi_i \downarrow & \searrow \psi & \\
 \prod \bar{B}_i & \xrightarrow{\bar{\psi}} & Z.
 \end{array} \quad (2.3)$$

In order to prove this theorem, we need some lemmas.

LEMMA 2.5. *Let Z be a weak H-space and let K be a compact space. Then Z^K is a weak H-space in a natural way. Moreover, if Z is WHA, WHC or with weak homotopy inverse, so is Z^K .*

Proof. The multiplication map of Z^K is the composition map

$$Z^K \times Z^K \longrightarrow (Z \times Z)^{K \times K} \xrightarrow{\Delta^*} (Z \times Z)^K \xrightarrow{\mu_*} Z^K. \quad (2.4)$$

In the following diagram, the two squares are commutative and the right triangle is WHC:

$$\begin{array}{ccccccc}
 Z^K \times * & \xlongequal{\quad} & Z^K \times *^K & \longrightarrow & (Z \times *)^{K \times K} & \longrightarrow & (Z \times *)^K \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Z^K \times Z^K & \longrightarrow & (Z \times Z)^{K \times K} & \longrightarrow & (Z \times Z)^K \longrightarrow Z^K.
 \end{array} \quad (2.5)$$

Then the diagram shows that μ_{Z^K} has the right unit.

In the same way, we can show that μ_{Z^K} has the left unit. The remainder of the statement can also be shown in the same way. $\square$

LEMMA 2.6. *Let Z be a weak H -space. For any compact spaces K and L , and for any map $f : K \rightarrow L$, the induced map $f^* : Z^L \rightarrow Z^K$ is a weak H -map.*

Proof. The following diagram shows that f^* is a weak H -map:

$$\begin{array}{ccccccc}
 Z^L \times Z^L & \longrightarrow & (Z \times Z)^{L \times L} & \longrightarrow & (Z \times Z)^L & \longrightarrow & Z^L \\
 f^* \times f^* \downarrow & & (f \times f)^* \downarrow & & f^* \downarrow & & f^* \downarrow \\
 Z^K \times Z^K & \longrightarrow & (Z \times Z)^{K \times K} & \longrightarrow & (Z \times Z)^K & \longrightarrow & Z^K.
 \end{array} \quad (2.6)$$

$\square$

LEMMA 2.7. *Let Z be a WHA, WHC H -space and let K and L be compact spaces. Define a map $(\mu_Z)_* : Z^K \times Z^L \rightarrow Z^{K \times L}$ by*

$$(\mu_Z)_*(f, g) : K \times L \xrightarrow{f \times g} Z \times Z \xrightarrow{\mu_Z} Z \quad (2.7)$$

for $(f, g) \in Z^L \times Z^L$. Then $(\mu_Z)_*$ is a weak H -map.

Proof. In the following diagram, the right lower square is WHC and the other squares

are commutative:

$$\begin{array}{ccccc}
 (Z^K \times Z^L)^2 & \longrightarrow & ((Z \times Z)^{K \times L})^2 & \xrightarrow{\mu_* \times \mu_*} & (Z^{K \times L})^2 \\
 \downarrow 1 \times T \times 1 & & \downarrow & & \downarrow \\
 (Z^K)^2 \times (Z^L)^2 & & (Z^4)^{(K \times L)^2} & \xrightarrow{(\mu \times \mu)_*} & (Z^2)^{(K \times L)^2} \\
 \downarrow & & \downarrow (1 \times T \times 1)^*(1 \times T \times 1)_* & & \downarrow (1 \times T \times 1)^* \\
 (Z^2)^{K^2} \times (Z^2)^{L^2} & \longrightarrow & (Z^4)^{K^2 \times L^2} & & (Z^2)^{K^2 \times L^2} \\
 \downarrow \Delta^* \times \Delta^* & & \downarrow (\Delta \times \Delta)^* & & \downarrow (\Delta \times \Delta)^* \\
 (Z^2)^K \times (Z^2)^L & \longrightarrow & (Z^4)^{K \times L} & & (Z^2)^{K \times L} \\
 \downarrow \mu_* \times \mu_* & & \downarrow (\mu \times \mu)_* & & \downarrow \mu_* \\
 Z^K \times Z^L & \longrightarrow & (Z^2)^{K \times L} & \xrightarrow{\mu_*} & Z^{K \times L}
 \end{array} \quad (2.8)$$

Then the diagram shows that $(\mu_Z)_*$ is a weak H-map. $\square$

Proof of Theorem 2.4. For a given weak multi H-map $\psi : B_1 \times \cdots \times B_n \rightarrow Z$, we consider the adjoint map $\text{Ad } \psi : B_1 \rightarrow Z^{B_2 \times \cdots \times B_n}$. Since ψ is a weak multi H-map, the composition map

$$B_1 \xrightarrow{\text{Ad } \psi} Z^{B_2 \times \cdots \times B_n} \xrightarrow{(i_2 \times \cdots \times i_n)^*} Z^{K_2 \times \cdots \times K_n} \quad (2.9)$$

is a weak H-map for any compact subspace $K_j \subset B_j$, where $i_j : K_j \rightarrow B_j$ is the inclusion map. By the definition of the group completion, there is a weak H-map $\overline{(i_2 \times \cdots \times i_n)^* \circ \text{Ad } \psi} : \overline{B_1} \rightarrow Z^{K_2 \times \cdots \times K_n}$, unique up to weak homotopy, such that the following diagram is WHC:

$$\begin{array}{ccc}
 B_1 & \xrightarrow{\text{Ad } \psi} & Z^{B_2 \times \cdots \times B_n} \\
 \varphi_1 \downarrow & & \downarrow (i_2 \times \cdots \times i_n)^* \\
 \overline{B_1} & \xrightarrow{\overline{(i_2 \times \cdots \times i_n)^* \circ \text{Ad } \psi}} & Z^{K_2 \times \cdots \times K_n}
 \end{array} \quad (2.10)$$

