THE STRUCTURE OF ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$
(THE CUBIC HYPERSURFACE CASE)

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1. Introduction

A holomorphic map $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ ($m > n \geq 1$) is called an algebraic embedding of $\mathbb{C}^n$ into $\mathbb{C}^m$ if $f$ is an injective polynomial map and $f(\mathbb{C}^n)$ is a smooth algebraic subvariety of $\mathbb{C}^m$. Let $f : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ be an algebraic embedding for $n \geq 1$. Identify $\mathbb{C}^{n+1}$ with an affine part of $\mathbb{P}^{n+1}$. Let $X_f$ be the closure of $f(\mathbb{C}^n)$ in $\mathbb{P}^{n+1}$ and put $Y_f := X_f - f(\mathbb{C}^n)$. Then, by construction, $Y_f$ is a hyperplane section of $X_f$ and $X_f - Y_f$ is biregular to $\mathbb{C}^n$, that is, $(X_f, Y_f)$ is a compactification of $\mathbb{C}^n$. The subvariety $Y_f$ is called the boundary of the compactification $(X_f, Y_f)$. Let $\text{Aut}(\mathbb{C}^n)$ denote polynomial automorphisms of $\mathbb{C}^n$ for $n \geq 1$.

Now let us consider the following problem.

**Problem.** Are any algebraic embeddings $f : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ equivalent to linear embeddings? Namely, does there exist a polynomial automorphism $\alpha$ of $\mathbb{C}^{n+1}$ such that $\alpha \circ f$ is a linear embedding?

For $n = 1$, this problem was solved affirmatively by Suzuki [Su1] (cf. [Su2]) and Abhyankar and Moh [AM] independently. For $n \geq 2$, the problem is still open. From now on to the end of this paper, we will consider the case when $n = 2$. It is obvious that the answer to the problem is affirmative when $\deg X_f \leq 2$. In the case where $X_f$ is a normal cubic or quartic hypersurface with the second Betti number $b_2(X_f) = 1$, the problem was solved affirmatively by Furushima [Fu2].

In this paper, we will treat the case when $\deg X_f = 3$. First we shall determine compactifications of $\mathbb{C}^2$ which are cubic hypersurfaces in $\mathbb{P}^3$ and whose boundaries are hyperplane sections. The result is the following.

**Theorem 1.** Let $X$ be a cubic hypersurface in $\mathbb{P}^3$ and $Y$ a closed subvariety of $X$ such that $X - Y$ is biholomorphic to $\mathbb{C}^2$. Assume that $Y$ is a hyperplane section of $X$, that is, there exists a hyperplane $H$ in $\mathbb{P}^3$ such that $Y = X \cap H$. Then $(X, Y)$ is one of the following (up to projective transformations of $\mathbb{P}^3$):

(I) \( X : z_0^3 + z_3(\lambda z_0^2 + z_1^2 + z_2 z_3) = 0; \)

(II) \( X : z_0(z_0 z_1 + z_2^2) + z_3(\lambda z_0 z_2 + z_2 z_3) = 0; \)

(III) \( X : z_0^2 z_1 + z_3(\lambda z_1^2 + z_2 z_3) = 0; \)

(IV) \( X : z_0^2 z_1 + z_3(z_1 z_2 + z_0 z_3) = 0; \)

(V) \( X : z_0 z_1(z_0 + z_1) + z_3(z_1 z_2 + z_0 z_3) = 0; \)

(VI) \( X : z_0 z_1(z_0 + z_1) + z_3(\lambda z_0 z_1 + z_2 z_3) = 0; \)

(VII) \( X : z_0^3 + z_3(\lambda z_0^2 + z_2 z_3) = 0; \)

(VIII) \( X : z_0^3 + z_3(z_0 z_1 + z_2 z_3) = 0; \)

(IX) \( X : z_0^2 z_1 + z_3(\lambda z_0 z_2 + z_2 z_3) = 0, \)

where \( H = \{z_3 = 0\} \) and \( \lambda \in \mathbb{C} \) is a parameter.

For each case, \( \mathcal{Y} := H|_X \) and \( x := \text{Sing } X \) are as follows:

(I) \( \mathcal{Y} = 3\text{line}, x = \{E_6\} = \{(0 : 0 : 1 : 0)\}; \)

(II) \( \mathcal{Y} = \text{line} + \text{conic}, x = \{D_5\} = \{(0 : 1 : 0 : 0)\}; \)

(III) \( \mathcal{Y} = 2\text{line} + \text{line}, x = \{D_5\} = \{(0 : 0 : 1 : 0)\}; \)

(IV) \( \mathcal{Y} = 2\text{line} + \text{line}, x = \{p, q\}, p = A_4 = \{(0 : 0 : 1 : 0)\}, \)

\( q = A_1 = \{(0 : 1 : 0 : 0)\}; \)

(V) \( \mathcal{Y} = \text{line} + \text{line} + \text{line}, x = \{A_4\} = \{(0 : 0 : 1 : 0)\}; \)

(VI) \( \mathcal{Y} = \text{line} + \text{line} + \text{line}, x = \{D_4\} = \{(0 : 0 : 1 : 0)\}; \)

(VII) \( \mathcal{Y} = 3\text{line}, x = \{z_0 = z_3 = 0\}; \)

(VIII) \( \mathcal{Y} = 3\text{line}, x = \{z_0 = z_3 = 0\}; \)

(IX) \( \mathcal{Y} = 2\text{line} + \text{line}, x = \{z_0 = z_3 = 0\}. \)

Remark. (1) If \( X \) is non-normal, we obtain (VII), (VIII) and (IX) directly by [BW] (cf. [Fu]). Hence we shall prove Theorem 1 in the case where \( X \) is normal. Then we obtain from (I) to (VI).

(2) One can summarize (II) and (III) and rewrite (IX) as follows:

\[ \text{(II) + (III)} \quad X : z_0^2 z_1 + z_3(z_1^2 + z_2 z_3) = 0 \]

\[ \text{(IX)} \quad X : z_0^2 z_1 + z_3 z_2 = 0, \]

where \( H = \{\lambda z_0 + \mu z_3 = 0\} \) and \( (\lambda : \mu) \in \mathbb{I}^1 \) is a parameter.

In (II) + (III), one obtains (II) if \( \lambda \neq 0 \) and (III) if \( \lambda = 0 \).

As a consequence of Theorem 1, we obtain the main result as follows.

Theorem 2. Let \( f : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \) be an algebraic embedding. Assume that \( \text{deg } X_f \leq 3. \) Then there exists an \( \alpha \in \text{Aut}(\mathbb{C}^3) \) such that \( \alpha \circ f \) is a linear embedding.

Remark. One can easily obtain Theorem 2 when \( \text{deg } X_f \leq 2. \)
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Notation.

$\omega_V$: dualizing sheaf of $V$.

$K_V$: canonical divisor of $V$.

$D|_V$: restriction of Cartier divisor $D$ to $V$.

$m_{V,v}$: maximal ideal of $O_{V,v}$.

$\mathbb{F}_t$: Hirzebruch surface of degree $t \geq 0$, where $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$.

$h^i(\mathcal{F}) = h^i(V, \mathcal{F}) := \dim_{\mathbb{C}} H^i(V, \mathcal{F})$.

$\operatorname{mult}_Z Y$: multiplicity of $Y$ at a general point of $Z$.

$\operatorname{Exc} f$: exceptional set of a birational morphism $f : V \longrightarrow W$.

$b_i(V) := \dim_{\mathbb{R}} H^i(V; \mathbb{R})$: the $i$th Betti number of $V$.

$\chi(V)$: topological Euler number of $V$.

$\sim$: linear equivalence.

$(V, v)$: normal two-dimensional singularity.

$p_g(v)$: geometric genus of $(V, v)$.

$p_g(v_1, \ldots, v_N) := \sum_{i=1}^N p_g(v_i)$.

$(-n)$-curve: smooth rational curve with the self-intersection number $-n$.

$\mathcal{O}$: $0$-curve.

$\bullet$: $(-1)$-curve.

$\circ$: $(-2)$-curve.

$\circ_n$: $(-n)$-curve.

$G_{\text{free}}$: free part of group $G$.

$G_{\text{tor}}$: torsion part of group $G$.

$\mathbb{N} := \{1, 2, 3, \ldots\}$.

2. Preliminaries

2.1. Compactifications of $\mathbb{C}^2$

Definition 1. Let $V$ be a two-dimensional compact complex space and $\Delta$ a closed subspace of $V$. Assume that $V$ is reduced and irreducible and $\Delta$ is reduced. The pair $(V, \Delta)$ is called a compactification of $\mathbb{C}^2$ if $V - \Delta$ is biholomorphic to $\mathbb{C}^2$. The closed subspace $\Delta$ is called the boundary of the compactification $(V, \Delta)$. A compactification $(V, \Delta)$ is said to be smooth if $V$ is a complex manifold. Two compactifications $(V, \Delta)$ and $(V', \Delta')$ are said to be isomorphic, we write simply as $(V, \Delta) \cong (V', \Delta')$, if there exists a biholomorphic mapping $\varphi : V \longrightarrow V'$ such that $\varphi(\Delta) = \Delta'$.

Let $(V, \Delta)$ be a compactification of $\mathbb{C}^2$ and $\Delta = \bigcup_i \Delta_i$ the irreducible decomposition of $\Delta$. Put $v := \text{Sing } V$. We first give some remarks.
Remark. (1) Since $V - \Delta \cong \mathbb{C}^2$ is smooth, one obtains $v \subset \Delta$.

(2) If $V$ is normal, then $\Delta$ has pure codimension one in $V$, that is, $\Delta$ is a curve by the Riemann extension theorem. Assume that $V$ is non-normal and let $v : V^v \longrightarrow V$ be the normalization of $V$. Note that $(V^v, v^{-1}(\Delta))$ is a compactification of $\mathbb{C}^2$ by the isomorphism $V^v - v^{-1}(\Delta) \cong V - \Delta \cong \mathbb{C}^2$. Hence $v^{-1}(\Delta)$ is a curve. Since $v$ is a finite holomorphic map, one has that $\Delta$ is also a curve.

(3) For every smooth compactification $(V, \Delta)$, $V$ is a rational surface by [Ko].

(4) If $V$ is projective, then $V - \Delta$ is biregular to $\mathbb{C}^2$ by (3) and [Mo].

**Proposition 1.** Assume that $V$ is normal. Then:

(i) $H_0(V, \mathbb{Z}) \cong H_0(\Delta, \mathbb{Z}) = \mathbb{Z}$;

(ii) $H_1(V, \mathbb{Z}) \cong H_1(\Delta, \mathbb{Z}) = 0$;

(iii) $H_2(V, \mathbb{Z}) \cong H_2(\Delta, \mathbb{Z}) = \bigoplus_i \mathbb{Z}. \Delta_i$;

(iv) $H_3(V, \mathbb{Z}) \cong H_3(\Delta, \mathbb{Z}) = 0$;

(v) $H^1(V, \mathcal{O}_V) = 0$;

(vi) $p_g(v) = h^2(V, \mathcal{O}_V)$.

**Proof.** Let us consider the following exact sequence:

$$
\longrightarrow H_{i+1}(V, \Delta; \mathbb{Z}) \longrightarrow H_i(\Delta; \mathbb{Z}) \longrightarrow H_i(V; \mathbb{Z}) \longrightarrow H_i(V, \Delta; \mathbb{Z}) \longrightarrow .
$$

Since the 4-sphere $S^4$ is the one point compactification of $V - \Delta \cong \mathbb{C}^2 \cong \mathbb{R}^4$, one has $H_i(V, \Delta; \mathbb{Z}) \cong H_i(S^4; \mathbb{Z}) = 0$ for $i \neq 4$. Thus one obtains $H_i(\Delta; \mathbb{Z}) \cong H_i(V; \mathbb{Z})$ for $i \leq 2$. This implies (i) and (iii). Since $\Delta$ is a curve, one obtains $H_3(\Delta; \mathbb{Z}) = 0$, which shows (iv).

Let $\pi : \tilde{V} \longrightarrow V$ be the minimal resolution of $V$. Let $\tilde{\Delta}_i$ be the proper transform of $\Delta_i$ by $\pi$ and put $\tilde{\Delta} := \bigcup_i \tilde{\Delta}_i$. Since $\tilde{V} - \tilde{\Delta} - \pi^{-1}(v) \cong V - \Delta \cong \mathbb{C}^2$, $\tilde{V}$ is a rational surface by Remark 3 of Definition 1. Hence one has $0 = H_1(\tilde{V}; \mathbb{Z}) \cong H_1(\tilde{\Delta} \cup \pi^{-1}(v); \mathbb{Z})$. In particular, one obtains $H_1(\Delta; \mathbb{Z}) = 0$, which shows (ii).

Next let us consider the following exact sequence:

$$
0 \longrightarrow H^1(V, \mathcal{O}_V) \longrightarrow H^1(\tilde{V}, \mathcal{O}_{\tilde{V}}) \longrightarrow H^0(V, R^1\pi_*\mathcal{O}_{\tilde{V}}) \longrightarrow H^2(V, \mathcal{O}_V) \longrightarrow H^2(\tilde{V}, \mathcal{O}_{\tilde{V}}).
$$

Since $\tilde{V}$ is a rational surface, this shows (v) and (vi).

Remark. By (i) and (ii), one sees that $\Delta$ is connected and $\Delta$ cannot have any cycles. In particular, one obtains that every $\Delta_i$ is a rational curve without nodes.

**Definition 2.** A smooth compactification $(V, \Delta)$ of $\mathbb{C}^2$ is said to be **minimal normal** if $\Delta$ satisfies the following two conditions:
(i) the boundary curve $\Delta = \bigcup_{i} \Delta_i$ has at most ordinary double points;
(ii) if $\Delta_j$ is a $(-1)$-curve, then there exist at least three irreducible components of $\Delta$ which are different from $\Delta_j$ and intersect $\Delta_j$.

Remark. From the resolution of singularities of curves embedded into smooth surfaces, it follows that every smooth compactification $(V, \Delta)$ can be transformed into a smooth compactification $(V', \Delta')$ such that $\Delta'$ has at most ordinary double points by blowing-up in the boundary $\Delta$ repeatedly, and hence into a minimal normal compactification $(V'', \Delta'')$ by blowing-down in the boundary $\Delta'$ repeatedly.

**Proposition 2.** ([Mo]) Let $(V, \Delta)$ be a minimal normal compactification of $\mathbb{C}^2$. Then the weighted dual graph of $\Delta$ is a linear tree of smooth rational curves which is one of the following:

\begin{align*}
(i) & \quad \begin{array}{c}
\circ \\
1
\end{array} \\
(ii) & \quad \begin{array}{cc}
0 & n \\
\circ & \circ
\end{array} \quad (n \neq -1) \\
(iii) & \quad \begin{array}{ccc}
\Delta' & n-1 & 0 \\
\cdots & \circ & \circ & \circ \\
& n & \Delta''
\end{array} \quad (n > 0),
\end{align*}

where $\Delta'$ (respectively $\Delta''$) is either an empty set or a linear tree of smooth rational curves whose self-intersection numbers are smaller than or equal to $-2$.

2.2. The hypersurface case

In this subsection, we assume the following.

Let $X$ be a normal hypersurface of degree $d \geq 1$ in $\mathbb{P}^3$ and $Y$ a closed subvariety of $X$ such that $X - Y$ is biholomorphic to $\mathbb{C}^2$. Assume that $Y$ is a hyperplane section of $X$, that is, there exists a hyperplane $H$ in $\mathbb{P}^3$ such that $Y = X \cap H$.

Now we define some notation as follows. Let $Y = \bigcup_{i=1}^{f} Y_i$ be the irreducible decomposition of $Y$ and $Y_i$ a plane curve of degree $d_i$ in $H \cong \mathbb{P}^2$ for each $i$. Put $Y := H|_X = \sum_{i=1}^{f} k_i Y_i$, where $\sum_{i=1}^{f} k_i d_i = d$. Put $x := \text{Sing } X = \{x_1, \ldots, x_n\}$ for $n \geq 0$. Let $\pi : M \rightarrow X$ be the minimal resolution of $X$ and $E = \bigcup_{i=1}^{f} E_i := \pi^{-1}(x)$ the exceptional set of $\pi$. Let $Z^{(i)}$ be the fundamental cycle of $\pi^{-1}(x_i)$ and
the proper transform of $Y_i$ by $\pi$. Set $\hat{Y} := \bigcup_{i=1}^r \hat{Y}_i$ and $A := \hat{Y} \cup E$. Let $\Gamma$ be a smooth hyperplane section of $X$ with $\Gamma \cap x = \emptyset$ and $H_\Gamma$ a hyperplane in $\mathbb{P}^3$ such that $\Gamma = X \cap H_\Gamma$. Let $\hat{\Gamma}$ be the proper transform of $\Gamma$ by $\pi$.