So we get the map

$$\text{Ad}^{-1}(\overline{i_1 \circ (i_2 \times \cdots \times i_n)^* \circ \text{Ad } \psi}) : \overline{K_1} \times K_2 \times \cdots \times K_n \longrightarrow Z \quad (2.11)$$

unique up to homotopy for any compact subspace $\overline{K_1} \subset \overline{B_1}$, where $\overline{i_1} : \overline{K_1} \rightarrow \overline{B_1}$ is the inclusion map. For another compact subspace $\overline{K'_1} \times K'_2 \times \cdots \times K'_n$ including

$\overline{K}_1 \times K_2 \times \cdots \times K_n$, the following diagram is homotopy commutative:

$$\begin{array}{ccc} \overline{K}_1 \times K_2 \times \cdots \times K_n & & \\ \downarrow & \searrow & \\ \overline{K}'_1 \times K'_2 \times \cdots \times K'_n & \longrightarrow & Z. \end{array} \quad (2.12)$$

Therefore we get the map

$$\psi_1 : \overline{B}_1 \times B_2 \times \cdots \times B_n \longrightarrow Z \quad (2.13)$$

unique up to weak homotopy. We want to show that ψ_1 is a weak multi H-map. Since $(i_2 \times \cdots \times i_n)^* \circ \text{Ad } \psi_1$ is a weak H-map, the following diagram

$$\begin{array}{ccc} \overline{B}_1 \times \overline{B}_1 & \xrightarrow{\mu_{\overline{B}_1}} & \overline{B}_1 \\ \text{Ad } \psi_1 \times \text{Ad } \psi_1 \downarrow & & \downarrow \text{Ad } \psi_1 \\ Z^{B_2 \times \cdots \times B_n} \times Z^{B_2 \times \cdots \times B_n} & & Z^{B_2 \times \cdots \times B_n} \\ (i_2 \times \cdots \times i_n)^* \times (i_2 \times \cdots \times i_n)^* \downarrow & & \downarrow (i_2 \times \cdots \times i_n)^* \\ Z^{K_2 \times \cdots \times K_n} \times Z^{K_2 \times \cdots \times K_n} & \xrightarrow{\mu_{Z^{K_2 \times \cdots \times K_n}}} & Z^{K_2 \times \cdots \times K_n} \end{array} \quad (2.14)$$

is WHC for any compact subspaces $K_j \subset B_j$. It follows that the diagram

$$\begin{array}{ccccc} \overline{K}_1^2 \times K_2 \times \cdots \times K_n & \longrightarrow & \overline{B}_1^2 \times B_2 \times \cdots \times B_n & \xrightarrow{\mu_{\overline{B}_1} \times 1} & \overline{B}_1 \times B_2 \times \cdots \times B_n \\ \downarrow & & \downarrow & & \downarrow \psi_1 \\ \overline{B}_1^2 \times B_2 \times \cdots \times B_n & & & & Z \\ \downarrow 1_{\overline{B}_1^2} \times \Delta_{B_2} \times \cdots \times \Delta_{B_n} & & & & \uparrow \mu_Z \\ \overline{B}_1^2 \times B_2^2 \times \cdots \times B_n^2 & \xrightarrow{T} & (\overline{B}_1 \times B_2 \times \cdots \times B_n)^2 & \xrightarrow{\psi_1^2} & Z \times Z \end{array} \quad (2.15)$$

is homotopy commutative for any compact subspace $\overline{K}_1 \subset \overline{B}_1$, where T is the obvious

shuffling map. It means that the diagram

$$\begin{array}{ccc}
 \overline{B}_1^2 \times \prod_{j=2}^n B_j & \xrightarrow{\mu_{B_1} \times \prod 1_{B_j}} & \overline{B}_1 \times \prod_{j=2}^n B_j \\
 \downarrow 1_{\overline{B}_1^2} \times \prod \Delta_{B_j} & & \downarrow \psi_1 \\
 \overline{B}_1^2 \times \prod_{j=2}^n B_j^2 & & Z \\
 \downarrow T & & \uparrow \mu_Z \\
 \left(\overline{B}_1 \times \prod_{j=2}^n B_j \right)^2 & \xrightarrow{\psi_1^2} & Z \times Z
 \end{array} \quad (2.16)$$

is WHC. Since ψ is a weak multi H-map, the following diagram is WHC for $i \geq 2$:

$$\begin{array}{ccc}
 \prod_{j=1}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j & \xrightarrow{\prod 1_{B_j} \times \mu_{B_i} \times \prod 1_{B_j}} & \prod_{j=1}^n B_j \\
 \downarrow \prod \Delta_{B_j} \times 1_{B_i^2} \times \prod \Delta_{B_j} & & \downarrow \psi \\
 \prod_{j=1}^n B_j^2 & & Z \\
 \downarrow T & & \uparrow \mu_Z \\
 \left(\prod_{j=1}^n B_j \right)^2 & \xrightarrow{\psi^2} & Z \times Z.
 \end{array} \quad (2.17)$$

Then, for any compact subspaces $K_j \subset B_j$, we have that

$$\begin{aligned}
 & (i_2 \times \cdots \times i_i^2 \times \cdots \times i_n)^* \circ \text{Ad} \left(\psi \circ \left(\prod 1_{B_j} \times \mu_{B_i} \times \prod 1_{B_j} \right) \right) \\
 & \simeq_w (i_2 \times \cdots \times i_i^2 \times \cdots \times i_n)^* \circ \text{Ad} \left(\mu_Z \circ \psi^2 \circ T \circ \left(\prod \Delta_{B_j} \times 1_{B_i^2} \times \prod \Delta_{B_j} \right) \right)
 \end{aligned}$$

which maps B_1 to $Z^{\prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j}$. This means that the following diagram is