Remark. (1) Since $X - Y \cong \mathbb{C}^2$ is smooth, one obtains $x \subset Y$.

(2) The pair $(M, A)$ is a smooth compactification of $\mathbb{C}^2$ since $\pi : M - E \cong X - x$. By [Ko], one sees that $M$ is rational and hence $X$ is also rational.

(3) Since $X$ is a hypersurface in $\mathbb{P}^3$, in particular, Gorenstein, the dualizing sheaf $\omega_X$ of $X$ is invertible. Then one obtains $\omega_X = \mathcal{O}_X(d - 4)$.

**Proposition 3.**

(i) $\quad H_0(X, \mathbb{Z}) \cong H_0(Y, \mathbb{Z}) = \mathbb{Z}$.

(ii) $\quad H_1(X, \mathbb{Z}) \cong H_1(Y, \mathbb{Z}) = 0$.

(iii) $\quad H_2(X, \mathbb{Z}) \cong H_2(Y, \mathbb{Z}) = \bigoplus_i \mathbb{Z}.Y_i$.

(iv) $\quad H_3(X, \mathbb{Z}) \cong H_3(Y, \mathbb{Z}) = 0$.

(v) $\quad H^1(X, \mathcal{O}_X) = 0$.

(vi) $\quad p_g(x) = (d - 1)(d - 2)(d - 3)/6$.

(vii) $\quad b_2(M) = b_2(Y) + b_2(E)$.

**Proof.** Apply Proposition 1 to the compactifications $(X, Y)$ and $(M, A)$. Then we have the assertions except (vi). By the Serre duality theorem for the projective Gorenstein surface $X$ and an easy computation, we obtain $p_g(x) = h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d - 4)) = (d - 1)(d - 2)(d - 3)/6$. Thus we obtain (vi).

**Remark.** (1) If $Y$ contains at least two lines, then $Y$ consists of lines which meet only one point. Indeed, this follows since $Y$ cannot have any cycles by (ii) and each $Y_i$ is a plane curve.

(2) In the case when $d = 3$, since $K_X \sim -\Gamma$, one obtains that $p_g(x) = 0$ and thus $x$ consists of rational double points. In the case when $d = 4$, since $K_X \sim 0$, one has that $p_g(x) = 1$ and hence $x$ contains a minimally elliptic singular point $x_1$ necessarily. If $x$ contains at least two points, $x - \{x_1\}$ consists of rational double points. By [Ar] and [La] (cf. [HW]), one obtains that $K_M \sim -\hat{\Gamma}$ for $d = 3$ and $K_M \sim -Z^{(1)}$ for $d = 4$.

**Proposition 4.** Assume that $d \geq 3$. Then:

(i) $\quad x \neq \emptyset$;

(ii) $\quad X$ is not a cone.

**Proof.** (i) Assume that $X$ is smooth. Note that $K_X \sim (d - 4)\Gamma$. Since $X$ is rational,
by the characterization of rational surfaces, we have that

\[ q(X) = h^1(\mathcal{O}_X) = 0 \]
\[ P_n(X) = h^0(nK_X) = 0 \quad (\forall n \geq 1). \]

If \( d \geq 5 \), then we obtain a non-zero holomorphic section of \( \mathcal{O}(K_X) \) corresponding to the Cartier divisor \((d-4)\Gamma\). Hence we have \( H^0(K_X) \neq 0 \), which is a contradiction. If \( d = 4 \), then we obtain \( H^0(K_X) = \mathbb{C} \) since \( K_X \sim 0 \). This is a contradiction. Let us consider the case when \( d = 3 \). Applying the Noether formula for rational surfaces, we obtain \( b_2(X) = 10 - K^2_X = 10 - 3 = 7 \). On the other hand, by Proposition 3(iii), we have \( b_2(X) = b_2(Y) \leq 3 \). This is absurd.

(ii) Assume that \( X \) is a cone. Take a vertex \( p \) of the cone \( X \). Let \( \sigma : \mathbb{P}^3 \to \mathbb{P}^3 \) be the blowing-up at \( p \) and \( \psi : \mathbb{P}^3 \to \mathbb{P}^2 \) the projection from \( p \). Let \( \overline{\psi} : \mathbb{P}^3 \to \mathbb{P}^2 \) be the resolution of indeterminacy of \( \psi \) and \( X \) the proper transform of \( X \) by \( \sigma \). Then \( \overline{\psi} : \overline{X} \to \overline{\psi}(\overline{X}) \) is a geometrically ruled surface over a smooth curve \( \overline{\psi}(\overline{X}) \cong \Gamma \) of genus \((d-1)(d-2)/2 \geq 1\). This implies that \( \overline{X} \) is not rational, which is a contradiction. \( \square \)

**Lemma 1.** Let \( M \) be a complex manifold with \( n := \dim M \geq 2 \). Let \( C \) be an irreducible curve in \( M \) and \( D \) an effective Cartier divisor of \( M \). Assume that \( C \cap D \) consists of only one point \( p \) and \( C \) is smooth at \( p \). Then

\[ \text{mult}_p D \leq \deg_C D|_C. \]

In particular, if \( \deg_C D|_C = 1 \), then \( D \) is smooth at \( p \).

**Proof.** Since the assertion is local, we may assume the following:

1. \( M \) is a domain of \( \mathbb{C}^n \) which contains \( p = (0, \ldots, 0) \), where \((z_1, \ldots, z_n)\) is the canonical coordinate system of \( \mathbb{C}^n \);
2. \( C = M \cap \{ z_1 = \cdots = z_{n-1} = 0 \} \);
3. \( D \) is given by a holomorphic function \( f(z_1, \ldots, z_n) \) on \( M \). Then we have an expansion \( f = f_r + f_{r+1} + \cdots \), where \( f_i \) are homogeneous polynomials of \( z_1, \ldots, z_n \) of degree \( i \) and \( r := \text{mult}_p D \).

Then the divisor \( D|_C \) of \( C \) is given by a holomorphic function of one variable \( z_n \)

\[ f(0, \ldots, 0, z_n) = f_r(0, \ldots, 0, z_n) + f_{r+1}(0, \ldots, 0, z_n) + \cdots = a_r z_n^r + a_{r+1} z_n^{r+1} + \cdots, \]

where \( a_i \in \mathbb{C} \) for each \( i \). Hence we obtain \( \deg_C D|_C \geq r = \text{mult}_p D \). \( \square \)
PROPOSITION 5. Let \( p \in Y = X \cap H \). Then
\[
\text{mult}_p X \leq \sum_i k_i \text{mult}_p Y_i.
\]

In particular, \( x \subset (\text{Sing } Y) \cup (\bigcup_{i: k_i \geq 2} Y_i) \).

Proof. Let \( l \) be a line in \( H \) through \( p \) such that \( l \not\subset Y \). Then there exists an open neighborhood \( U_l \) of \( p \) in \( \mathbb{P}^3 \) such that \( (X \cap l) \cap U_l = \{ p \} \). Applying Lemma 1, we obtain
\[
\text{mult}_p X \leq \deg_{l \cap U_l} X|_{l \cap U_l} = \deg_{l \cap U_l} (X|_H)|_{l \cap U_l} = \sum_i k_i \deg_{l \cap U_l} Y_i|_{l \cap U_l}.
\]
Taking a minimum over \( l \) in this inequality and using the characterization of multiplicities (cf. [Mu, p. 75]), we obtain \( \text{mult}_p X \leq \sum_i k_i \text{mult}_p Y_i \).

PROPOSITION 6. \( \gcd(d_1, \ldots, d_t) = 1 \). In particular, if \( t = 1 \), then
\[\mathcal{Y} = dY_1 \ (Y_1 : \text{line}).\]

Proof. Let us consider the following exact sequence:
\[
\rightarrow H_2(X; \mathbb{Z}) \xrightarrow{\iota_*} H_2(\mathbb{P}^3; \mathbb{Z}) \rightarrow H_2(\mathbb{P}^3, X; \mathbb{Z}) \rightarrow .
\]
If we have \( H_2(\mathbb{P}^3, X; \mathbb{Z}) = 0 \), then \( \iota_* \) is surjective. Since \( H_2(X; \mathbb{Z}) = \bigoplus_{i=1}^t \mathbb{Z}Y_i \) and \( H_2(\mathbb{P}^3; \mathbb{Z}) = \mathbb{Z}l \) where \( l \) is a line in \( \mathbb{P}^3 \), there exist \( a_i \in \mathbb{Z} \) such that \( \iota_*(\sum_i a_i Y_i) \sim l \) (homologous). By computing the intersection number of a general hyperplane \( H' \) and \( l \), we obtain \( 1 = (H' \cdot l) = (H' \cdot \iota_*(\sum_i a_i Y_i)) = \sum_i a_i (H' \cdot Y_i) = \sum_i a_i d_i \), that is, \( \gcd(d_1, \ldots, d_t) = 1 \).

Now we show \( H_2(\mathbb{P}^3, X; \mathbb{Z}) = 0 \) by using the following two facts:

1. the universal coefficient theorem
   \[H^i(\mathbb{P}^3, X; \mathbb{Z}) \cong H_i(\mathbb{P}^3, X; \mathbb{Z})_{\text{free}} \oplus H_{i-1}(\mathbb{P}^3, X; \mathbb{Z})_{\text{tor}} \ (\forall i);\]

2. the fact that \( \mathbb{P}^3 - X \) is Stein implies
   \[H^i(\mathbb{P}^3, X; \mathbb{Z}) \cong H_{6-i}(\mathbb{P}^3 - X; \mathbb{Z}) = 0 \ (i = 1, 2)\]
   \[H^3(\mathbb{P}^3, X; \mathbb{Z}) \cong H_3(\mathbb{P}^3 - X; \mathbb{Z}) : \text{free}.\]

By (2) and the case when \( i = 3 \) in (1), we have \( H_2(\mathbb{P}^3, X; \mathbb{Z})_{\text{tor}} = 0 \). Similarly, by (2) and the case when \( i = 2 \) in (1), we obtain \( H_2(\mathbb{P}^3, X; \mathbb{Z})_{\text{free}} = 0 \).
From now on to the end of this subsection, we assume that \( d \geq 3 \). By Proposition 4(i), we have that \( x \neq \emptyset \). Now we consider projections of \( \mathbb{P}^3 \) from a point and blowing-up of \( \mathbb{P}^3 \) at a point. Here we give some notation as follows.

Take a singular point \( p \) of \( X \). Put \( m := \text{mult}_p X \). Let \( \sigma : \mathbb{P}^3 \to \mathbb{P}^3 \) be the blowing-up at \( p \) and \( \Delta \subset \mathbb{P}^3 \) the exceptional divisor of \( \sigma \). Note that \( \sigma : \mathbb{P}^3 - \Delta \cong \mathbb{P}^3 - \{p\}, \Delta \cong \mathbb{P}^2 \) and \( \mathcal{O}_{\mathbb{P}^3}(\Delta)|_\Delta \cong \mathcal{O}_{\mathbb{P}^2}(-1) \). Let \( \overline{V} \) be the proper transform of a subvariety \( V \) of \( \mathbb{P}^3 \) by \( \sigma \). Put \( \overline{x} := \text{Sing } \overline{X} \). Set \( \overline{E} = \bigcup_i \overline{E}_i := \overline{X} \cap \Delta \) and \( \overline{E} := \overline{X}|_\Delta = \sum_i e_i \overline{E}_i \). Note that \( \mathcal{O}_\Delta(\overline{E}) \cong \mathcal{O}_{\mathbb{P}^2}(m) \). Let \( \psi : \mathbb{P}^3 \to \mathbb{P}^2 \) be the proper transform from \( p \) and \( \overline{\psi} : \mathbb{P}^3 \to \mathbb{P}^2 \) the resolution of indeterminacy of \( \psi \).

Note that \( \overline{\psi}|_\Delta : \Delta \cong \mathbb{P}^2 \). Put \( \overline{V}^* := \overline{\psi}(\overline{V}), \overline{E}_i^* := \overline{\psi}(\overline{E}_i) \) and \( \overline{E}^* := \bigcup_i \overline{E}_i^* \). We denote simply \( \sigma|_{\overline{X}}, \psi|_{\overline{X}} \) and \( \overline{\psi}|_{\overline{X}} \) by \( \sigma, \psi \) and \( \overline{\psi} \) respectively. Thus we obtain the commutative diagram as in Figure 1.

\[
\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\psi} & \mathbb{P}^2 \\
\overline{\psi} \downarrow & & \downarrow \overline{\psi} \\
\mathbb{P}^3 - \{p\} & \xrightarrow{\psi} & \mathbb{P}^2 \\
\sigma \downarrow & & \downarrow \sigma \\
X & \xrightarrow{\psi} & \mathbb{P}^2 \\
\end{array}
\]

(1) \hspace{2cm} (2)

\textbf{Figure 1.}

\textbf{Remark.} (1) The hypersurface \( \overline{X} \) is non-normal in general.
(2) The pair \((\overline{X}, \overline{Y} \cup \overline{E})\) is a compactification of \( \mathbb{C}^2 \) since \( \sigma : \overline{X} - \overline{E} \cong X - \{p\} \).
(3) Since \( \overline{X} - \overline{Y} - \overline{E} \cong \mathbb{C}^2 \) is smooth, one obtains \( \overline{x} \subset \overline{Y} \cup \overline{E} \).
(4) Since \( \overline{X} \) is a hypersurface in \( \mathbb{P}^3 \), in particular, Gorenstein, the dualizing sheaf \( \omega_{\overline{X}} \) of \( \overline{X} \) is invertible. Then one obtains \( \omega_{\overline{X}} = \mathcal{O}_{\overline{X}}((d - 4)(\sigma^*H)|_{\overline{X}} - (m - 2)\Delta|_{\overline{X}}) \).

\textbf{PROPOSITION 7.} One has the following.
(i) The morphism \( \overline{\psi} : \overline{X} \to \mathbb{P}^2 \) is surjective and generically of degree \( d - m \).
(ii) Assume that each \( Y_i \) is a line through \( p \). Then there exists a line \( \overline{E}_j \subset \overline{E} \) in \( \Delta \cong \mathbb{P}^2 \) such that \( \overline{H}^* = \overline{E}_j^* \).
(iii) Let \( q \in \overline{E} = \overline{X} \cap \Delta \). Then
\[
\text{\text{mult}}_q \overline{X} \leq \sum_i e_i \text{mult}_q \overline{E}_i.
\]

In particular, \( \overline{x} \cap \Delta \subset (\text{Sing } \overline{E}) \cup (\bigcup_{i; e_i \geq 2} \overline{E}_i) \).
Proof. (i) Since $X$ is not a cone by Proposition 4(ii), the morphism $\overline{\psi} : \overline{X} \rightarrow \mathbb{P}^2$ is surjective and there exist at most finite many lines in $X$ through $p$. Take a line $l$ through $p$ such that $l \not\subset X$. Then we have that $\overline{\psi}$ is generically of degree $d - m$ since $(\overline{X} \cdot \overline{l}) = d - m$.

(ii) We obtain the assertion since $\overline{\psi} : \overline{X} \rightarrow \mathbb{P}^2$ is surjective by (i).

(iii) Take a line $l$ in $\Delta \cong \mathbb{P}^2$ through $q$ such that $l \not\subset \overline{E}$. Then we obtain the assertions by the same argument as that in the proof of Proposition 5. \qed

Remark. By (iii), one sees that $\overline{x}$ is a finite set if $E$ is a reduced divisor. Then, since $\overline{X}$ is a hypersurface, one has that $\overline{X}$ is normal.

PROPOSITION 8. Assume that $\overline{X}$ is normal. Then:

(i) $H_0(\overline{X}, \mathbb{Z}) \cong H_0(\overline{Y} \cup \overline{E}, \mathbb{Z}) = \mathbb{Z}$;

(ii) $H_1(\overline{X}, \mathbb{Z}) \cong H_1(\overline{Y} \cup \overline{E}, \mathbb{Z}) = 0$;

(iii) $H_2(\overline{X}, \mathbb{Z}) \cong H_2(\overline{Y} \cup \overline{E}, \mathbb{Z}) = (\bigoplus_i \mathbb{Z} \overline{Y}_i) \bigoplus (\bigoplus_j \mathbb{Z} \overline{E}_j)$;

(iv) $H_3(\overline{X}, \mathbb{Z}) \cong H_3(\overline{Y} \cup \overline{E}, \mathbb{Z}) = 0$;

(v) $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$;

(vi) $p_g(\overline{x}) = h^2(\overline{X}, \mathcal{O}_{\overline{X}}) = h^0(\overline{X}, \omega_{\overline{X}})$.

Proof. Since $\overline{X}$ is normal and $(\overline{X}, \overline{Y} \cup \overline{E})$ is a compactification of $\mathbb{C}^2$, by applying Proposition 1 and using the Serre duality theorem for the projective Gorenstein surface $\overline{X}$, we obtain the assertions. \qed

From now on to the end of this subsection, we assume that $\text{mult}_p X = d - 1$. Here we put $k := \#(\{\text{lines in } X \text{ through } p\})$. We denote these lines by $\ell_1, \ldots, \ell_k$ and set $\ell := \ell_1 \cup \cdots \cup \ell_k$. In order to describe detailed properties of $\overline{X}$, we need some lemmas from normal two-dimensional singularities.

LEMMA 2. ([Ar]) Let $(X, x)$ be a rational singularity. Then

$$\pi^* m_{X, x} = \mathcal{O}_M(-Z),$$

where $\pi : M \rightarrow X$ is the minimal resolution of the singularity $x$ and $Z$ is the fundamental cycle of the exceptional set $\pi^{-1}(x)$.

LEMMA 3. Let $X$ be a two-dimensional normal space and $x \in \text{Sing } X$. Let $C$ be a compact irreducible curve in $X$ through $x$. Let $\pi : \overline{X} \rightarrow X$ be the minimal resolution of the singularity $x$. Assume that $C$ is smooth at $x$ and $\pi^* m_{X, x}$ is invertible. Then

$$\langle \overline{C} \cdot \pi^* m_{X, x}, \overline{X} \rangle = -1,$$

where $\overline{C}$ is the proper transform of $C$ by $\pi$. 
Structure of algebraic embeddings $\mathbb{C}^2$ into $\mathbb{C}^3$

**Proof.** Let $i : C \hookrightarrow X$ and $\tilde{i} : \tilde{C} \hookrightarrow \tilde{X}$ be the canonical injections. Since $C$ is smooth at $x$, we see that $\pi|_{\tilde{C}} : \tilde{C} \rightarrow C$ is an isomorphism. By this isomorphism, we obtain $\tilde{x}^*\pi^*m_{x,x} = (\pi|_{\tilde{C}})^*i^*m_{x,x} = (\pi|_{\tilde{C}})^*m_{C,x} \cong m_{C,x} = \mathcal{O}_C(-x)$. This implies the assertion. \qed

**Lemma 4.** Let $(V, v)$ be a normal two-dimensional singularity. Let $\tilde{V}$ be a two-dimensional normal space and $\tau : \tilde{V} \rightarrow V$ a proper surjective holomorphic map. Let $\tilde{\pi} : \tilde{\tilde{V}} \rightarrow \tilde{V}$ be the minimal resolution of $\tilde{V}$. Put $\tilde{v} := \text{Sing} \tilde{V}$ and $E = \bigcup_i E_i := \tilde{\pi}^{-1}(\tilde{v})$. Assume the following conditions:

(i) $C := \tau^{-1}(v)$ is a smooth irreducible curve.

(ii) $\tau$ induces an isomorphism $\tilde{V} - C \cong V - \{v\}$.

(iii) $\emptyset \neq \tilde{v} \subset C$ and $\tilde{v}$ consists of rational double points.

(iv) The proper transform $\tilde{C}$ of $C$ by $\tilde{\pi}$ is a $(-1)$-curve in $\tilde{V}$.

Then $\tilde{v}$ consists of only one $A_*$-type singularity and the weighted dual graph of $\tilde{C} \cup E$ is a linear tree $\bullet \longrightarrow \cdots \longrightarrow \circ$, where $\bullet$ is the $(-1)$-curve $\tilde{C}$ and $\circ$ is a $(-2)$-curve $E_i$. In particular, $v$ is a smooth point of $V$.

![Figure 2](image)

**Proof.** Note that $\tilde{v}$ is a finite set and put $\{\tilde{v}_1, \ldots, \tilde{v}_n\} := \tilde{v}$. Let $Z_i$ be the fundamental cycle of $\tilde{\pi}^{-1}(\tilde{v}_i)$ for each $i$. Since $\tilde{v}$, which consists of rational double points, lies on a smooth curve $C$, by Lemmas 2 and 3, we obtain $\tilde{C} \cdot Z_i = 1$ for each $i$. Hence we have that the curve $\tilde{C} \cup E$ is of normal-crossing type. By blowing-down $\tilde{C}$ and some irreducible components of $E$ successively, we obtain the minimal resolution $\pi : \tilde{V} \rightarrow V$ of $V$. Here we denote by $\phi : \tilde{\tilde{V}} \rightarrow V$ the blowing-down of $\tilde{\tilde{V}}$ to $\tilde{V}$ and thus we obtain the commutative diagram as in Figure 2. Note that $\phi(\tilde{C} \cup E) = \pi^{-1}(v)$ and the intersection matrix of $\pi^{-1}(v)$ is negative definite.

Assume that $\# \tilde{v} \geq 2$. Then the weighted dual graph of $\tilde{C} \cup E$ contains $\circ \longrightarrow \circ$ as a subgraph. Thus $\phi(\tilde{C} \cup E) = \pi^{-1}(v)$ contains a curve whose self-intersection number is positive, which is a contradiction. Hence we have that $\tilde{v}$ consists of only one point.

Assume that the weighted dual graph of $\tilde{C} \cup E$ is not a linear tree. Then, by the list of fundamental cycles of rational double points in the Appendix, the weighted dual graph of $\tilde{C} \cup E$ contains
as a subgraph and thus, similarly, we obtain a contradiction. Hence we have that $\overline{\psi}$ is an $A_*$-type singularity and that the weighted dual graph of $\overline{C} \cup E$ is a linear tree as required.

PROPOSITION 9. One has the following.

(i) $k > 0$ and $\overline{\psi} : \overline{X} \dashv \overline{\ell} \cong \mathbb{P}^2 - \overline{\ell}^*$. In particular, one has that $\overline{x} \subset \overline{\ell}$, $x \subset \ell$ and $\overline{X}$ is normal since $X$ is normal. Let $\overline{\pi} : \overline{X} \longrightarrow \overline{X}$ be the minimal resolution of $\overline{X}$. Then one obtains the commutative diagram as in Figure 3(1).

(ii) $b_2(\overline{X}) = b_2(\mathbb{P}^2) + k$.

(iii) If $\overline{x} \neq \emptyset$ and $q \in \overline{x}$, then there exists a unique line $l$ in $X$ through $p$ such that $q \in \overline{l}$.

(iv) $p_q(\overline{x}) = 0$, that is, $\overline{x}$ consists of at most rational double points.

(v) The resolution $\sigma \circ \overline{\pi} : \overline{\pi} \longrightarrow X$ of $X$ is the minimal resolution $\pi : M \longrightarrow X$ of $X$. Thus one obtains the commutative diagram as in Figure 3(2).

(vi) If $l$ is a line in $X$ through $p$, then $\overline{l}$ is a $(-1)$-curve in $M$.

(vii) If $l$ is a line in $X$ through $p$, then $\overline{x} \cap \overline{l}$ consists of at most one $A_*$-type singularity and, in particular, $x \cap l$ consists of at most two points. Moreover, if $\overline{x} \cap \overline{l} \neq \emptyset$, then the weighted dual graph of $\overline{l} \cup \overline{\pi}^{-1}(\overline{x} \cap \overline{l})$ is a linear tree $\bullet \longrightarrow \cdots \longrightarrow \bullet$, where $\bullet$ is the $(-1)$-curve $\overline{l}$ and $\circ$ is a $(-2)$-curve which is an irreducible component of $\overline{\pi}^{-1}(\overline{x} \cap \overline{l})$.

(viii) $\overline{x}$ consists of $A_*$-type singularities and $\#\overline{x} \leq k$.

\begin{figure}
\centering
\begin{tikzpicture}
\node (X) at (0,0) {$\overline{X}$};
\node (X1) at (0,-2) {$X$};
\node (X2) at (2,-2) {$\mathbb{P}^2$};
\node (M) at (4,0) {$\overline{X}$};
\node (M1) at (4,-2) {$X$};
\node (M2) at (6,-2) {$\mathbb{P}^2$};
\draw[->] (X) to node [above] {$\overline{\pi}$} (X1);
\draw[->] (X1) to node [below] {$\overline{\psi}$} (X2);
\draw[->] (X) to node [right] {$\pi$} (M);
\draw[->] (M) to node [below] {$\sigma$} (M1);
\draw[->] (M1) to node [right] {$\overline{\psi}$} (M2);
\draw[->] (X1) to node [left] {$\sigma \circ \overline{\pi}$} (X).
\end{tikzpicture}
\caption{}
\end{figure}

Proof. (i) By Proposition 7(i), we obtain an isomorphism $\overline{\psi} : \overline{X} \dashv \overline{\ell} \cong \mathbb{P}^2 - \overline{\ell}^*$. Assume that $k = 0$, that is, $\ell = \emptyset$. Then we obtain the isomorphism $\overline{\psi} : \overline{X} \cong \mathbb{P}^2$ and thus $b_2(\overline{X}) = b_2(\mathbb{P}^2) = 1$. On the other hand, by Proposition 8(iii), we obtain $b_2(\overline{X}) = b_2(\overline{Y}) + b_2(\overline{E}) \geq 2$, which is a contradiction. Hence we obtain $k > 0$. 


(ii) Let us consider the following exact sequence:
\[ \rightarrow H^i(\overline{X}, \overline{c}; \mathbb{Z}) \rightarrow H^i(\overline{X}; \mathbb{Z}) \rightarrow H^i(\overline{c}; \mathbb{Z}) \rightarrow H^{i+1}(\overline{X}, \overline{c}; \mathbb{Z}) \rightarrow . \]

Since \( \overline{\psi} : \overline{X} - \overline{c} \cong \mathbb{P}^2 - \overline{c}^* \) and \( \overline{c}^* \) consists of finite many points, we have \( H^i(\overline{X}, \overline{c}; \mathbb{Z}) \cong H^i(\mathbb{P}^2, \overline{c}^*; \mathbb{Z}) \cong H^i(\mathbb{P}^2; \mathbb{Z}) \) for \( i \geq 2 \). Since \( \overline{c} \) is a disjoint union of \( k \) smooth rational curves, we have that \( H^1(\overline{c}; \mathbb{Z}) = H^1(\mathbb{P}^1; \mathbb{Z})^\oplus k = 0 \) and \( H^2(\overline{c}; \mathbb{Z}) = H^2(\mathbb{P}^1; \mathbb{Z})^\oplus k = \mathbb{Z}^\oplus k \). Since \( H^1(\overline{c}; \mathbb{Z}) = H^3(\mathbb{P}^2; \mathbb{Z}) = 0 \), we obtain \( H^2(\overline{X}, \overline{c}; \mathbb{Z}) \cong H^2(\mathbb{P}^2; \mathbb{Z}) \oplus H^2(\overline{c}; \mathbb{Z}) \). This shows (ii).

(iii) The uniqueness of \( l \) is obvious. If there do not exist such lines, by the isomorphism \( \overline{\psi} : \overline{X} - \overline{c} \cong \mathbb{P}^2 - \overline{c}^* \), \( q \) is a smooth point of \( \overline{X} \). This is a contradiction.

(iv) Since \( \overline{X} \) is normal, we have \( p_8(\overline{x}) = h^0(\overline{X}, \mathcal{O}(K_{\overline{X}})) \) by Proposition 8(vi). Note that \( K_{\overline{X}} \sim (d - 3)\overline{\psi}^* L - \overline{\Gamma} \), where \( L \) is a line in \( \mathbb{P}^2 \).

Let us consider the exact sequence
\[ 0 \rightarrow \mathcal{O}(K_{\overline{X}}) \rightarrow \mathcal{O}((d - 3)\overline{\psi}^* L) \rightarrow \mathcal{O}_{\overline{\Gamma}}((d - 3)\overline{\psi}^* L) \rightarrow 0 \]
of sheaves on \( \overline{X} \). Since \( H^1(\overline{X}, \mathcal{O}(K_{\overline{X}})) = H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0 \) by Proposition 8(v), we obtain the following exact sequence of \( \mathbb{C} \)-vector spaces:
\[
0 \rightarrow H^0(\overline{X}, \mathcal{O}(K_{\overline{X}})) \rightarrow H^0(\overline{X}, \mathcal{O}((d - 3)\overline{\psi}^* L)) \rightarrow H^0(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}((d - 3)\overline{\psi}^* L)) \rightarrow 0.
\]
If we have \( h^0(\overline{X}, \mathcal{O}((d - 3)\overline{\psi}^* L)) = h^0(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}((d - 3)\overline{\psi}^* L)) \), then we obtain \( p_8(\overline{x}) = h^0(\overline{X}, \mathcal{O}(K_{\overline{X}})) = 0 \) by the exact sequence.

Now by the isomorphism \( \overline{\psi} : \overline{X} - \overline{c} \cong \mathbb{P}^2 - \overline{c}^* \), we have \( h^0(\overline{X}, \mathcal{O}((d - 3)\overline{\psi}^* L)) = h^0(\mathbb{P}^2, \mathcal{O}(d - 3)) = (d - 1)(d - 2)/2 \). On the other hand, by the isomorphism \( \overline{\psi} |_{\overline{\Gamma}} : \overline{\Gamma} \cong \overline{\Gamma}^* \), we obtain \( h^0(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}((d - 3)\overline{\psi}^* L)) = h^0(\overline{\Gamma}^*, \mathcal{O}_{\overline{\Gamma}^*}(d - 3)) = (d - 1)(d - 2)/2 \).

Thus we obtain (iv).

(v) By (iv) and [Ar] (cf. [HW]), we obtain \( K_{\overline{X}} = \overline{\pi}^* K_{\overline{X}} \). Now we show that there exist no \((-1)\)-curves in \( \text{Exc}(\sigma \circ \overline{\pi}) \). Note that \( \text{Exc}(\sigma \circ \overline{\pi}) \) is a union of \( \text{Exc} \overline{\pi} \) and the proper transform of \( \overline{E} \) by \( \overline{\pi} \) and that \( \text{Exc} \overline{\pi} \) consists of \((-2)\)-curves. Assume that the proper transform \( \overline{E}_i \) of some \( E_i \subset \overline{E} \) by \( \overline{\pi} \) is a \((-1)\)-curve in \( \overline{X} \). Then, by projection formula, we obtain \( -1 = (K_{\overline{X}} \cdot \overline{E}_i)_{\overline{X}} = (K_{\overline{X}} \cdot \overline{E}_i)|_{\overline{X}} = (d - 4)(\sigma^* H \cdot \overline{E}_i)|_{\overline{\Gamma}^3} + (d - 3)(-\Delta \cdot \overline{E}_i)|_{\overline{\Gamma}^3} \geq 0 \). This is a contradiction.

(vi) By (iv) and (v), we obtain \( K_M = \overline{\pi}^* K_{\overline{X}} \). By (i), we can take a line \( l \) in \( X \) through \( p \). Then, by projection formula, we obtain \( (K_M \cdot \overline{l})_M = (K_{\overline{X}} \cdot \overline{l})_{\overline{X}} = -1 \). Since \( \overline{l} \cong \overline{\Gamma} \cong \mathbb{P}^1 \), we have that \( \overline{l} \) is \((-1)\)-curve in \( M \).
(vii) We may assume that $\bar{x} \cap \bar{t} \neq \emptyset$. Since $\bar{x} \cap \bar{t}$ consists of rational double points by (iv) and $\bar{t}$ is a $(−1)$-curve in $M$ by (vi), we can apply Lemma 4 to a normal two-dimensional singularity $(\mathbb{P}^2, \bar{t}^*)$, $\bar{t}$ and $M \xrightarrow{\bar{\pi}} \bar{X} \xrightarrow{\bar{\psi}} \mathbb{P}^2$. Hence we have that $\bar{x} \cap \bar{t}$ consists of only one $A_1$-type singularity and we obtain the weighted dual graph of $\bar{t} \cup \bar{\pi}^{-1}(\bar{x} \cap \bar{t})$ as required.