WHC:

$$\begin{array}{ccccc}
 B_1 & \xrightarrow{\text{Ad } \psi} & \prod_{j=2}^n B_j & \xrightarrow{\left(\prod_{j=2}^{i-1} 1_{B_j} \times \mu_{B_i} \times \prod_{j=i+1}^n 1_{B_j} \right)^*} & \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \\
 \Delta_{B_1} \downarrow & & & & \downarrow \\
 B_1 \times B_1 & & & & \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \\
 \text{Ad } \psi \times \text{Ad } \psi \downarrow & & & & \uparrow \\
 \prod_{j=2}^n B_j \times \prod_{j=2}^n B_j & & & & \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \\
 (\mu_Z)_* \downarrow & & & & \uparrow \\
 Z \left(\prod_{j=2}^n B_j \right)^2 & \xrightarrow{T^*} & \prod_{j=2}^n B_j^2 & \xrightarrow{\left(\prod_{j=2}^{i-1} \Delta_{B_j} \times 1_{B_i^2} \times \prod_{j=i+1}^n \Delta_{B_j} \right)^*} & \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j
 \end{array} \quad (2.18)$$

We consider the following two WHC diagrams:

$$\begin{array}{ccccc}
 B_1 & \xrightarrow{\text{Ad } \psi} & \prod_{j=2}^n B_j & \longrightarrow & \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \\
 \downarrow & & \downarrow & & \downarrow \\
 \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j & \longrightarrow & \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j & & \\
 \uparrow & & \uparrow & & \uparrow \\
 \overline{B}_1 & \xrightarrow{\text{Ad } \psi_1} & \prod_{j=2}^n B_j & \longrightarrow & \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j
 \end{array} \quad (2.19)$$

$$\begin{array}{ccccccc}
B_1 & \xrightarrow{\alpha_B} & \left(\prod_{j=2}^n B_j \right)^2 & \xrightarrow{(\mu_Z)_*} & Z \left(\prod_{j=2}^n B_j \right)^2 & \xrightarrow{T^*} & \prod_{j=2}^n B_j^2 \xrightarrow{\beta_B} \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \\
\downarrow \varphi_1 & & \downarrow & & \downarrow & & \downarrow \\
& & \left(\prod_{j=2}^n K_j \right)^2 & \xrightarrow{(\mu_Z)_*} & Z \left(\prod_{j=2}^n K_j \right)^2 & \xrightarrow{T^*} & \prod_{j=2}^n K_j^2 \xrightarrow{\beta_K} \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \\
& & \uparrow & & \uparrow & & \uparrow \\
\bar{B}_1 & \xrightarrow{\alpha_{\bar{B}}} & \left(\prod_{j=2}^n B_j \right)^2 & \xrightarrow{(\mu_Z)_*} & Z \left(\prod_{j=2}^n B_j \right)^2 & \xrightarrow{T^*} & \prod_{j=2}^n B_j^2 \xrightarrow{\beta_B} \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j
\end{array}
\quad (2.20)$$

where $\alpha_B, \alpha_{\bar{B}}, \beta_B$ and β_K are the following maps, and $K'_i = \mu_{B_i}(K_i \times K_i)$:

$$\alpha_B = (\text{Ad } \psi \times \text{Ad } \psi) \circ \Delta_{B_1} : B_1 \longrightarrow B_1 \times B_1 \longrightarrow \left(\prod_{j=2}^n B_j \right)^2 \quad (2.21)$$

$$\alpha_{\bar{B}} = (\text{Ad } \psi_1 \times \text{Ad } \psi_1) \circ \Delta_{\bar{B}_1} : \bar{B}_1 \longrightarrow \bar{B}_1 \times \bar{B}_1 \longrightarrow \left(\prod_{j=2}^n B_j \right)^2 \quad (2.22)$$

$$\beta_B = \left(\prod_{j=2}^{i-1} \Delta_{B_j} \times 1_{B_i^2} \times \prod_{j=i+1}^n \Delta_{B_j} \right)^* : Z \left(\prod_{j=2}^n B_j \right)^2 \longrightarrow \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \quad (2.23)$$

$$\beta_K = \left(\prod_{j=2}^{i-1} \Delta_{K_j} \times 1_{K_i^2} \times \prod_{j=i+1}^n \Delta_{K_j} \right)^* : Z \left(\prod_{j=2}^n K_j \right)^2 \longrightarrow \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \quad (2.24)$$

Then the following diagram is WHC since the composition map in diagram (2.19),

$$B_1 \xrightarrow{\text{Ad } \psi} \prod_{j=2}^n B_j \longrightarrow \prod_{j=2}^{i-1} K_j \times K'_i \times \prod_{j=i+1}^n K_j \longrightarrow \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \quad (2.25)$$

$$\bar{B}_1 \xrightarrow{\text{Ad } \psi_1} \prod_{j=2}^n B_j \longrightarrow \prod_{j=2}^{i-1} K_j \times K'_i \times \prod_{j=i+1}^n K_j \longrightarrow \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \quad (2.26)$$

and the composition map in diagram (2.20),

$$B_1 \xrightarrow{\alpha_B} \left(\prod_{j=2}^n B_j \right)^2 \longrightarrow \left(Z \prod_{j=2}^n K_j \right)^2 \xrightarrow{\beta_K \circ T^* \circ (\mu_Z)^*} Z \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \quad (2.27)$$

$$\overline{B}_1 \xrightarrow{\alpha_{\overline{B}}} \left(\prod_{j=2}^n B_j \right)^2 \longrightarrow \left(Z \prod_{j=2}^n K_j \right)^2 \xrightarrow{\beta_K \circ T^* \circ (\mu_Z)^*} Z \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \quad (2.28)$$

are weak H-maps and since $\varphi_1 : B_1 \rightarrow \overline{B}_1$ is a group completion:

$$\begin{array}{ccc} \overline{B}_1 & \xrightarrow{\text{Ad } \psi_1} & Z \prod_{j=2}^n B_j \xrightarrow{\left(\prod_{j=2}^{i-1} 1_{B_j} \times \mu_{B_i} \times \prod_{j=i+1}^n 1_{B_j} \right)^*} Z \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \\ \downarrow \Delta_{\overline{B}_1} & & \downarrow \\ \overline{B}_1 \times \overline{B}_1 & & \\ \downarrow \text{Ad } \psi_1 \times \text{Ad } \psi_1 & & \\ Z \prod_{j=2}^n B_j \times Z \prod_{j=2}^n B_j & & Z \prod_{j=2}^{i-1} K_j \times K_i^2 \times \prod_{j=i+1}^n K_j \\ \downarrow (\mu_Z)^* & & \uparrow \\ Z \left(\prod_{j=2}^n B_j \right)^2 \xrightarrow{T^*} Z \prod_{j=2}^n B_j^2 \xrightarrow{\left(\prod_{j=2}^{i-1} \Delta_{B_j} \times 1_{B_i^2} \times \prod_{j=i+1}^n \Delta_{B_j} \right)^*} Z \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j \end{array} \quad (2.29)$$