(viii) By (iii) and (vii), we have the assertion. □

**Proposition 10.** If there exists $Y_i \subset Y$ such that $\bar{Y}_i^*$ is a line in $\mathbb{P}^2$, that is, $Y_i$ is not a line through $p$, then the following holds:

(i) $Y = Y_i \cup Y_j$, where $Y_j$ is a line through $p$;

(ii) $\bar{Y}^* \not\subset \bar{E}^*$ and $\bar{Y}^* \cup \bar{E}^*$ does not have any cycles;

(iii) if $\bar{E}^*$ contains a line in $\mathbb{P}^2$, then $\bar{E}^*$ consists of lines in $\mathbb{P}^2$.

**Proof.** Note that $\bar{H}^* = \bar{Y}_i^*$.

(i) Assume that $Y = Y_i$. Then we see that $Y_i$ is a line by Proposition 6. Since $p \in x \subset Y = Y_i$, this contradicts the assumption. Hence we have that $Y$ contains at least two irreducible components. By the isomorphism $\bar{\psi} : \bar{X} - \bar{\xi} \cong \mathbb{P}^2 - \bar{\xi}^*$ and $\bar{H}^* = \bar{Y}_i^*$, we have that $\bar{Y}_j^*$ is a point, that is, $Y_j$ is a line through $p$ for each $j \neq i$.

If $Y$ contains two lines $Y_{j_1}$ and $Y_{j_2}$ through $p$, then we have that $Y_i$ is a line through $p$ by Remark 1 of Proposition 3. This is a contradiction. Hence we obtain $Y = Y_i \cup Y_j$, where $Y_j$ is a line through $p$.

(ii) By the isomorphism $\bar{\psi} : \bar{X} - \bar{\xi} \cong \mathbb{P}^2 - \bar{\xi}^*$ and $\bar{H}^* = \bar{Y}_i^*$, we obtain $\bar{Y}^* \not\subset \bar{E}^*$. Since $\bar{X}$ is normal and $\bar{X} - \bar{Y} - \bar{E} \cong X - Y \cong \mathbb{C}^2$, by Proposition 8(ii), we have that $H_1(\bar{Y} \cup \bar{E}; \mathbb{Z}) = 0$ and, in particular, $H_1(\bar{E}; \mathbb{Z}) = 0$. From the isomorphism $\bar{\psi}|_{\Delta} : \Delta \cong \mathbb{P}^2$, it follows that $H_1(\bar{E}^*; \mathbb{Z}) = 0$ and hence that $\bar{E}^*$ does not have any cycles. If $\bar{Y}^* \cup \bar{E}^*$ has a cycle, that is, $\bar{Y}^* \cup \bar{E}^*$ contains at least two points, then there exists a point $q \in \bar{Y}^* \cap \bar{E}^*$ such that $q$ is not the point $\bar{Y}_j^*$. Then, since $X \cap H = Y = Y_i \cup Y_j$ and $\bar{H}^* = \bar{Y}^* = \bar{Y}_i^*$, we obtain $q \in \mathbb{P}^2 - \bar{\xi}^*$. Hence $(\bar{\psi}|_{\bar{Y}})^{-1}(q)$ contain at least two points. On the other hand, since $\bar{\psi} : \bar{X} - \bar{\xi} \cong \mathbb{P}^2 - \bar{\xi}^* \ni q$, $(\bar{\psi}|_{\bar{X}})^{-1}(q)$ consists of only one point. This is a contradiction.

(iii) If $\bar{E}^*$ contains a line in $\mathbb{P}^2$, then by (ii) we see that $\bar{Y}^* \cup \bar{E}^*$ is a plane curve without cycles and contains at least two lines. Hence we have that $\bar{Y}^* \cup \bar{E}^*$ consists of lines and, in particular, $\bar{E}^*$ consists of lines. □

Now we continue to give some notation as follows. Let $\hat{V}$ be the proper transform of $V$ by $\pi$ and $E_i$ the proper transform of $\bar{E}_i$ by $\bar{\pi}$. Put $\phi := \bar{\psi} \circ \pi$ and $\Phi := \pi \circ \phi^{-1}$. Thus we obtain the commutative diagram as in Figure 4. Let $\delta'$ be the linear system associated to $\pi : M \rightarrow X$. Put $\delta := \phi_* \delta'$. Then $\delta$ is the linear system associated
to $\Phi : \mathbb{P}^2 \to X$. Let $\mathbb{M}_\delta$ and $\mathbb{M}_{\delta'}$ be the $\mathbb{C}$-vector spaces associated to $\delta$ and $\delta'$ respectively. Note that $\dim \delta = \dim \delta' = 3$. Let $H_0, H_1, H_2$ and $H_3$ be any hyperplanes in $\mathbb{P}^3$ which are independent over $\mathbb{C}$ as elements of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and such that $H_0, H_1, H_2$ passes through $p$ and $H_3$ does not pass through $p$. We can put $H_3 := H_\Gamma$. Set $L_i := \overline{H_i}$ for $0 \leq i \leq 2$.

![Diagram](image)

**Figure 4.**

**Proposition 11.** One has the following.

(i) $\delta' = |\Gamma|$ and $\mathbb{M}_{\delta'}$ is spanned by the holomorphic sections corresponding to the divisors $\pi^*(H_i|_X)$ for $0 \leq i \leq 3$.

(ii) $\delta \subset |\Gamma^*| = |\mathcal{O}_{\mathbb{P}^2}(d)|$ and $\mathbb{M}_\delta$ is spanned by the holomorphic sections corresponding to the divisors $L_i + \overline{\psi}_*(\Delta|_X)$ for $0 \leq i \leq 2$ and $\Gamma^*$. In particular, the birational map $\Phi$ is given by

$$
\Phi : \begin{cases}
  z_0 = w_0 G(w_0, w_1, w_2) \\
  z_1 = w_1 G(w_0, w_1, w_2) \\
  z_2 = w_2 G(w_0, w_1, w_2) \\
  z_3 = F(w_0, w_1, w_2)
\end{cases}
$$

and the image $X$ of $\Phi$ is given by $F(z_0, z_1, z_2) - z_3 G(z_0, z_1, z_2) = 0$, where $F$ (respectively $G$) is a homogeneous polynomial of $w_0, w_1, w_2$ of degree $d$ (respectively $d - 1$) which defines the divisor $\Gamma^*$ (respectively $\overline{\psi}_* (\Delta|_X)$).

**Proof.** (i) Let $\iota : X \hookrightarrow \mathbb{P}^3$ be the natural inclusion. Since $\dim \delta' = 3$ and $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \cong H^0(X, \mathcal{O}_X(1)) \cong H^0(M, \mathcal{O}(\Gamma))$, we obtain $\delta' = |\Gamma|$ and $\mathbb{M}_{\delta'} = H^0(M, \mathcal{O}(\Gamma))$. By the isomorphisms $\iota^*$ and $\pi^*$, the vector space $\mathbb{M}_{\delta'}$ is spanned by the holomorphic sections corresponding to $\pi^*(H_i|_X)$ for $0 \leq i \leq 3$.

(ii) Note that $\Gamma^* \cong \Gamma$ is a smooth plane curve of degree $d$. Since $\Gamma^* = \phi_* \overline{\Gamma} \in \phi_* \delta' = \delta$, we see that $\delta \subset |\Gamma^*| = |\mathcal{O}_{\mathbb{P}^2}(d)|$. By $\delta = \phi_* \delta'$, we have that $\mathbb{M}_\delta$ is spanned.
by the holomorphic sections corresponding to \( \phi_*(\pi^*(H_i|_X)) \) for \( 0 \leq i \leq 3 \). Note that

\[
\phi_*(\pi^*(H_i|_X)) = \overline{\psi_*}\overline{\pi_*}\overline{\alpha^*(H_i|_X)} = \overline{\psi_*}(H_i|_X + \Delta|_X) = L_i + \overline{\psi_*}(\Delta|_X)
\]

for \( 0 \leq i \leq 2 \) and \( \phi_*(\pi^*(H_3|_X)) = \phi_*(\Gamma) = \Gamma^* \).

\[\Box\]

3. Proof of Theorem 1

First we consider the case where \( X \) is non-normal. Then we can write down the defining equation of \( X \) directly by [BW, p. 252], and find a hyperplane \( H \) in \( \mathbb{P}^3 \) such that \( H \) contains \( x = \text{Sing} \cdot X \), which is a line, and \( X - H \) is biregular to \( \mathbb{C}^2 \). Thus we obtain (VII), (VIII) and (IX).

Next we consider the case where \( X \) is normal. We use the same notation as that in Section 2. By Remark 2 of Proposition 3 and Proposition 4(i), we have that \( x \neq \emptyset \), \( x \) consists of rational double points and \( K_M \sim -\Gamma \). See the Appendix for rational double points and their fundamental cycles. Here we prove some lemmas needed later.

**Lemma 5.** We have the following:

\[-K_M \cdot \Gamma = \Gamma \cdot \Gamma = \Gamma \cdot l = 1, \quad \Gamma^2 = -1 \quad (\text{if } l \text{ : line in } X);\]

\[-K_M \cdot \Gamma = \Gamma \cdot \Gamma = \Gamma \cdot C = 2, \quad \Gamma^2 = 0 \quad (\text{if } C \text{ : conic in } X);\]

\[\Gamma \cdot Z^{(i)} = 1 \quad (\text{if } l \text{ : line in } X \text{ through } x_i);\]

\[\Gamma \cdot Z^{(i)} = 1 \quad (\text{if } C \text{ : conic in } X \text{ through } x_i);\]

\[K_M^2 = \Gamma^2 = \Gamma^2 = 3, \quad E_j^2 = -2 \quad (\forall j),\]

where \( \Gamma \) (respectively \( C \)) is the proper transform of \( l \) (respectively \( C \)) by the minimal resolution \( \pi : M \rightarrow X \) and \( Z^{(i)} \) is the fundamental cycle of \( \pi^{-1}(x_i) \) for each \( i \).

**Proof.** Since \( \Gamma \) is a general hyperplane section of \( X \) such that \( \Gamma \cap x = \emptyset \), we have that \( \Gamma \cdot \Gamma = \Gamma \cdot l = 1 \), \( \Gamma \cdot C = \Gamma \cdot C = 2 \) and \( \Gamma^2 = \Gamma^2 = 3 \). Since \( \Gamma \cong l \cong \mathbb{P}^1 \) and \( C \cong C \cong \mathbb{P}^1 \), by adjunction formula, we obtain \( \Gamma^2 = -1 \) and \( \Gamma^2 = 0 \). Since \( x \) consists of rational double points, every \( E_j \) is a \((-2)\)-curve. By Lemmas 2 and 3, we obtain the assertions for fundamental cycles. \[\Box\]

**Remark.** Applying the Noether formula for rational surfaces and Proposition 3(vii), we obtain \( b_2(Y) + b_2(E) = b_2(M) = 10 - K_M^2 = 7 \).
LEMMA 6. Assume that \(\#(x \cap Y_i) \geq 2\) for a line \(Y_i\). Then \(x \cap Y_i\) consists of exactly two \(A_*\)-type singularities.

Proof. Apply Proposition 9(vii).

PROPOSITION 12. There exist the following four cases for the divisor \(\mathcal{Y}\).

(i) \(\mathcal{Y} = 3Y_1 (Y_1 : \text{line}), x \subset Y_1\). In this case, \(x\) consists of one point or two \(A_*\)-type singularities.

(ii) \(\mathcal{Y} = Y_1 + Y_2 (Y_1 : \text{line}, Y_2 : \text{conic}), x = \{p\}\). In this case, \(Y_1\) and \(Y_2\) meet tangentially to second order at \(p\).

(iii) \(\mathcal{Y} = 2Y_1 + Y_2 (Y_1, Y_2 : \text{line}), x \subset Y_1\). In this case, \(x\) consists of one point or two \(A_*\)-type singularities.

(iv) \(\mathcal{Y} = Y_1 + Y_2 + Y_3 (Y_1, Y_2, Y_3 : \text{line}), x = \{p\}\). In this case, \(Y_1, Y_2\) and \(Y_3\) meet only at \(p\).

Proof. Note that the divisor \(\mathcal{Y} = H|_X\) is a plane cubic curve. By Proposition 6, it cannot be the case that \(\mathcal{Y}\) is an irreducible cubic curve. Hence we obtain the above four cases for the divisor \(\mathcal{Y}\). By Proposition 3(ii), Proposition 5 and Lemma 6, the proof is completed.

From now on, we consider the four cases for the divisor \(\mathcal{Y}\) in Proposition 12. For each case, we consider projections of \(\mathbb{P}^3\) from a point and blowing-up of \(\mathbb{P}^3\) at a point to determine the weighted dual graph of \(A\). Let \(\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^3\) be the blowing-up at a singular point \(p\) of \(X\) and consider the commutative diagrams as in Figure 1. Since \(p\) is a rational double point, by Lemma 2, the \(\mathcal{O}_M\)-module \(\pi^*\mathcal{N}_{X,p}\) is invertible. Hence, by [Ha, Proposition 7.14, pp. 164–165], there exists a morphism \(\overline{\pi} : M \longrightarrow \overline{X}\) such that \(\sigma \circ \overline{\pi} = \pi\). From Proposition 9(v), it also follows the existence of such morphism \(\overline{\pi}\). Thus we get the commutative diagrams as in Figure 5. Note that \(M\) is obtained by blowing-up \(\mathbb{P}^3\) at a point (cf. the appendix).

If we obtain the weighted dual graph of \(A\), then we check the following two conditions for \((M, A)\).

1. The smooth compactification \((M, A)\) can be transformed into a minimal normal compactification by blowing-up and blowing-down in the boundary \(A\) repeatedly.

2. The linear equation \(-K_M \sim \hat{\Gamma} \sim \sum_i k_i \hat{Y}_i + \sum_j m_j E_j\) have a solution \(m_j \in \mathbb{N}\). If \((M, A)\) satisfies (1) and (2), then we can write down the defining equation of \((X, Y)\) by the weighted dual graph of \(A\) and Proposition 11(ii).
3.1. The case $\mathcal{Y} = 3$ line (cf. [Fu2])

Let $\mathcal{Y} = 3Y_1$ ($Y_1 : \text{line}$) be the hyperplane section of $X$. By Proposition 12(i), we have that $x \subset Y_1$ and $x$ consists of either one point or two $A_*$-type singularities.

**Lemma 7.** $x$ consists of one point.

**Proof.** Assume that $\#x = 2$. Then $x = \{p, q\}$ consists of two $A_*$-type singularities. Since $b_2(E) = 7 - b_2(Y) = 6$, we obtain the following three cases: (1) $p = A_3$, $q = A_3$ (2) $p = A_4$, $q = A_2$ (3) $p = A_5$, $q = A_1$. Let $\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the blowing-up at $p$ and consider the commutative diagrams as in Figure 5. Since $p$ is an $A_N$-type singularity of $X$ ($N = 3, 4, 5$), we have that $\overline{X} \cap \Delta$ consists of two lines $\overline{E_1}$ and $\overline{E_2}$ in $\Delta \cong \mathbb{P}^2$ and that $\{\overline{p}\} := \overline{E_1} \cap \overline{E_2}$ is an $A_*$-type singularity of $\overline{X}$. Put $\overline{q} := \sigma^{-1}(q)$. By Propositions 8(iii) and 9(ii), there exist two lines in $X$ through $p$. Let $Y_1$ and $\overline{l}$ be the two lines in $X$ through $p$. By Proposition 7(ii), we may assume that $H^* = \overline{E_1}^*$ and hence that $\overline{Y}_1 \cap \Delta$ lies on $\overline{E_1}$ and $\overline{l} \cap \Delta$ lies on $\overline{E_2} - \{\overline{p}\}$. By Proposition 9(iii), $\overline{Y}_1$ passes through $\overline{p}$. Thus we obtain Figure 6. Since two $A_*$-type singularities $\overline{p}$ and $\overline{q}$ lie on $\overline{Y}_1$, by Proposition 9(vii), this is a contradiction. 