It is clear that (2.29) induces the following WHC diagram:

$$\begin{array}{ccc} \overline{B}_1 \times \prod_{j=2}^{i-1} B_j \times B_i^2 \times \prod_{j=i+1}^n B_j & \xrightarrow{1_{\overline{B}_1} \times \prod 1_{B_j} \times \mu_{B_i} \times \prod 1_{B_j}} & \overline{B}_1 \times \prod_{j=2}^n B_j \\ \downarrow \Delta_{\overline{B}_1} \times \prod \Delta_{B_j} \times 1_{B_i^2} \times \prod \Delta_{B_j} & & \downarrow \psi_1 \\ \overline{B}_1^2 \times \prod_{j=2}^n B_j^2 & & Z \\ \downarrow T & & \uparrow \mu_Z \\ \left(\overline{B}_1 \times \prod_{j=2}^n B_j \right)^2 & \xrightarrow{\psi_1^2} & Z \times Z. \end{array} \quad (2.30)$$

Therefore we see that ψ_1 is a weak multi H-map.

If there are two weak multi H-maps ψ_1, ψ_1' such that $\psi \underset{w}{\simeq} \psi_1 \circ (\varphi_1 \times 1_{B_2 \times \dots \times B_n}) \underset{w}{\simeq} \psi_1' \circ (\varphi_1 \times 1_{B_2 \times \dots \times B_n})$, then $(\prod i_j)^* \circ \text{Ad } \psi_1$ and $(\prod i_j)^* \circ \text{Ad } \psi_1'$ are both extensions of $(\prod i_j)^* \circ \text{Ad } \psi$. Therefore, by the uniqueness of the group completion, we see that $(\prod i_j)^* \circ \text{Ad } \psi_1 \underset{w}{\simeq} (\prod i_j)^* \circ \text{Ad } \psi_1'$ and hence $\psi_1 \underset{w}{\simeq} \psi_1'$. Therefore there is a unique weak multi H-map ψ_1 such that $\psi \underset{w}{\simeq} \psi_1 \circ (\varphi_1 \times 1_{B_2 \times \dots \times B_n})$. Inductively we obtain a weak multi H-map

$$\psi_i : \overline{B}_1 \times \dots \times \overline{B}_i \times B_{i+1} \times \dots \times B_n \longrightarrow Z \quad (2.31)$$

and define $\overline{\psi} = \psi_n : \overline{B}_1 \times \dots \times \overline{B}_n \rightarrow Z$. $\square$

3. An application

Let G be a finite group and let X be a finite G -complex. For any G -space E , a (G, E) -covering over X was defined in [4] as follows: it is a pair of G -map $(p, f) = (X \xleftarrow{p} \tilde{X} \xrightarrow{f} E)$, where $p : \tilde{X} \rightarrow X$ is a finite covering. Two (G, E) -coverings $(p, f) = (X \xleftarrow{p} \tilde{X} \xrightarrow{f} E)$ and $(p', f') = (X \xleftarrow{p'} \tilde{X}' \xrightarrow{f'} E)$ are equivalent if there is a G -homeomorphism $\phi : \tilde{X} \rightarrow \tilde{X}'$ such that $p = p' \circ \phi$ and $f \underset{G}{\simeq} f' \circ \phi$. The set of equivalent classes of (G, E) -coverings over X is denoted by $C_G(X, E)$ which is evidently an abelian monoid. If $G = e$ or $E = *$, $C_G(X, E)$ is denoted by $C(X, E)$ or $C_G(X)$ respectively.

For G -spaces X and Y , $\{X_+, Y_+\}_G$ denotes the group of stable G -homotopy classes of based G -maps, where X_+ is the based G -space obtained by adding one disjoint fixed point to X as its base point. A homomorphism $\omega : C_G(X, E) \rightarrow \{X_+, E_+\}_G$ was defined in [4] as follows: for a (G, E) -covering (p, f) , a stable G -map $\omega(p, f)$ is the composition $X_+ \xrightarrow{t} \tilde{X}_+ \xrightarrow{\sigma(f_+)} E_+$, where t is the equivariant transfer.

Let H be a subgroup of G . $W(H)$ denotes the Weyl group of H , that is, $W(H) = N(H)/H$ where $N(H)$ is the normalizer of H .

From now on, we assume that X is a finite complex with trivial G -action.

A homomorphism $\xi_H : \{X_+, EW(H)_+\}_{W(H)} \rightarrow \{X_+, S^0\}_G$ is defined in [1] as follows: for a map $f \in \{X_+, EW(H)_+\}_{W(H)}$, $\xi_H(f)$ is a composition map

$$X_+ = (* \times X)_+ \xrightarrow{t \times 1} \left(G \times_{N(H)} X \right)_+ \xrightarrow{1 \times f} \left(G \times_{N(H)} EW(H) \right)_+ \xrightarrow{\pi} S^0, \quad (3.1)$$

where t is a transfer of the covering $G/N(H) \rightarrow *$ and π is the collapsing map. We define a map

$$\xi = \prod \xi_H : \prod \{X_+, EW(H)_+\}_{W(H)} \longrightarrow \{X_+, S^0\}_G, \quad (3.2)$$

where the product is taken over the conjugacy classes of subgroups of G .