![Figure 5](image1)

![Figure 6](image2)

Put $\{p\} := x$. Let $\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the blowing-up at $p$ and consider the commutative diagrams as in Figure 5. Let $Z$ be the fundamental cycle of $\pi^{-1}(p)$. 


Note that $\hat{Y}_1 \cdot Z = 1$ by Lemma 5.

**Lemma 8.** There exist the following two cases.

(i) $x = \{p\} \ (p = A_6)$.
(ii) $x = \{p\} \ (p = E_6)$.

**Proof.** Since $b_2(E) = 7 - b_2(Y) = 6$, we see that $p$ is an $A_6$-type singularity, a $D_6$-type singularity or an $E_6$-type singularity. Assume that $p$ is a $D_6$-type singularity of $X$. Note that $E_1 := \overline{X} \cap \Delta$ is a line in $\Delta \cong \mathbb{P}^2$ and that there exist a $D_4$-type singularity $\overline{p_1}$ and an $A_1$-type singularity $\overline{p_2}$ of $\overline{X}$ on $\overline{E_1}$. Thus we have $\overline{x} = \{\overline{p_1}, \overline{p_2}\}$. On the other hand, by Proposition 9(viii), we have that $\overline{x}$ consists of $A_4$-type singularities. This is a contradiction. \hfill $\square$

3.1.1. The case $p = A_6$. Since $p$ is an $A_6$-type singularity of $X$, we see that $\overline{X} \cap \Delta$ consists of two lines $\overline{E_1}$ and $\overline{E_2}$ in $\Delta \cong \mathbb{P}^2$ and that $\{\overline{p}\} := \overline{E_1} \cap \overline{E_2}$ is an $A_4$-type singularity of $\overline{X}$. Then we have $\overline{x} = \{\overline{p}\}$. Put $E_3 \cup \cdots \cup E_6 := \text{Exc} \overline{x}$ as in Figure 7(a).

**Lemma 9.** There exist two lines in $X$ through $p$.

Let $Y_1$ and $l$ be the two lines in $X$ through $p$. By Proposition 7(ii), we may assume that $H^* = E_1^*$ and hence that $Y_1 \cap \Delta$ lies on $E_1$ and $l \cap \Delta$ lies on $E_2 - \{\overline{p}\}$. By Proposition 9(iii), $Y_1$ passes through $\overline{p}$. Thus we obtain Figure 7(b).

**Proposition 13.** The case $p = A_6$ cannot occur.

**Proof.** Since $Y_1$ passes through $\overline{p}$, by Proposition 9(vii), we may assume that $\hat{Y}_1$ intersects $E_3$ transversally at one point. Hence we have the weighted dual graph of $A$ as in Figure 7(a). We transform $(M, A)$ into a minimal normal compactification.
by blowing-down in the boundary $A$ repeatedly. However the weighted dual graph of $A'$ cannot be found in Proposition 2, which is a contradiction. □

3.1.2. The case $p = E_6$. Since $p$ is an $E_6$-type singularity of $X$, we see that $\overline{E_1} := \overline{X} \cap \Delta$ is a line in $\Delta \cong \mathbb{P}^2$ and that there exists an $A_5$-type singularity $\overline{p}$ of $\overline{X}$ on $\overline{E_1}$. Then we have $\overline{x} = [\overline{p}]$. Put $E_2 \cup \cdots \cup E_6 := \text{Exc } \overline{x}$ as in Figure 8(a). By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 10.** There exists a unique line in $X$ through $p$.

The boundary $Y_1$ is the line in $X$ through $p$. By Proposition 9(iii), $\overline{Y_1}$ passes through $\overline{p}$. Note that $\overline{H}^* = \overline{E_1}^*$ by Proposition 7(ii). Thus we obtain Figure 8(b).

**Proposition 14.** The case $p = E_6$ can occur. One has the weighted dual graph of $A$ as in Figure 8(a) and obtains $\mathbb{P}^2$ by blowing-down $\widehat{Y_1}, E_2, E_3, E_4, E_5, E_6$ successively.

![Figure 8](image)

**Proof.** Since $\overline{p} \in \overline{Y_1}$ and $\overline{Y_1} \cdot Z = 1$, we may assume that $\widehat{Y_1}$ intersects $E_2$ transversally at one point. Hence we have the weighted dual graph of $A$ as in Figure 8(a) and we obtain $\widehat{\Gamma} \sim 3\widehat{Y_1} + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6$. □

Let $\phi_6$ (respectively $\phi_5, \phi_4, \phi_3, \phi_2, \phi_1$) be the blowing-down of $\widehat{Y_1}$ (respectively $E_2, E_3, E_4, E_5, E_6$). Note that $\phi = \phi_1 \circ \cdots \circ \phi_6$ and $\text{Exc } \phi = \widehat{Y_1} \cup E_2 \cup \cdots \cup E_6$.

Put $P := \phi(\widehat{Y_1})$. Note that $\overline{\Gamma}^*$ is a smooth elliptic curve and that $\overline{\Gamma}^*$ and $\overline{E_1}^*$ meet tangentially to third order at $P$. Let $S$ be a line through $P$ such that $S$ and $\overline{\Gamma}^*$ meet transversally at exactly three points. Let $Q \in S \cap \overline{\Gamma}^*$ be a point ($\neq P$) and $T$ a tangent line of $\overline{\Gamma}^*$ at $Q$. We may assume that $P = (0 : 1 : 0)$, $Q = (0 : 0 : 1)$, $\overline{E_1}^* = L_2 = \{w_2 = 0\}$, $S = L_0 = \{w_0 = 0\}$ and $T = L_1 = \{w_1 = 0\}$.

Now we obtain the defining equations of $\phi$ and $(X, Y)$ as follows.
PROPOSITION 15. The birational map $\Phi$ is given by

$$
\Phi : \begin{cases}
    z_0 = w_0 w_2^2 \\
    z_1 = w_1 w_2^2 \\
    z_2 = w_2^3 \\
    z_3 = a_1 w_0^3 + a_3 w_0^2 w_2 + a_5 w_0 w_1 w_2 + a_8 w_1^2 w_2,
\end{cases}
$$

where $a_i \in \mathbb{C}$, $a_1 \neq 0$ and $a_8 \neq 0$. The compactification $(X, Y)$ is given by:

$$
X : z_2^2 z_3 = a_1 z_0^3 + a_3 z_0^2 z_2 + a_5 z_0 z_1 z_2 + a_8 z_1^2 z_2;
$$

$$
Y : z_2 = z_0^3 = 0.
$$

Considering a projective transformation of $\mathbb{P}^3$, we obtain (I).

Proof. Since $\overline{\nu}_*(\Delta|_{\overline{X}}) = \overline{\nu}_*(\overline{E}_1) = 2\overline{E}_1^* = 2L_2$, by Proposition 11(ii), we obtain $M_8 = \mathbb{C} < w_0 w_2^2, w_1 w_2^2, w_2^3, F >$, where $F$ is the defining equation of $\overline{\Gamma}^*$ as follows:

$$
0 = F := a_1 w_0^3 + a_2 w_0^2 w_1 + a_3 w_0^2 w_2 + a_4 w_0 w_1^2 + a_5 w_0 w_1 w_2 + a_6 w_0 w_2^2 \\
+ a_7 w_1^3 + a_8 w_1^2 w_2 + a_9 w_1 w_2^2 + a_{10} w_2^3.
$$

Since $\overline{\Gamma}^*$ passes through $P$ and $Q$, we have $a_7 = a_{10} = 0$. Since $\overline{\Gamma}^*$ and $L_2$ meet tangentially to third order at $P$, we have $a_1 \neq 0$ and $a_2 = a_4 = 0$. Since $\overline{\Gamma}^*$ is smooth at $P$, we have $a_8 \neq 0$. Since $\overline{\Gamma}^*$ and $L_1$ meet tangentially at $Q$, we have $a_6 = 0$. Since $\overline{\Gamma}^*$ is smooth at $Q$, we have $a_9 \neq 0$. Hence we obtain

$$
M_8 = \mathbb{C}(w_0 w_2^2, w_1 w_2^2, w_2^3, F)
$$

$$
= \mathbb{C}(w_0 w_2^2, w_1 w_2^2, w_2^3, a_1 w_0^3 + a_3 w_0^2 w_2 + a_5 w_0 w_1 w_2 + a_8 w_1^2 w_2)
$$

with $a_1 \neq 0$ and $a_8 \neq 0$. Thus we obtain the birational map $\Phi$ as desired.

Since $\overline{H}^* = \overline{E}_1^* = L_2$, by the defining equation of $\Phi$, we obtain $H = \{z_2 = 0\}$ and the compactification $(X, Y)$ as desired. We deform the defining equation of $X$ as follows:

$$
z_2^2 z_3 = a_1 z_0^3 + a_3 z_0^2 z_2 + z_2 a_8 \left( z_1 + \frac{a_5}{2a_8} z_0 \right)^2 - \frac{a_5^2}{4a_8} z_0^3 z_2.
$$

Considering the projective transformation of $\mathbb{P}^3$ such that

$$
z_0' := \sqrt{a_1} z_0, \quad z_1' := \sqrt{a_8} \left( z_1 + \frac{a_5}{2a_8} z_0 \right), \quad z_2' := -z_3, \quad z_3' := z_2
$$
and putting
\[ \lambda := \frac{1}{(\sqrt[3]{a_1})^2} \left( a_3 - \frac{a_5^2}{4a_8} \right), \]
we obtain (I).

\[ \square \]

3.2. The case \( \mathcal{Y} = \text{line} + \text{conic} \)

Let \( \mathcal{Y} = Y_1 + Y_2 (Y_1 : \text{line}, Y_2 : \text{conic}) \) be the hyperplane section of \( X \). By Proposition 12(ii), we obtain \( x = Y_1 \cap Y_2 = \{p\} \). Let \( \sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) be the blowing-up at \( p \) and consider the commutative diagrams as in Figure 5. Since \( Y_1 \) and \( Y_2 \) are smooth curves and meet tangentially to second order at \( p \), \( \overline{Y}_1 \) intersects \( \overline{Y}_2 \) transversally at one point \( \overline{Y}_1 \cap \Delta = \overline{Y}_2 \cap \Delta \). Since \( b_2(E) = 7 - b_2(Y) = 5 \), we have that \( p \) is an \( A_5 \)-type singularity or a \( D_5 \)-type singularity. Let \( Z \) be the fundamental cycle of \( \pi^{-1}(p) \). Note that \( \overline{Y}_2^2 = 0 \) and \( \overline{Y}_1 \cdot Z = \overline{Y}_2 \cdot Z = 1 \) by Lemma 5.

3.2.1. The case \( p = A_5 \). Since \( p \) is an \( A_5 \)-type singularity of \( X \), we see that \( \overline{X} \cap \Delta \) consists of two lines \( \overline{E}_1 \) and \( \overline{E}_2 \) in \( \Delta \cong \mathbb{P}^2 \) and that \( \{\overline{p}\} := \overline{E}_1 \cap \overline{E}_2 \) is an \( A_3 \)-type singularity of \( \overline{X} \). Then we have \( \overline{X} = \{\overline{p}\} \). Put \( E_3 \cup E_4 \cup E_5 := \text{Exc} \pi \) as in Figure 9(a). By Proposition 8(iii) and 9(ii), we obtain the following.

**Lemma 11.** There exist three lines in \( X \) through \( p \).

Let \( Y_1, l_1 \) and \( l_2 \) be the three lines in \( X \) through \( p \). Note that one of \( \overline{Y}_1, \overline{l}_1 \) and \( \overline{l}_2 \) passes through \( \overline{p} \) by Proposition 9(iii). Since \( \overline{Y}^* = \overline{Y}_2^* \) is a line in \( \mathbb{P}^2 \), by Proposition 10(ii), \( \overline{Y}_1 \) passes through \( \overline{p} \). Thus we obtain Figure 9(b).

**Proposition 16.** The case \( p = A_5 \) cannot occur.

![Figure 9](image-url)
Proof. Since $\overline{Y_1}$ passes through $\overline{p}$, by Proposition 9(vii), $\hat{Y}_1$ intersects either $E_3$ or $E_4$ transversally. We may assume that $\hat{Y}_1$ meets $E_3$ by symmetry. Since $\overline{Y_2}$ passes through $\overline{p}$ and $\overline{Y_2} \cdot Z = 1$, we have that $\hat{Y}_2$ intersects one of $E_3$, $E_4$ and $E_5$ transversally. Now we transform the smooth compactification $(M, A)$ into a minimal normal compactification $(M', A')$ by blowing-down in the boundary $A$ repeatedly. However the weighted dual graph of $A'$ cannot be found in Proposition 2 in all the cases, which is a contradiction. □

3.2.2. The case $p = D_5$. Since $p$ is a $D_5$-type singularity of $X$, we see that $\overline{E_1} := \overline{X} \cap \Delta$ is a line in $\Delta \cong \mathbb{P}^2$ and that there exist an $A_1$-type singularity $\overline{p_1}$ and an $A_3$-type singularity $\overline{p_2}$ of $\overline{X}$ on $E_1$. Then we have $\overline{x} = (\overline{p_1}, \overline{p_2})$. Put $E_2 := \overline{\pi}^{-1}(\overline{p_1})$ and $E_3 \cup E_4 \cup E_5 := \overline{\pi}^{-1}(\overline{p_2})$ as in Figure 10(a). By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 12.** There exist two lines in $X$ through $p$.

Let $l_1$ and $l_2$ be the two lines in $X$ through $p$. Let $H_l$ be the hyperplane spanned by $l_1$ and $l_2$. Note that one of $\overline{l_1}$ and $\overline{l_2}$ passes through $\overline{p_1}$ and the other one passes through $\overline{p_2}$ by Proposition 9(iii). We may assume that $l_i$ is the line such that $\overline{l_i}$ passes through $\overline{p_1}$ for $i = 1, 2$. Note that $\overline{H_l}^* = \overline{E_1}^*$. Since $Y_1$ is a line in $X$ through $p$, we have that $Y_1 = l_1$ or $l_2$. Thus we obtain Figure 10(b).

**Lemma 13.** The pairs $(X, l_1 \cup l_2)$, $(\overline{X}, \overline{l_1} \cup \overline{l_2} \cup \overline{E_1})$ and $(M, \hat{l_1} \cup \hat{l_2} \cup E)$ are compactifications of $\mathbb{C}^2$ and we have $H_l|_X = l_1 + l_2$.

**Proof.** By Proposition 9(i), we obtain the isomorphism $\overline{\psi} : \overline{X} - \overline{l_1} - \overline{l_2} - E_1 \cong \mathbb{P}^2 - \overline{E_1}^* \cong \mathbb{C}^2$. Since $M - \hat{l_1} - E \cong \overline{X} - \overline{l_1} - \overline{l_2} - \overline{E_1}$, $\overline{X} - l_1 - l_2$, we obtain the first assertion.

Assume that there exists a line $l_3$ such that $H_l|_X = l_1 + l_2 + l_3$ and $p \notin l_3$. Then we have $\overline{l_3}^* = \overline{E_1}^*$. Since $\overline{\psi} : \overline{X} - \overline{l_1} - \overline{l_2} \cong \mathbb{P}^2 - \overline{l_1} - \overline{l_2}$, this is a contradiction. Hence we obtain $H_l|_X = 2l_1 + l_2$ or $l_1 + 2l_2$. If $H_l|_X = 2l_1 + l_2$, then we cannot get a solution of $\hat{\Gamma} \sim 2\hat{l_1} + \hat{l_2} + \sum m_i E_i$ for $m_i \in \mathbb{N}$. This is absurd. Thus we obtain $H_l|_X = l_1 + 2l_2$ and $\hat{\Gamma} \sim \hat{l_1} + 2\hat{l_2} + 3E_1 + 2E_2 + 3E_3 + 2E_4 + 4E_5$. □

**Proposition 17.** One has the weighted dual graph of $\hat{l_1} \cup \hat{l_2} \cup E$ as in Figure 10(a) and obtains $\mathbb{P}^2$ by blowing-down $\hat{l_1}, E_2, \hat{l_2}, E_3, E_5, E_4$ successively.
Proof. Since \( I_1 \) passes through \( p_1 \) and \( I_1 \cdot Z = 1 \), we see that \( I_1 \) intersects \( E_2 \) transversally. Since \( I_2 \) passes through \( p_2 \) and \( I_2 \cdot Z = 1 \), we may assume that \( I_2 \) intersects \( E_3 \) transversally. Thus we obtain the weighted dual graph of \( I_1 \cup I_2 \cup E \) as in Figure 10(a).

Since \( Y_1 = I_1 \) or \( I_2 \), we have the following two cases:

(i) \( Y_1 \) passes through \( p_1 \);

(ii) \( Y_1 \) passes through \( p_2 \).

**Proposition 18.** The case (i) cannot occur.

Proof. Assume the case (i). Since \( Y_1 \) and \( Y_2 \) pass through \( p_1 \) and \( Y_1 \cdot Z = Y_2 \cdot Z = 1 \), we see that \( Y_1 \) and \( Y_2 \) intersect \( E_2 \) transversally. Thus we obtain the weighted dual graph of \( A \) as in Figure 10(a), (i). However we cannot get a linear tree from this graph by blowing-down in the boundary \( A \) repeatedly. This is a contradiction.

**Proposition 19.** The case (ii) can occur. One has the weighted dual graph of \( A \) as in Figure 10(a), (ii)-(1), where \( Y_1 = I_2 \), and obtains \( F_1 \) by blowing-down \( Y_1, E_3, E_5, E_1, E_2 \) successively.

Proof. Assume the case (ii). Since \( Y_1 \) passes through \( p_2 \) and \( Y_1 \cdot Z = 1 \), we may assume that \( Y_1 \) intersects \( E_3 \) transversally. Since \( Y_2 \) passes through \( p_2 \) and \( Y_2 \cdot Z = 1 \), we see that \( Y_2 \) intersects either \( E_3 \) or \( E_4 \) transversally.

1. Assume that \( Y_2 \) intersects \( E_3 \). We have the weighted dual graph of \( A \) as in Figure 10(a),(ii)-(1). However we cannot get a linear tree from this graph by blowing-down in the boundary \( A \) repeatedly. This is a contradiction.

2. Assume that \( Y_2 \) intersects \( E_4 \). Then we have the weighted dual graph of \( A \) as in Figure 10(a), (ii)-(2) and obtain \( \tilde{F} \sim \tilde{Y}_1 + \tilde{Y}_2 + 2E_1 + E_2 + 2E_3 + 2E_4 + 3E_5 \).
Let \( \phi_6 \) (respectively \( \phi_5, \phi_4, \phi_3, \phi_2, \phi_1 \)) be the blowing-down of \( \bar{Y}_1 \) (respectively \( E_3, E_5, E_4, \bar{I}_1, E_2 \)). Note that \( \phi = \phi_1 \circ \cdots \circ \phi_6 \) and \( \text{Exc} \phi = \bar{Y}_1 \cup \bar{I}_1 \cup E_2 \cup \cdots \cup E_5 \). Put \( P_1 := \phi(\bar{Y}_1) \) and \( P_2 := \phi(\bar{I}_1) \). Note that \( \bar{\Gamma}^* \) is a smooth elliptic curve and \( \bar{Y}_2^* \) is a line. We see that \( \bar{\Gamma}^* \) and \( \bar{E}_1^* \) meet tangentially to second order at \( P_1 \) and meet transversally at \( P_2 \). Let \( T \) be a tangent line of \( \bar{\Gamma}^* \) at \( P_2 \). We may assume that \( P_1 = (0 : 1 : 0), P_2 = (1 : 0 : 0), \bar{E}_1^* = L_2 = \{ w_2 = 0 \}, \bar{Y}_2^* = L_0 = \{ w_0 = 0 \}, \) and \( T = L_1 = \{ w_1 = 0 \} \).

Now we obtain the defining equations of \( \Phi, (X, Y) \) and \( (X, Y_1 \cup I_1) \) as follows.

**Proposition 20.** The birational map \( \Phi \) is given by

\[
\Phi : \begin{cases} 
  z_0 = w_0 w_2^2 \\
  z_1 = w_1 w_2^2 \\
  z_2 = w_2^3 \\
  z_3 = w_1 (a_2 w_0^2 + a_5 w_0 w_2 + a_8 w_1 w_2),
\end{cases}
\]

where \( a_1 \in \mathbb{C}, a_2 \neq 0 \) and \( a_8 \neq 0 \). The compactifications \( (X, Y) \) and \( (X, Y_1 \cup I_1) \) are given by:

\[
X : z_2 z_3 = z_1 (a_2 z_0^2 + a_5 z_0 z_2 + a_8 z_1 z_2);
\]

\[
Y : z_0 = z_2 (z_2 z_3 - a_8 z_1^2) = 0;
\]

\[
Y_1 \cup I_1 : z_2 = z_0 z_1 = 0.
\]

Considering projective transformations of \( \mathbb{P}^3 \), we obtain (II), (III) and (II) + (III).

**Proof.** Since \( \overline{\psi}_* (\Delta |X) = \overline{\psi}_* (2E_1) = 2 \bar{E}_1^* = 2L_2 \), by Proposition 11(ii), we obtain \( \mathbb{M}_3 = \mathbb{C}(w_0 w_2^2, w_1 w_2^2, w_2^3, F) \), where \( F \) is the defining equation of \( \bar{\Gamma}^* \) as follows:

\[
0 = F := a_1 w_3^3 + a_2 w_0^2 w_1 + a_3 w_0^2 w_2 + a_4 w_0 w_1^2 + a_5 w_0 w_1 w_2 + a_6 w_0 w_2^2 + a_7 w_1^3 + a_8 w_1 w_2 + a_9 w_1 w_2^2 + a_{10} w_2^3.
\]

Since \( \bar{\Gamma}^* \) passes through \( P_1 \) and \( P_2 \), we have \( a_1 = a_7 = 0 \). Since \( \bar{\Gamma}^* \) and \( L_2 \) meet tangentially to second order at \( P_1 \), we have \( a_2 \neq 0 \) and \( a_4 = 0 \). Since \( \bar{\Gamma}^* \) is smooth at \( P_1 \), we have \( a_8 \neq 0 \). Since \( \bar{\Gamma}^* \) and \( L_1 \) meet tangentially at \( P_2 \), we have \( a_3 = 0 \). Hence we obtain

\[
\mathbb{M}_3 = \mathbb{C}(w_0 w_2^2, w_1 w_2^2, w_2^3, F)
\]

\[
= \mathbb{C}(w_0 w_2^2, w_1 w_2^2, w_2^3, a_2 w_0^2 w_1 + a_5 w_0 w_1 w_2 + a_8 w_1 w_2^2)
\]

with \( a_2 \neq 0 \) and \( a_8 \neq 0 \). Thus we have the birational map \( \Phi \) as desired.
Since $\overrightarrow{H}^* = \overrightarrow{Y}^* = L_0$ and $\overrightarrow{H_1}^* = \overrightarrow{E}^* = L_2$, by the defining equation of $\Phi$, we obtain $H = \{z_0 = 0\}$ and $H_1 = \{z_2 = 0\}$. Hence we have the compactifications $(X, Y)$ and $(X, Y_1 \cup l_1)$ as desired. We deform the defining equation of $X$ as follows:

$$z_2^2 \left( z_3 + \frac{a_5^2}{4a_2} z_1 \right) = z_1 a_2 \left( z_0 + \frac{a_5}{2a_2} z_2 \right)^2 + a_8 z_1^2 z_2.$$

Considering the projective transformation of $\mathbb{P}^3$ such that

$$z'_0 := \sqrt{\frac{a_2}{a_8}} \left( z_0 + \frac{a_5}{2a_2} z_2 \right), \quad z'_1 := z_1, \quad z'_2 := -\frac{1}{a_8} \left( z_3 + \frac{a_5^2}{4a_2} z_1 \right), \quad z'_3 := z_2$$

and putting

$$\mu := -\frac{a_5}{2\sqrt{2a_2a_8}},$$

we have

$$X : z_0^2 z_1 + z_3(z_1^2 + z_2 z_3) = 0$$

$$Y : z_0 + \mu z_3 = 0$$

$$Y_1 \cup l_1 : z_3 = 0.$$

Hence we have (III) and, by summarizing these equations, we obtain (II) + (III).

On the other hand, by considering the projective transformation of $\mathbb{P}^3$ such that

$$z'_0 := \frac{a_8}{a_2} z_2, \quad z'_1 := -\frac{a_2}{a_8} z_3, \quad z'_2 := z_1, \quad z'_3 := z_0$$

and putting

$$\beta := \frac{a_5}{a_8},$$

we obtain (II):

$$X : z_0(z_0 z_1 + z_2^2) + z_3(z_2 z_3 + \beta z_0 z_2) = 0$$

$$Y : z_3 = z_0(z_0 z_1 + z_2^2) = 0.$$

3.3. The case $\mathcal{Y} = 2$ line + line

Let $\mathcal{Y} = 2Y_1 + Y_2$ ($Y_i : line$) be the hyperplane section of $X$. By Proposition 12(iii), we have that $x \subset Y_1$ and $x$ consists of either one point or two $A_\epsilon$-type singularities. Let $p$ be the intersection point of $Y_1$ and $Y_2$. Note that $\widehat{Y}_1$ and $\widehat{Y}_2$ are $(-1)$-curves by Lemma 5.
Lemma 14. \( p \in x \).

Proof. Assume that \( p \not\in x \). By Lemma 5, we obtain \( \hat{Y}_1 \cdot Z^{(i)} = 1 \) for each \( i \). Hence we see that the boundary \( \hat{Y} \cup E \) has at most ordinary double points and note that each irreducible component of \( E \) is a \((-2)\)-curve. Thus we have the weighted dual graph of \( \hat{Y} \cup E \) and blow-down \( \hat{Y}_2 \):

\[
\begin{array}{ccc}
E' & \hat{Y}_2 & E'' \\
\cdots & \circ & \cdots \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
E' & 0 & E'' \\
\cdots & \circ & \cdots \\
\end{array}
\]

where \( E = E' \cup E'' \) and either \( E' \) or \( E'' \) is empty if \( \#x = 1 \). Since \( E = E' \cup E'' \) consist of five \((-2)\)-curves, we obtain the weighted dual graph of a minimal normal compactification of \( \mathbb{C}^2 \). However this dual graph cannot be found in Proposition 2, which is a contradiction.

Lemma 15. There exist the following three cases:

(i) \( x = \{ p \} \ (p = A_5) \);

(ii) \( x = \{ p \} \ (p = D_5) \);

(iii) \( x = \{ p, q \} \ (p = A_4, q = A_1) \), where \( q(\neq p) \) lies on \( Y_1 \).

Proof. Assume that \( \#x = 1 \). Since \( b_2(E) = 7 - b_2(Y) = 5 \), we see that \( p \) is an \( A_5 \)-type singularity or a \( D_5 \)-type singularity.

Assume that \( \#x = 2 \). Then we obtain \( x = \{ p, q \} \), where \( q(\neq p) \) lies on \( Y_1 \) and where \( p \) and \( q \) are \( A_4 \)-type singularities. Since \( b_2(E) = 7 - b_2(Y) = 5 \), we obtain the following four cases:

(1) \( p = A_1, q = A_4 \);

(2) \( p = A_2, q = A_3 \);

(3) \( p = A_3, q = A_2 \);

(4) \( p = A_4, q = A_1 \).

Let us consider the cases (1), (2) and (3). Let \( \sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \) be the blowing-up at \( q \) and consider the commutative diagrams as in Figure 5. Let \( Z \) (respectively \( Z' \)) be the fundamental cycle of \( \pi^{-1}(p) \) (respectively \( \pi^{-1}(q) \)). Note that \( \hat{Y}_1 \cdot Z' = 1 \) by Lemma 5. Since \( q \) is an \( A_N \)-type singularity of \( X \) for \( N = 2, 3, 4 \), we see that \( \overline{X} \cap \Delta \) consists of two lines \( \overline{E_1} \) and \( \overline{E_2} \) in \( \Delta \cong \mathbb{P}^2 \). Put \( \overline{q} := \overline{E_1} \cap \overline{E_2} \) and \( \overline{p} := \sigma^{-1}(p) \). Since \( \overline{Y}_2 \) is a line in \( \mathbb{P}^2 \), by Proposition 10(ii), \( \overline{Y}_1 \) passes through \( \overline{q} \). Thus we obtain Figure 11. In the cases (1) and (2), we have that \( \overline{p} \) and \( \overline{q} \) are \( A_4 \)-type singularities of \( \overline{X} \). Since \( \overline{p} \) and \( \overline{q} \) lie on \( \overline{Y}_1 \), by Proposition 9(vii), this is a contradiction. In the
case (3), since \( \overline{q} \) is a smooth point of \( \overline{X} \), we obtain \( \overline{Y}_1 \cdot Z' = 2 \). This is a contradiction since \( \overline{Y}_1 \cdot Z' = 1 \). Hence the cases (1), (2) and (3) cannot occur. \( \square \)

![Diagram](image)

**Figure 11.**

**Remark.** Let us consider the case (ii) in Lemma 15. Since \( x = Y_1 \cap Y_2 = \{ p \} \) and \( p \) is a \( D_5 \)-type singularity, we can use the same argument in Section 3.2.2. Note that \( Y_1 = l_2 \) and \( Y_2 = l_1 \) in the notation of Section 3.2.2. Hence we obtain the weighted dual graph of \( \overline{Y} \cup E \) as in Proposition 17 and the compactification \( (X, Y) \) as in Proposition 20. Thus we have (III). Hence it is sufficient to consider the cases (i) and (iii) in Lemma 15.

### 3.3.1. The case \( x = \text{one point} = A_5 \)

By Lemma 15(i), we have \( x = Y_1 \cap Y_2 = \{ p \} \). Let \( \sigma : \mathbb{P}^3 \to \mathbb{P}^3 \) be the blowing-up at \( p \) and consider the commutative diagrams as in Figure 5. Since \( p \) is an \( A_5 \)-type singularity of \( X \), we see that \( \overline{X} \cap \Delta \) consists of two lines \( \overline{E}_1 \) and \( \overline{E}_2 \) in \( \Delta \cong \mathbb{P}^2 \) and that \( \{ \overline{p} \} := \overline{E}_1 \cap \overline{E}_2 \) is an \( A_3 \)-type singularity of \( \overline{X} \). Then we have \( \overline{x} = \{ \overline{p} \} \). Let \( Z \) be the fundamental cycle of \( \pi^{-1}(p) \). Note that \( \overline{Y}_1 \cdot Z = \overline{Y}_2 \cdot Z = 1 \) by Lemma 5. By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 16.** There exist three lines in \( X \) through \( p \).

Let \( Y_1, Y_2 \) and \( l \) be the three lines in \( X \) through \( p \). By Proposition 7(ii), we may assume that \( \overline{H}^* = \overline{E}_1^* \) and hence that \( \overline{Y}_1 \cap \Delta \) and \( \overline{Y}_2 \cap \Delta \) lie on \( \overline{E}_1 \) and \( l \) lies on \( \overline{E}_2 \) (\( \overline{p} \)). Note that one of \( \overline{Y}_1 \) and \( \overline{Y}_2 \) passes through \( \overline{p} \) by Proposition 9(iii).

Let \( Y_i \) be the line such that \( \overline{Y}_i \) passes through \( \overline{p} \) and \( Y_j \) the line different from \( Y_i \) for \( \{ i, j \} = \{ 1, 2 \} \). Thus we obtain Figure 12(b).

**Lemma 17.** The pairs \( (X, Y_i \cup l) \), \( (\overline{X}, \overline{Y}_i \cup \overline{l} \cup \overline{E}) \) and \( (M, \overline{Y}_i \cup \overline{l} \cup E) \) are compactifications of \( \mathbb{C}^2 \).

**Proof.** Since \( \overline{x} \cap \overline{Y}_j = \emptyset \), we see that the restriction of \( \overline{\psi} : \overline{X} \to \mathbb{P}^2 \) to a neighborhood of the \((-1)\)-curve \( \overline{Y}_j \) is a blowing-down. Let \( \tau : \mathbb{P}_1 \to \mathbb{P}^2 \) be a blowing-up at the point \( \overline{Y}_j^* \in \mathbb{P}^2 \). Then there exists a birational morphism \( f : \overline{X} \to \mathbb{P}_1 \)
such that $\tau \circ f = \overline{\psi}$ and $f : \overline{X - \overline{Y}_i - \overline{l} - \overline{E}} \cong \mathbb{P}^1 - f(\overline{E}) \cong \mathbb{C}^2$. Since $X - Y_i - l \cong \overline{X - \overline{Y}_i - \overline{l} - \overline{E}} \cong M - \overline{Y}_i - \overline{l} - E$, we have the assertion.