THEOREM 3.1. ([1]) ξ is an isomorphism.

A homomorphism $\lambda_G : \{X_+, EG_+\}_G \rightarrow \{X_+, BG_+\}$ was defined in [2] as follows: let $f \in \{X_+, EG_+\}_G$ be represented by a G -map $k : X_+ \wedge V^c \rightarrow EG_+ \wedge V^c$ which is factored, for sufficiently large r , as follows:

$$X_+ \wedge V^c \xrightarrow{1 \wedge \pi} X_+ \wedge V_G^c \xrightarrow{k'} EG_+^{(r)} \wedge V^c \xrightarrow{j \wedge 1} EG_+ \wedge V^c, \quad (3.3)$$

where V^c and V_G^c are the one-point-compactifications, $V_G = \{v \in V; G_v = \{e\}\}$, π is the natural projection, $EG^{(r)}$ is the r -skeleton of EG and j is the natural inclusion. There is a real G -module W such that $EG^{(r)} \times (V \oplus W) \cong EG^{(r)} \times R^N$, where R^N is the trivial representation. Moreover, we may assume that V_G/G can be embedded in R^N such that the normal bundle is $V_G \times_G W$. The embedding induces the Thom map $c : S^N \rightarrow (V_G \times_G W)^c$. Then the homomorphism $\lambda_G(f)$ is defined as the composition map

$$X_+ \wedge S^N \xrightarrow{1 \wedge c} X_+ \wedge (V_G \times_G W)^c \xrightarrow{k' \wedge 1/G} BG_+^{(r)} \wedge S^N \longrightarrow BG_+ \wedge S^N. \quad (3.4)$$

THEOREM 3.2. ([2]) λ_G is an isomorphism.

A homomorphism $\tilde{\xi}_H : C_{W(H)}(X, EW(H)) \rightarrow C_G(X)$ was defined in [4] as follows: for a $(W(H), EW(H))$ -covering $(X \xleftarrow{p} \tilde{X} \rightarrow EW(H))$, we get the composition map

$$G \times_{N(H)} \tilde{X} \xrightarrow{\pi} \tilde{X}/W(H) \xrightarrow{p/W(H)} X, \quad (3.5)$$

where $\tilde{X}$ is regarded as an $N(H)$ -space. Then the composite map $(p/W(H)) \circ \pi$ is a G -covering. We define a map

$$\tilde{\xi} = \prod \tilde{\xi}_H : \prod C_{W(H)}(X, EW(H)) \longrightarrow C_G(X), \quad (3.6)$$

where the product is taken over the conjugacy classes of subgroups of G .

A homomorphism $\tilde{\lambda}_G : C_G(X, EG) \rightarrow C(X, BG)$ was defined in [4] as follows: for a (G, EG) -covering $(X \leftarrow \tilde{X} \rightarrow EG)$, $\lambda_G(X \leftarrow \tilde{X} \rightarrow EG) = (X \leftarrow \tilde{X}/G \rightarrow BG)$.

THEOREM 3.3. ([4]) $\tilde{\xi}$ and $\tilde{\lambda}_G$ are isomorphisms of monoids.

Let N be a finite G -set and let $E(N, V)$ be the space of G -embedding maps $e : N \rightarrow V$, where V is a G -vector space. The inclusion $V \rightarrow V \oplus W$ induces a map $E(N, V) \rightarrow E(N, V \oplus W)$ so that we get the space $E(N) = \varinjlim_V E(N, V)$.

LEMMA 3.4. ([2]) $E(N)$ is a free, contractible $A(N)$ -space, where $A(N)$ is the G -automorphism group of N .

Obviously, $C_G(X) \cong [X, \coprod B(N)]$ and $\{X_+, S^0\} \cong \left[X, \varinjlim_V \Omega^V S^V \right]$, where $B(N) = E(N)/A(N)$ and the limit is taken over the category of G -modules and embeddings.

THEOREM 3.5. ([2]) There is an H -map

$$T : \coprod B(N) \longrightarrow \varinjlim_V \Omega^V S^V \quad (3.7)$$

such that it is a group completion.

It is well known that T induces a map $\omega : C_G(X) \rightarrow \{X_+, S^0\}_G$.

In the same way, the Barratt–Priddy–Quillen theorem states that there is a group completion $T : \coprod B\Sigma_n \int G \rightarrow QBG_+$. It is well known that it induces $\omega : C_G(X, EG) \rightarrow \{X_+, EG_+\}_G$ and $\omega : C(X, BG) \rightarrow \{X_+, BG_+\}$.

THEOREM 3.6. ([4]) The following diagram is commutative:

$$\begin{array}{ccccc} C_G(X) & \xleftarrow{\tilde{\xi}} & \prod C_{W(H)}(X, EW(H)) & \xrightarrow{\prod \tilde{\lambda}_{W(H)}} & \prod C(X, BW(H)) \\ \downarrow \omega & & \downarrow \prod \omega & & \downarrow \prod \omega \\ \{X_+, S^0\}_G & \xleftarrow{\xi} & \prod \{X_+, EW(H)_+\}_{W(H)} & \xrightarrow{\prod \lambda_{W(H)}} & \prod \{X_+, BW(H)_+\}. \end{array} \quad (3.8)$$

Let G' be another finite group and let $\gamma : \{X_+, S^0\}_G \rightarrow \{X_+, S^0\}_{G'}$ be a natural transformation of functors on X . We call γ admissible if $\gamma(\{X_+, BG_+\}) \subset \{X_+, BG'_+\}$ when we identify $\{X_+, S^0\}_G$ with $\prod \{X_+, BW(H)_+\}$.