**Proposition 21.** The case $x = 1$ point $= A_5$ cannot occur.

**Proof.** From Figure 12(b), we obtain the weighted dual graph of $\overline{\Gamma} \cup \overline{\overline{Y}}_j \cup E$ as in Figure 12(a). Since $\overline{Y}_i$ passes through $\overline{p}$, by Proposition 9(vii), $\overline{Y}_i$ intersects either $E_3$ or $E_4$ transversally.

1. Assume that $\overline{Y}_i$ intersects $E_3$. Then we transform $(M, A)$ into a minimal normal compactification $(M', A')$ by blowing-down in the boundary $A$ repeatedly. However the weighted dual graph of $A'$ cannot be found in Proposition 2, which is a contradiction.

2. Assume that $\overline{Y}_i$ intersects $E_4$. Note that $(M, \overline{Y}_i \cup \overline{l} \cup E)$ is a smooth compactification of $\mathbb{C}^2$. We transform $(M, \overline{Y}_i \cup \overline{l} \cup E)$ into a minimal normal compactification $(M', A')$ by blowing-down in the boundary $\overline{Y}_i \cup \overline{l} \cup E$ repeatedly. However the weighted dual graph of $A'$ cannot be found in Proposition 2, which is a contradiction.

**3.3.2. The case $x = 2$ points.** By Lemma 15(iii), we have $x = \{p, q\}$ ($q \in \overline{Y}_1$, $p \in \overline{Y}_1 \cap \overline{Y}_2$), where $p$ is an $A_4$-type singularity of $X$ and $q$ is an $A_1$-type singularity of $X$. Let $\sigma : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the blowing-up at $p$ and consider the commutative diagrams as in Figure 5. Put $\overline{q} := \sigma^{-1}(q)$. Since $p$ is an $A_4$-type singularity of $X$, we see that $\overline{X} \cap \Delta$ consists of two lines $\overline{E}_1$ and $\overline{E}_2$ in $\Delta \cong \mathbb{P}^2$ and that $(\overline{p}) := \overline{E}_1 \cap \overline{E}_2$ is an $A_2$-type singularity of $\overline{X}$. Then we have $\overline{x} = \{\overline{p}, \overline{q}\}$. Let $Z$ (respectively $Z'$) be the fundamental cycle of $\pi^{-1}(p)$ (respectively $\pi^{-1}(q)$). Note that $\overline{Y}_1 \cdot Z = \overline{Y}_2 \cdot Z = \overline{Y}_1 \cdot Z' = 1$ by Lemma 5. By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 18.** There exist three lines in $X$ through $p$ and two lines in $X$ through $q$. 
Let $Y_1$, $Y_2$ and $I_1$ be the three lines in $X$ through $p$. Let $Y_1$ and $I_2$ be the two lines in $X$ through $q$. By Proposition 7(ii), we may assume that $H^* = E_1^*$ and hence that $\overline{Y_1} \cap \Delta$ and $\overline{Y_2} \cap \Delta$ lie on $\overline{E_1}$ and $\overline{I_1} \cap \Delta$ lies on $\overline{E_2} - \{\overline{p}\}$. Note that one of $\overline{Y_1}$ and $\overline{Y_2}$ passes through $\overline{p}$ by Proposition 9(iii). Since $\overline{Y_1}$ contains $\overline{q}$, by Proposition 9(vii), $\overline{Y_2}$ passes through $\overline{p}$. Thus we obtain Figure 13(b).

**Proposition 22.** The case $x =$ two points can occur. One has the weighted dual graph of $\tilde{I} \cup \tilde{Y}_1 \cup E$ as in Figure 13(a) and obtains $\mathbb{F}_0$ by blowing-down $\tilde{Y}_1$, $E_5$, $\tilde{Y}_2$, $E_4$, $E_2$ successively.

![Diagram](image)

**(a)**

**Figure 13.**

**Proof.** From Figure 13(b), we obtain the weighted dual graph of $\tilde{I} \cup \tilde{Y}_1 \cup E$ as in Figure 13(a). Since $\tilde{Y}_2$ passes through $\overline{p}$ and $\tilde{Y}_2 \cdot Z = 1$, we see that $\tilde{Y}_2$ intersects either $E_3$ or $E_4$ transversally.

(1) Assume that $\tilde{Y}_2$ intersects $E_3$. Then we transform $(M, A)$ into a minimal normal compactification $(M', A')$ by blowing-down in the boundary $A$ repeatedly. However the weighted dual graph of $A'$ cannot be found in Proposition 2, which is a contradiction.

(2) Assume that $\tilde{Y}_2$ intersects $E_4$. Then we have the weighted dual graph of $A$ as in Figure 13(a) and obtain $\tilde{\Gamma} \sim 2\tilde{Y}_1 + \tilde{Y}_2 + 2E_1 + E_2 + 2E_3 + 2E_4 + E_5$. $\square$

Let $\phi_6$ (respectively $\phi_5, \phi_4, \phi_3, \phi_2, \phi_1$) be the blowing-down of $\tilde{Y}_1$ (respectively $E_5, \tilde{Y}_2, E_4, E_3, \tilde{I}_1$). Note that $\phi = \phi_1 \circ \cdots \circ \phi_6$ and $\text{Exc } \phi = \overline{Y}_1 \cup \overline{I}_1 \cup E_3 \cup E_4 \cup E_5$. Put $P_1 := \phi(\overline{Y}_2), P_2 := \phi(\overline{Y}_1)$ and $Q := \phi(\overline{I}_1)$. Note that $\overline{\Gamma}^*$ is a smooth elliptic curve. We see that $\overline{\Gamma}^*$ and $\overline{E_1}^*$ meet tangentially to second order at $P_2$ and that $\overline{\Gamma}^*$ and $\overline{E_2}^*$ meet tangentially to second order at $P_1$. We may assume that $\overline{E_1}^* = L_2 = \{w_2 = 0\}$, $\overline{E_2}^* = L_0 = \{w_0 = 0\}$, $P_1 = (0 : 1 : 0)$, $P_2 = (1 : 0 : 0)$ and $Q = (0 : 0 : 1)$.

Now we obtain the defining equations of $\Phi$ and $(X, Y)$ as follows.
PROPOSITION 23. The birational map $\Phi$ is given by

$$\Phi : \begin{cases} 
  z_0 = w_0^2 w_2 \\
  z_1 = w_0 w_1 w_2 \\
  z_2 = w_0 w_2^2 \\
  z_3 = w_1 (a_4 w_0 w_1 + a_9 w_2^2),
\end{cases}$$

where $a_1 \in \mathbb{C}$, $a_4 \neq 0$ and $a_9 \neq 0$. The compactification $(X, Y)$ is given by:

$$X : z_0 z_2 z_3 = a_4 z_2 z_1^2 + a_9 z_1 z_2^2;$$
$$Y : z_2 = z_0 z_1^2 = 0.$$  

Considering a projective transformation of $\mathbb{P}^3$, we obtain (IV).

Proof. Since $\bar{\psi}_* (\Delta | \bar{X}) = \bar{\psi}_* (\bar{E}_1 + \bar{E}_2) = \bar{E}_1^* + \bar{E}_2^* = L_0 + L_2$, by Proposition 11(ii), we obtain $M_8 = \mathcal{C}(w_0^3 w_2, w_0 w_1 w_2, w_0 w_2^2, F)$, where $F = 0$ is the defining equation of $\Gamma^*$ as follows:

$$0 = F := a_1 w_0^3 + a_2 w_0^2 w_1 + a_3 w_0^2 w_2 + a_4 w_0 w_1^2 + a_5 w_0 w_1 w_2 + a_6 w_0 w_2^2$$
$$+ a_7 w_1^3 + a_8 w_1^2 w_2 + a_9 w_1 w_2^2 + a_{10} w_2^3.$$  

Since $\Gamma^*$ passes through $P_1$, $P_2$ and $Q$, we have $a_1 = a_7 = a_{10} = 0$. Since $\Gamma^*$ and $L_0$ meet tangentially to second order at $P_1$, we have $a_8 = 0$ and $a_9 \neq 0$. Since $\Gamma^*$ and $L_2$ meet tangentially to second order at $P_2$, we have $a_2 = 0$ and $a_4 \neq 0$. Hence we obtain

$$M_8 = \mathcal{C}(w_0^3 w_2, w_0 w_1 w_2, w_0 w_2^2, F)$$
$$= \mathcal{C}(w_0^3 w_2, w_0 w_1 w_2, w_0 w_2^2, a_4 w_0 w_1^2 + a_9 w_1 w_2^2)$$

with $a_4 \neq 0$ and $a_9 \neq 0$. Thus we have the birational map $\Phi$ as desired.

Since $\bar{H}^* = \bar{E}_1^* = L_2$, by the defining equation of $\Phi$, we obtain $H = \{ z_2 = 0 \}$ and the compactification $(X, Y)$ as desired. Considering the projective transformation of $\mathbb{P}^3$ such that

$$z_0' := z_0, \quad z_1' := z_1, \quad z_2' := - \frac{1}{\sqrt{a_4 a_9}} z_3, \quad z_3' := \sqrt{\frac{a_9}{a_4}} z_2,$$

we obtain (IV). \qed

3.4. The case $\mathcal{Y} = \text{line} + \text{line} + \text{line}$

Let $\mathcal{Y} = Y_1 + Y_2 + Y_3$ ($Y_i : \text{line}$) be the hyperplane section of $X$. By Proposition 12(iv), we have $x = \bigcap_i Y_i = \{ p \}$. Let $\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$ be the blowing-up at $p$ and consider
the commutative diagrams as in Figure 5. Since \( b_2(E) = 7 - b_2(Y) = 4 \), we see that \( p \) is an \( A_4 \)-type singularity or a \( D_4 \)-type singularity. Let \( Z \) be the fundamental cycle of \( \pi^{-1}(p) \). Note that \( \tilde{Y}_1 \cdot Z = \tilde{Y}_2 \cdot Z = \tilde{Y}_3 \cdot Z = 1 \) by Lemma 5.

3.4.1. The case \( p = A_4 \). Since \( p \) is an \( A_4 \)-type singularity of \( X \), we see that \( \overline{X} \cap \Delta \) consists of two lines \( \overline{E_1} \) and \( \overline{E_2} \) in \( \Delta \cong \mathbb{P}^2 \) and that \( \{ \overline{p} \} \) is an \( E_1 \cap E_2 \) is an \( A_2 \)-type singularity of \( \overline{X} \). Then we have \( \overline{x} = \{ \overline{p} \} \). Put \( E_3 \cup E_4 := \text{Exc} \overline{\pi} \) as in Figure 14(a). By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 19.** There exist four lines in \( X \) through \( p \).

Let \( Y_1, Y_2, Y_3 \) and \( l \) be the four lines in \( X \) through \( p \). By Proposition 10(ii), we may assume that \( \overline{H} = \overline{E_1} \) and hence that \( Y_i \cap \Delta \) lies on \( E_1 \) for \( i = 1, 2, 3 \) and \( l \cap \Delta \) lies on \( E_2 \setminus \{ \overline{p} \} \). Note that one of \( Y_1, Y_2 \) and \( Y_3 \) passes through \( \overline{p} \) by Proposition 9(iii). We may assume that \( Y_3 \) is such one. Thus we obtain Figure 14(b).

**Proposition 24.** The case \( p = A_4 \) can occur. One has the weighted dual graph of \( l \cup Y_1 \cup Y_2 \cup E \) as in Figure 14(a) and obtains \( \mathbb{F}_0 \) by blowing-down \( Y_3, E_4, E_2, Y_2, Y_1 \) successively.

\[ \begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
Y_1 & E_1 & E_2 \\
Y_2 & E_3 & E_4 \\
Y_3 & & \\
\end{array}
\end{array} \]

(a)

\[ \begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
Y_1 & Y_2 & Y_3 \\
E_1 & E_2 & E_3 \\
\overline{p} & \overline{\pi} & \Delta \\
\end{array}
\end{array} \]

(b)

**Figure 14.**

**Proof.** From Figure 14(b), we obtain the weighted dual graph of \( l \cup \tilde{Y}_1 \cup \tilde{Y}_2 \cup E \) as in Figure 14(a). Since \( \tilde{Y}_3 \) passes through \( \overline{p} \) and \( \tilde{Y}_3 \cdot Z = 1 \), we see that \( \tilde{Y}_3 \) intersects either \( E_3 \) or \( E_4 \) transversally.

1. Assume that \( \tilde{Y}_3 \) intersects \( E_3 \). Then we transform \( (M, A) \) into a minimal normal compactification \( (M', A') \) by blowing-down in the boundary \( A \) repeatedly. However the weighted dual graph of \( A' \) cannot be found in Proposition 2, which is a contradiction.

2. Assume that \( \tilde{Y}_3 \) intersects \( E_4 \). Then we have the weighted dual graph of \( A \) as in Figure 14(a) and obtain \( \tilde{\Gamma} \sim \tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3 + 2E_1 + E_2 + 2E_3 + 2E_4 \). \( \square \)
Let $\phi_6$ (respectively $\phi_5$, $\phi_4$, $\phi_3$, $\phi_2$, $\phi_1$) be the blowing-down of $\tilde{Y}_3$ (respectively $E_4$, $E_3$, $\tilde{Y}_2$, $\tilde{Y}_1$, $\tilde{I}$). Note that $\phi = \phi_1 \circ \cdots \circ \phi_6$ and $\text{Exc} \phi = \tilde{Y} \cup \tilde{I} \cup E_3 \cup E_4$. Put $P_1 := \phi(\tilde{Y}_3)$, $P_2 := \phi(\tilde{Y}_2)$, $P_3 := \phi(\tilde{Y}_1)$ and $Q := \phi(\tilde{I})$. Note that $\Gamma^*$ is a smooth elliptic curve. We see that $\Gamma^*$ and $\tilde{E}_2^*$ meet tangentially to second order at $P_1$ and that $\tilde{E}_1^*$ and $\tilde{E}_2^*$ meet transversally at $P_1$, $P_2$ and $P_3$. We may assume that $\tilde{E}_1^* = L_2$, $\tilde{E}_2^* = L_0$, $P_1 = (0 : 1 : 0)$, $P_2 = (1 : 0 : 0)$ and $Q = (0 : 0 : 1)$.

Now we obtain the defining equations of $\Phi$ and $(X, Y)$ as follows.

**Proposition 25.** The birational map $\Phi$ is given by

$$
\Phi : \begin{cases}
    z_0 = w_0^2w_2 \\
    z_1 = w_0w_1w_2 \\
    z_2 = w_0w_2^2 \\
    z_3 = w_1(a_2w_0^2 + a_4w_0w_1 + a_9w_2^2),
\end{cases}
$$

where $a_i \in \mathbb{C}$, $a_2 \neq 0$, $a_4 \neq 0$ and $a_9 \neq 0$. The compactification $(X, Y)$ is given by:

$$
X : z_0z_2z_3 = a_2z_0^2z_1 + a_4z_0z_1^2 + a_9z_1z_2^2;
$$

$$
Y : z_2 = z_0z_1(a_2z_0 + a_4z_1) = 0.
$$

Considering a projective transformation of $\mathbb{P}^3$, we obtain (V).

**Proof.** Since $\overline{\psi}_*(\Delta|_{\tilde{X}}) = \overline{\psi}_*(\tilde{E}_1 + \tilde{E}_2) = \tilde{E}_1^* + \tilde{E}_2^* = L_0 + L_2$, by Proposition 11(ii), we obtain $M_5 = \mathbb{C}(w_0^2w_2, w_0w_1w_2, w_0w_2^2, F)$, where $F = 0$ is the defining equation of $\Gamma^*$ as follows:

$$
0 = F := a_1w_0^3 + a_2w_0^2w_1 + a_3w_0^2w_2 + a_4w_0w_1^2 + a_5w_0w_1w_2 + a_6w_0w_2^2
$$

$$
+ a_7w_1^3 + a_8w_1^2w_2 + a_9w_1w_2^2 + a_{10}w_2^3.
$$

Since $\Gamma^*$ passes through $P_1$, $P_2$ and $Q$, we have $a_1 = a_7 = a_{10} = 0$. Since $\Gamma^* \cap L_2$ consists of three points, we have $a_2 \neq 0$ and $a_4 \neq 0$. Since $\Gamma^*$ and $L_0$ meet tangentially to second order at $P_1$, we obtain $a_8 = 0$ and $a_9 \neq 0$. Hence we obtain

$$
M_5 = \mathbb{C}(w_0^2w_2, w_0w_1w_2, w_0w_2^2, F)
$$

$$
= \mathbb{C}(w_0^2w_2, w_0w_1w_2, w_0w_2^2, a_2w_0^2w_1 + a_4w_0w_1^2 + a_9w_1w_2^2)
$$

with $a_2 \neq 0$, $a_4 \neq 0$ and $a_9 \neq 0$. Thus we have the birational map $\Phi$ as desired.