Let p be a prime number and let $\mathbb{Z}/p$ be a cyclic group of order p . Let S_q be the set of all functions $\sigma : \{1, 2, \dots, p\} \rightarrow \{1, 2, \dots, q\}$ for $1 \leq q \leq p$. Since $\mathbb{Z}/p$ is a

subgroup of Σ_p , $\mathbb{Z}/p$ acts on S_q such that, for $\alpha \in \mathbb{Z}/p$, $(\alpha\sigma)(r) = \sigma(\alpha^{-1}(r))$. For $q \geq 2$, we put

$$\begin{aligned} S'_q &= \{\sigma \in S_q \mid \sigma \text{ is not a constant map}\}, \\ S''_q &= \{\sigma \in S_q \mid \sigma \text{ is surjective}\}. \end{aligned} \quad (3.9)$$

Similarly $\mathbb{Z}/p$ acts on S'_q and S''_q . Let $[\sigma] = \sigma\mathbb{Z}/p$ be an element of $S_q/(\mathbb{Z}/p)$ which is the orbit of $\sigma \in S_q$. We give here natural transformations

$$\begin{aligned} \theta_{[\sigma]} : (\{X_+, S^0\}_G)^q &\longrightarrow \{X_+, S^0\}_{\mathbb{Z}/p \times G} \\ \theta'_{[\sigma]} : (C_G(X))^q &\longrightarrow C_{\mathbb{Z}/p \times G}(X) \end{aligned} \quad (3.10)$$

as follows. First, we define them in the case where σ is not a constant map. For $(x_1, \dots, x_q) \in (\{X_+, S^0\}_G)^q$, the sum of the smash products

$$\sum_{\tau \in \sigma\mathbb{Z}/p} (x_{\tau(1)} \wedge \cdots \wedge x_{\tau(p)}) : X_+ \wedge \cdots \wedge X_+ \longrightarrow S^0 \quad (3.11)$$

can be regarded as a stable $(\Sigma_p \int G)$ -map, where $\Sigma_p \int G$ is the wreath product. Let $d : X_+ \rightarrow X_+ \wedge \cdots \wedge X_+$ be the diagonal map and regard it as a stable $(\mathbb{Z}/p \times G)$ -map, where $\mathbb{Z}/p \times G \subset \Sigma_p \int G$. So we get a stable $(\mathbb{Z}/p \times G)$ -map

$$\theta_{[\sigma]}(x_1, \dots, x_q) = \sum_{\tau \in \sigma\mathbb{Z}/p} (x_{\tau(1)} \wedge \cdots \wedge x_{\tau(p)}) \circ d : X_+ \longrightarrow S^0. \quad (3.12)$$

For $(p_i : \tilde{X}_i \rightarrow X)_{1 \leq i \leq q} \in (C_G(X))^q$, a finite covering

$$\coprod_{\tau \in \sigma\mathbb{Z}/p} \tilde{X}_{\tau(1)} \times \cdots \times \tilde{X}_{\tau(p)} \longrightarrow \prod^p X \quad (3.13)$$

can be regarded as a finite $(\Sigma_p \int G)$ -covering. So we get a finite $(\mathbb{Z}/p \times G)$ -covering

$$\theta'_{[\sigma]}(\tilde{X}_1, \dots, \tilde{X}_q) = \coprod_{\tau \in \sigma\mathbb{Z}/p} \tilde{X}_{\tau(1)} \cdot \tilde{X}_{\tau(2)} \cdot \cdots \cdot \tilde{X}_{\tau(p)}, \quad (3.14)$$

where $\tilde{X}_{\tau(1)} \cdot \tilde{X}_{\tau(2)} \cdot \cdots \cdot \tilde{X}_{\tau(p)} = \tilde{X}_{\tau(1)} \times \tilde{X}_{\tau(2)} \times \cdots \times \tilde{X}_{\tau(p)} \mid d(X)$.

Second, we define them in the case that σ is a constant map. For $(x_1, \dots, x_q) \in (\{X_+, S^0\}_G)^q$, the smash product

$$(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) : X_+ \wedge \cdots \wedge X_+ \longrightarrow S^0 \quad (3.15)$$

can be regarded as a stable $(\Sigma_p \int G)$ -map. So we get a stable $(\mathbb{Z}/p \times G)$ -map

$$\theta_{[\sigma]}(x_1, \dots, x_q) = (x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) \circ d : X_+ \longrightarrow S^0 \quad (3.16)$$

in the same way. For $(p_i : \tilde{X}_i \rightarrow X)_{1 \leq i \leq q} \in (C_G(X))^q$, a finite covering

$$\tilde{X}_{\sigma(1)} \times \cdots \times \tilde{X}_{\sigma(p)} \longrightarrow X \times \cdots \times X \quad (3.17)$$

can be regarded as a finite $(\Sigma_p \int G)$ -covering. So we get a finite $(\mathbb{Z}/p \times G)$ -covering

$$\theta'_{[\sigma]}(\tilde{X}_1, \dots, \tilde{X}_q) = \tilde{X}_{\sigma(1)} \cdot \tilde{X}_{\sigma(2)} \cdot \cdots \cdot \tilde{X}_{\sigma(p)}. \quad (3.18)$$

In the case when $q = 1$, $\theta_{[\sigma]}$ and $\theta'_{[\sigma]}$ are called power operations which are denoted by P and P' respectively (see [5]).

In the proof of Lemma 4.6 in [5], there is a definition of a natural transformation

$$\theta : (\{X_+, S^0\}_G)^p \longrightarrow \{X_+, S^0\}_{\mathbb{Z}/p \times G} \quad (3.19)$$

by $\theta(x_1, \dots, x_p) = P(x_1 + \cdots + x_p) - \sum_{j=1}^p P(x_j)$. It defines a natural transformation

$$\theta' : (C_G(X))^p \longrightarrow C_{\mathbb{Z}/p \times G}(X) \quad (3.20)$$

by $\theta(\tilde{X}_1, \dots, \tilde{X}_p) = \sum_{\tau \in S'_p} \tilde{X}_{\tau(1)} \cdot \tilde{X}_{\tau(2)} \cdot \cdots \cdot \tilde{X}_{\tau(p)}$.

LEMMA 3.7. *The following diagram is commutative:*

$$\begin{array}{ccc} (C_G(X))^q & \xrightarrow{\theta'_{[\sigma]}} & C_{\mathbb{Z}/p \times G}(X) \\ \downarrow \Pi \omega & & \downarrow \omega \\ (\{X_+, S^0\}_G)^q & \xrightarrow{\theta_{[\sigma]}} & \{X_+, S^0\}_{\mathbb{Z}/p \times G}. \end{array} \quad (3.21)$$

Proof. First we consider the case when σ is not a constant map. For $(\tilde{X}_1, \dots, \tilde{X}_q) \in (C_G(X))^q$, the following diagram is commutative because of the naturality of the transfer:

$$\begin{array}{ccc} X_+ & \longrightarrow & \left(\coprod_{\tau \in \sigma \mathbb{Z}/p} \tilde{X}_{\tau(1)} \cdot \cdots \cdot \tilde{X}_{\tau(p)} \right)_+ \\ \downarrow & & \downarrow \quad \searrow \\ X_+ \wedge \cdots \wedge X_+ & \longrightarrow & \left(\coprod_{\tau \in \sigma \mathbb{Z}/p} \tilde{X}_{\tau(1)} \times \cdots \times \tilde{X}_{\tau(p)} \right)_+ \longrightarrow S^0. \end{array} \quad (3.22)$$

The diagram shows that diagram (3.21) is commutative in the case when σ is not a constant map.

The case when σ is a constant map can be proved in the same way. $\square$

LEMMA 3.8. *For any $\sigma \in S'_q$, $\theta_{[\sigma]}$ is admissible.*

Proof. It is enough to prove for the case $\sigma \in S''_q$.

Define $s_t([\sigma]) = \#\{r \mid \#\sigma^{-1}(r) = t\}$ and $w([\sigma]) = (s_{p-1}([\sigma]), \dots, s_1([\sigma]))$. We define an order in the set $\coprod S''_q/(\mathbb{Z}/p)$. If $s_{p-1}([\sigma]) = s_{p-1}([\sigma']), \dots, s_{t+1}([\sigma]) = s_{t+1}([\sigma'])$ and $s_t([\sigma]) > s_t([\sigma'])$, we define $[\sigma] > [\sigma']$.

We use an induction on the order in $\coprod S''_q/(\mathbb{Z}/p)$ to prove the lemma. If $w([\sigma]) = (0, \dots, 0, p)$, $\theta_{[\sigma]}$ and $\theta'_{[\sigma]}$ are multilinear maps and $\theta'_{[\sigma]}$ is admissible. Then $\theta_{[\sigma]}$ is admissible by Theorem 2.4 and Lemma 3.7. Suppose that, for any q and for any natural transformation $\theta_{[\sigma']}$ which satisfies $[\sigma] > [\sigma']$, the lemma is correct. For any $1 \leq r \leq q$, we define a natural transformation

$$\theta_{[\sigma],r} : (\{X_+, S^0\}_G)^{q+1} \longrightarrow \{X_+, S^0\}_{\mathbb{Z}/p \times G} \quad (3.23)$$

by $\theta_{[\sigma],r}(x_1, \dots, x_{q+1}) = \theta_{[\sigma]}(x_1, \dots, x_r + x_{r+1}, \dots, x_{q+1}) - \theta_{[\sigma]}(x_1, \dots, x_r, x_{r+2}, \dots, x_{q+1}) - \theta_{[\sigma]}(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_{q+1})$. If $\#\sigma^{-1}(r) = 1$, evidently $\theta_{[\sigma],r} = 0$. If $\#\sigma^{-1}(r) > 1$, $\theta_{[\sigma],r}$ is a sum of some natural transformations $\theta_{[\sigma']}$ which satisfy $[\sigma] > [\sigma']$. Because of the inductive hypothesis, $\theta_{[\sigma],r}$ is admissible. Then the composition map

$$(\{X_+, BG_+\})^{q+1} \xrightarrow{i^{q+1}} (\{X_+, S^0\}_G)^{q+1} \xrightarrow{\theta_{[\sigma],r}} \{X_+, S^0\}_{\mathbb{Z}/p \times G} \xrightarrow{\pi_H} \{X_+, BW(H)_+\} \quad (3.24)$$

is a 0-map, where $\{e\} \neq H \subset \mathbb{Z}/p \times G$. Therefore $\pi_H \circ \theta_{[\sigma]} \circ i^q$ is multilinear. Obviously the composition map

$$(C(X, BG))^q \xrightarrow{(i')^q} (C_G(X))^q \xrightarrow{\theta'_{[\sigma]}} C_{\mathbb{Z}/p \times G}(X) \xrightarrow{\pi'_H} C(X, BW(H)) \quad (3.25)$$

is a 0-map and a multilinear map. Then $\pi_H \circ \theta_{[\sigma]} \circ i^q$ is a 0-map by Theorem 2.4 and Lemma 3.7. $\square$

Obviously we have $\theta = \sum_{[\sigma] \in S'(\mathbb{Z}/p)} \theta_{[\sigma]}$ and $\theta' = \sum_{[\sigma] \in S'(\mathbb{Z}/p)} \theta'_{[\sigma]}$, so we get the following corollary.

COROLLARY 3.9. *The composition map*

$$(\{X_+, BG_+\})^p \xrightarrow{i^p} (\{X_+, S^0\}_G)^p \xrightarrow{\theta} \{X_+, S^0\}_{\mathbb{Z}/p \times G} \xrightarrow{\pi_H} \{X_+, BW(H)_+\} \quad (3.26)$$

is a 0-map if H is not the trivial subgroup $\{e\} \in \mathbb{Z}/p \times G$.

Lastly, we give another proof of the Kahn–Priddy theorem.

THEOREM 3.10. (Kahn–Priddy) For $q > 0$, the composition map

$$t_* \circ j_* : \{S^q, B\mathbb{Z}/p\} \longrightarrow \{S^q, B\mathbb{Z}/p_+\} \longrightarrow \{S^q, S^0\} \quad (3.27)$$

is a split epimorphism of the p -primary components, where $j : B\mathbb{Z}/p \rightarrow B\mathbb{Z}/p \vee S^0 \xrightarrow{S} B\mathbb{Z}/p_+$ is the inclusion map and t is the transfer.

Proof. We consider the diagram

$$\begin{array}{ccc} \{X_+, S^0\} & \xrightarrow{P} & \{X_+, S^0\}_{\mathbb{Z}/p} \\ & & \downarrow \pi_{\mathbb{Z}/p} \oplus \pi_{\{e\}} \cong \\ & & \{X_+, S^0\} \oplus \{X_+, B\mathbb{Z}/p_+\} \xrightarrow{1+t_*} \{X_+, S^0\}, \end{array} \quad (3.28)$$

λ

where, for $x \in \{X_+, S^0\}_{\mathbb{Z}/p}$, $\lambda(x)$ is the same map without group action and P is the power operation. It is easy to prove the commutativity of the diagram by using the definition of group completion and the commutative diagram

$$\begin{array}{ccc} C_{\mathbb{Z}/p}(X) & & \\ \pi'_{\mathbb{Z}/p} \oplus \pi'_{\{e\}} \downarrow \cong & \searrow \lambda' & \\ C(X) \oplus C(X, B\mathbb{Z}/p) & \xrightarrow{1+t'_*} & C(X), \end{array} \quad (3.29)$$

where λ' forgets about the group action and, for $(X \leftarrow \tilde{X} \xrightarrow{f} B\mathbb{Z}/p) \in C(X, \mathbb{Z}/p)$, $t'_*(X \leftarrow \tilde{X} \xrightarrow{f} B\mathbb{Z}/p)$ is the composition covering $f^*E\mathbb{Z}/p \rightarrow \tilde{X} \rightarrow X$.

First, we show that $\pi_{\mathbb{Z}/p} \circ P = 1$:

$$\{X_+, S^0\} \xrightarrow{P} \{X_+, S^0\}_{\mathbb{Z}/p} \xrightarrow{\pi_{\mathbb{Z}/p}} \{X_+, S^0\}. \quad (3.30)$$

Define a natural transformation

$$\eta : \{X_+, S^0\} \times \{X_+, S^0\} \longrightarrow \{X_+, S^0\}_{\mathbb{Z}/p} \quad (3.31)$$

by $\eta(x_1, x_2) = P(x_1 + x_2) - P(x_1) - P(x_2)$. Clearly, $\eta = \sum_{[\sigma] \in S'_2/(\mathbb{Z}/p)} \theta_{[\sigma]}$ and

$\pi_{\mathbb{Z}/p} \circ \eta = 0$ by Lemma 3.8. Then $\pi_{\mathbb{Z}/p} \circ P$ is additive, since

$$\begin{aligned} \pi_{\mathbb{Z}/p} \circ P(x_1 + x_2) &= \pi_{\mathbb{Z}/p} \circ P(x_1) + \pi_{\mathbb{Z}/p} \circ P(x_2) + \pi_{\mathbb{Z}/p} \circ \eta(x_1 + x_2) \\ &= \pi_{\mathbb{Z}/p} \circ P(x_1) + \pi_{\mathbb{Z}/p} \circ P(x_2). \end{aligned} \quad (3.32)$$

We consider the next commutative diagram:

$$\begin{array}{ccccc}
 C(X) & \xrightarrow{P'} & C_{\mathbb{Z}/p}(X) & \xrightarrow{\pi'_{\mathbb{Z}/p}} & C(X) \\
 \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\
 \{X_+, S^0\} & \xrightarrow{P} & \{X_+, S^0\}_{\mathbb{Z}/p} & \xrightarrow{\pi_{\mathbb{Z}/p}} & \{X_+, S^0\}.
 \end{array} \quad (3.33)$$

Since $\pi'_{\mathbb{Z}/p} \circ P'$ is clearly the identity map, the composition $\pi_{\mathbb{Z}/p} \circ P$ is the identity map by the definition of group completion.

Since $B\mathbb{Z}/p_+ \cong_S B\mathbb{Z}/p \vee S^0$, it induces an isomorphism $h : \{X_+, B\mathbb{Z}/p_+\} \rightarrow \{X_+, B\mathbb{Z}/p\} \oplus \{X_+, S^0\}$, and we consider the inclusion maps $i : S^0 \rightarrow B\mathbb{Z}/p \vee S^0 \cong_S B\mathbb{Z}/p_+$ and $j : B\mathbb{Z}/p \rightarrow B\mathbb{Z}/p \vee S^0 \cong_S B\mathbb{Z}/p_+$. We put $h \circ \pi_{[e]} \circ P(x) = (\alpha(x), \beta(x)) \in \{X_+, B\mathbb{Z}/p\} \oplus \{X_+, S^0\}$. We consider the following diagram:

$$\begin{array}{ccccc}
 C(X) & \xrightarrow{i'} & C(X, B\mathbb{Z}/p) & \xrightarrow{t'} & C(X) \\
 \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\
 \{X_+, S^0\} & \xrightarrow{i_*} & \{X_+, B\mathbb{Z}/p_+\} & \xrightarrow{t_*} & \{X_+, S^0\},
 \end{array} \quad (3.34)$$

where i' is induced by $* \rightarrow B\mathbb{Z}/p$. The diagram induces $t_* \circ i_*(y) = py$ since $t' \circ i'(\tilde{X}) = p\tilde{X}$.

Since $\lambda \circ P(x) = x^p$, we have $t_* \circ j_* \circ \alpha(x) = x^p - x - p\beta(x)$. For any $x \in \{S^q, S^0\}$ where $q > 0$, we have that $x^p = 0$. Then

$$t_* \circ j_* \circ \alpha : \{S^q, S^0\} \longrightarrow \{S^q, S^0\} \quad (3.35)$$

is an automorphism of the p -primary components of $\{S^q, S^0\}$. $\square$

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Tetsu Nishimoto
Department of Mathematics
Faculty of Science
Okayama University
Japan
(E-mail: nisimoto@math.okayama-u.ac.jp)