Since $\overline{H}^* = \overline{E}_1^* = L_2$, by the defining equation of $\Phi$, we obtain $H = \{z_2 = 0\}$ and the compactification $(X, Y)$ as desired. Considering the projective transformation
of $\mathbb{P}^3$ such that

$$z_0' := cz_1, \quad z_1' := \frac{a_2c}{a_4} z_0, \quad z_2' := z_3, \quad z_3' := -\frac{c^2}{a_4} z_2,$$

where $c = \sqrt{a_2a_9}$, we obtain (V).

\[\square\]

3.4.2. The case $p = D_4$. Since $p$ is a $D_4$-type singularity of $X$, we see that $E_1 := \overline{X} \cap \Delta$ is a line in $\Delta \cong \mathbb{P}^2$ and there exist three $A_1$-type singularities $\overline{p_1}, \overline{p_2}$ and $\overline{p_3}$ of $\overline{X}$ on $E_1$. Then we have $\overline{x} = \{\overline{p_1}, \overline{p_2}, \overline{p_3}\}$.

Put $E_{i+1} := \overline{x}^{-1}(\overline{p_i})$ for $i = 1, 2, 3$ as in Figure 15(a). By Propositions 8(iii) and 9(ii), we obtain the following.

**Lemma 20.** There exist three lines in $X$ through $p$.

The three lines are $Y_1, Y_2$ and $Y_3$. By Proposition 9(iii), we may assume that $\overline{Y_i}$ passes through $\overline{p_i}$ for $i = 1, 2, 3$. Thus we obtain Figure 15(b).

**Proposition 26.** The case $p = D_4$ can occur. One has the weighted dual graph of $A$ as in Figure 15(a) and obtains $\mathbb{P}^2$ by blowing-down $\widehat{Y_1}, E_4, \widehat{Y_2}, E_3, \widehat{Y_3}, E_2$ successively.

![Figure 15](image)

**Proof.** Since $\overline{Y_i}$ passes through $\overline{p_i}$ and $\overline{Y_i} \cdot Z = 1$ for each $i$, we have the weighted dual graph of $A$ as in Figure 15(a) and obtain $\widehat{Y} \sim \widehat{Y_1} + \widehat{Y_2} + \widehat{Y_3} + 3E_1 + 2E_2 + 2E_3 + 2E_4$. \[\square\]

Let $\phi_6$ (respectively $\phi_5, \phi_4, \phi_3, \phi_2, \phi_1$) be the blowing-down of $\widehat{Y_3}$ (respectively $E_4, \widehat{Y_2}, E_3, \widehat{Y_1}, E_2$). Note that $\phi = \phi_1 \circ \cdots \circ \phi_6$ and $\text{Exc } \phi = \widehat{Y} \cup E_2 \cup E_3 \cup E_4$.

Put $P_i := \phi(\widehat{Y_i})$ for each $i$. Note that $\widehat{E}^*$ is a smooth elliptic curve and that $\widehat{E}^*$ and $\overline{E_1}^*$ meet transversally at three points $P_1, P_2$ and $P_3$. Let $T_i$ be a tangent line of $\overline{E_1}^*$ at $P_i$ for $i = 1, 2$. We may assume that $P_1 = (0 : 1 : 0), P_2 = (1 : 0 : 0), \overline{E_1}^* = L_2 = \{w_2 = 0\}, T_1 = L_0 = \{w_0 = 0\}$ and $T_2 = L_1 = \{w_1 = 0\}$.

Now we obtain the defining equations of $\Phi$ and $(X, Y)$ as follows.
PROPOSITION 27. The birational map $\Phi$ is given by

$$
\begin{align*}
\Phi : \\
z_0 &= w_0 w_2^2 \\
z_1 &= w_1 w_2^2 \\
z_2 &= w_2^3 \\
z_3 &= w_0 w_1 (a_2 w_0 + a_4 w_1 + a_5 w_2),
\end{align*}
$$

where $a_i \in \mathbb{C}$, $a_2 \neq 0$ and $a_4 \neq 0$. The compactification $(X, Y)$ is given by:

$$
\begin{align*}
X : z_2^2 z_3 &= z_0 z_1 (a_2 z_0 + a_4 z_1 + a_5 z_2); \\
Y : z_2 &= z_0 z_1 (a_2 z_0 + a_4 z_1) = 0.
\end{align*}
$$

Considering a projective transformation of $\mathbb{P}^3$, we obtain (VI).

Proof. Since $\overline{\psi}_* (\Delta | X) = \overline{\psi}_* (2 \overline{E}_1) = 2 \overline{E}_1^* = 2 L_2$, by Proposition 11(ii), we obtain $M_5 = \mathbb{C} (w_0 w_2^2, w_1 w_2^2, w_3^2, F)$, where $F = 0$ is the defining equation of $\overline{\Gamma}^*$ as follows:

$$
0 = F := a_1 w_0^3 + a_2 w_0^2 w_1 + a_3 w_0 w_1^2 + a_4 w_1 w_2^2 + a_5 w_0 w_1 w_2 + a_6 w_0 w_2^2 \\
+ a_7 w_1 + a_8 w_1 w_2 + a_9 w_1 w_2^2 + a_10 w_2^3.
$$

Since $\overline{\Gamma}^*$ passes through $P_1$ and $P_2$, we have $a_1 = a_7 = 0$. Since $\overline{\Gamma}^* \cap L_2$ consists of three points, we have $a_2 \neq 0$ and $a_4 \neq 0$. Since $\overline{\Gamma}^*$ and $L_0$ meet tangentially at $P_1$, we obtain $a_8 = 0$. Since $\overline{\Gamma}^*$ and $L_1$ meet tangentially at $P_2$, we obtain $a_3 = 0$. Hence we obtain

$$
M_5 = \mathbb{C} (w_0 w_2^2, w_1 w_2^2, w_3^2, F)
= \mathbb{C} (w_0 w_2^2, w_1 w_2^2, w_3^2, a_2 w_0 w_2^3 + a_4 w_0 w_1^2 + a_5 w_0 w_1 w_2)
$$

with $a_2 \neq 0$ and $a_4 \neq 0$. Thus we have the birational map $\Phi$ as desired.

Since $\overline{H}^* = \overline{E}_1^* = L_2$, by the defining equation of $\Phi$, we obtain $H = \{ z_2 = 0 \}$ and the compactification $(X, Y)$ as desired. Considering the projective transformation of $\mathbb{P}^3$ such that

$$
\begin{align*}
z'_0 &:= \sqrt[n]{\frac{a_4}{a_2}} z_0, \\
z'_1 &:= \sqrt[n]{\frac{a_2}{a_4}} z_1, \\
z'_2 &:= -z_3, \\
z'_3 &:= z_2,
\end{align*}
$$

we obtain (VI). \hfill \Box

4. Proof of Theorem 2

Now we may assume that $\deg X_f = 3$. Since $X_f$ is a cubic hypersurface in $\mathbb{P}^3$ and $Y_f$ is a hyperplane section of $X_f$ such that $X_f - Y_f$ is birational to $\mathbb{C}^2$, by Theorem 1,
we see that \((X_f, Y_f)\) is, up to projective transformations of \(\mathbb{P}^3\), one of (I) \(~\) (IX). Since \(Y_f = X_f \cap \{z_3 = 0\}\), we take the affine part \(\mathbb{P}^3 - \{z_3 = 0\} \cong \mathbb{C}^3\). Put \(x := z_0/z_3, y := z_1/z_3\) and \(z := z_2/z_3\). For each case (I) \~ (IX), we can transform \(f(\mathbb{C}^2) = X_f - Y_f\) into a hyperplane of \(\mathbb{C}^3\) by the following polynomial automorphism \(\alpha\) of \(\mathbb{C}^3\). Note that except (II) one can find elementary automorphisms \(\alpha\) and that for (II) one needs the so called Nagata’s automorphism. The proof is completed.

(I) \[
X_f : z_0^3 + \lambda z_0^2 z_3 + z_1^2 z_3 + z_2 z_3^2 = 0.
\]
\(f(\mathbb{C}^2) : x^3 + \lambda x^2 + y^2 + z = 0.\)
\[
\alpha : \begin{cases} 
  x' = x \\
  y' = y \\
  z' = z + \lambda x^2 + x^3 + y^2.
\end{cases}
\]

(II) \[
X_f : z_0^2 z_1 + z_0 z_2^2 + \lambda z_0 z_2 z_3 + z_2 z_3^2 = 0.
\]
\(f(\mathbb{C}^2) : x^2 y + x z^2 + \lambda x z + z = 0.\)

We deform as follows:
\[
x \left\{ xy + \left( z + \frac{\lambda}{2} \right)^2 \right\} + \left( z + \frac{\lambda}{2} \right) - \frac{\lambda^2}{4} x - \frac{\lambda}{2} = 0.
\]

Put \(\alpha := \alpha_2 \circ \alpha_1\), where
\[
\alpha_1 : \begin{cases} 
  x' = x \\
  y' = y \\
  z' = z + \frac{\lambda}{2}
\end{cases}, \quad \alpha_2 : \begin{cases} 
  x'' = x' \\
  y'' = y' - 2 z' (x'y' + z'^2) - x'(x'y' + z'^2)^2 \\
  z'' = z' + x'(x'y' + z'^2).
\end{cases}
\]

The map \(\alpha_2\) is called Nagata’s automorphism [Na].

(III) \[
X_f : z_0^2 z_1 + z_1^2 z_3 + z_2 z_3^2 = 0.
\]
\(f(\mathbb{C}^2) : x^2 y + y^2 + z = 0.\)
\[
\alpha : \begin{cases} 
  x' = x \\
  y' = y \\
  z' = z + y^2 + x^2 y.
\end{cases}
\]

(IV) \[
X_f : z_0^2 z_1 + z_1 z_2 z_3 + z_0 z_3^2 = 0.
\]
\(f(\mathbb{C}^2) : x^2 y + y z + x = 0.\)
Put \( \alpha := \alpha_2 \circ \alpha_1 \), where

\[
\begin{align*}
\alpha_1 : & \quad \begin{cases} 
x' = x \\
y' = y \\
z' = z + x^2 
\end{cases} \\
\alpha_2 : & \quad \begin{cases} 
x'' = x' + y'z' \\
y'' = y' \\
z'' = z'. 
\end{cases}
\end{align*}
\]

(V) \( X_f : z_0^2z_1 + z_0z_1^2 + z_1z_2z_3 + z_0z_3^2 = 0. \)
\( f(\mathbb{C}^2) : x^2y + xy^2 + yz + x = 0. \)

Put \( \alpha := \alpha_2 \circ \alpha_1 \), where

\[
\begin{align*}
\alpha_1 : & \quad \begin{cases} 
x' = x \\
y' = y \\
z' = z + xy + x^2 
\end{cases} \\
\alpha_2 : & \quad \begin{cases} 
x'' = x' + y'z' \\
y'' = y' \\
z'' = z'. 
\end{cases}
\end{align*}
\]

(VI) \( X_f : z_0^2z_1 + z_0z_1^2 + \lambda z_0z_1z_3 + z_2z_3^2 = 0. \)
\( f(\mathbb{C}^2) : x^2y + xy^2 + \lambda xy + z = 0. \)

\[
\begin{align*}
\alpha : & \quad \begin{cases} 
x' = x \\
y' = y \\
z' = z + \lambda xy + x^2y + xy^2. 
\end{cases}
\end{align*}
\]

(VII) \( X_f : z_0^3 + \lambda z_0^2z_3 + z_2z_3^2 = 0. \)
\( f(\mathbb{C}^2) : x^3 + \lambda x^2 + z = 0. \)

\[
\begin{align*}
\alpha : & \quad \begin{cases} 
x' = x \\
y' = y \\
z' = z + \lambda x^2 + x^3. 
\end{cases}
\end{align*}
\]

(VIII) \( X_f : z_0^3 + z_0z_1z_3 + z_2z_3^2 = 0. \)
\( f(\mathbb{C}^2) : x^3 + xy + z = 0. \)

\[
\begin{align*}
\alpha : & \quad \begin{cases} 
x' = x \\
y' = y \\
z' = z + xy + x^3. 
\end{cases}
\end{align*}
\]

(IX) \( X_f : z_0^2z_1 + \lambda z_0z_2z_3 + z_2z_3^2 = 0. \)
\( f(\mathbb{C}^2) : x^2y + \lambda xz + z = 0. \)
(1) If $\lambda = 0$, then
\[
\alpha : \begin{cases} 
  x' = x \\
  y' = y \\
  z' = z + x^2 y.
\end{cases}
\]

(2) If $\lambda \neq 0$, then put $\alpha := \alpha_3 \circ \alpha_2 \circ \alpha_1$, where
\[
\alpha_1 : \begin{cases} 
  x' = x \\
  y' = y \\
  z' = z + \frac{1}{\lambda} xy
\end{cases}, \quad \alpha_2 : \begin{cases} 
  x'' = x' \\
  y'' = y' - \lambda^2 z' \\
  z'' = z'
\end{cases}, \quad \alpha_3 : \begin{cases} 
  x''' = x'' \\
  y''' = y'' \\
  z''' = z'' - \frac{1}{\lambda} x'' y''
\end{cases}.
\]

Appendix. Rational double points

Let $(X, 0) \dashrightarrow (\mathbb{C}^3, 0)$ be a rational double point. It is well-known that $(X, 0)$ is equivalent to one of the following:

- **Type $A_n$** $(n \geq 1)$ : $x^2 + y^2 + z^{n+1} = 0$
- **Type $D_n$** $(n \geq 4)$ : $x^2 + y^{n-1} + yz^2 = 0$
- **Type $E_6$** : $x^2 + y^3 + z^4 = 0$
- **Type $E_7$** : $x^2 + y^3 + yz^3 = 0$
- **Type $E_8$** : $x^2 + y^3 + z^5 = 0$.

Let $\pi : M \rightarrow X$ be the minimal resolution of $X$. Let $\sigma : \overline{\mathbb{C}^3} \rightarrow \mathbb{C}^3$ be the blowing-up at $0$, $\Delta \subset \overline{\mathbb{C}^3}$ the exceptional divisor of $\sigma$ and $\overline{X}$ the proper transform of $X$ by $\sigma$. Since $0 \in X$ is a double point, $\overline{X}|_{\Delta}$ is a double line, two lines or a conic. Let $\overline{X} \cap \Delta = \bigcup_i \overline{C_i}$ be the irreducible decomposition of $\overline{X} \cap \Delta$. Since $0 \in X$ is a rational double point, by Lemma 2, the $O_M$-module $\pi^*m_{X, 0}$ is invertible. Hence, by [Ha, Proposition 7.14, pp. 164–165], there exists a morphism $\overline{\pi} : M \rightarrow \overline{X}$ such that $\sigma \circ \overline{\pi} = \pi$. Let $\overline{C_i}$ be the proper transform of $\overline{C_i}$ by $\overline{\pi}$. Thus we get the commutative diagrams as in Figure 16. Note that the minimal resolution $\pi : M \rightarrow X$ can be obtained by blowing-up of $\mathbb{C}^3$ at points since Sing $\overline{X}$ consists of rational double points again. In Figure 17, we give the dual graph of $\pi^{-1}(0)$, which consists of $(-2)$ curves, for each rational double point and write the weight of its fundamental cycles adjacent to each vertex. Furthermore, in Figure 17, we give the figure of $\overline{X} \cap \Delta$ and Sing $\overline{X}$, where an $A_0$-type singularity means a smooth point of $\overline{X}$ and a $D_3$-type singularity means an $A_3$-type singularity.
Structure of algebraic embeddings $\mathbb{C}^2$ into $\mathbb{C}^3$

\[ M \xrightarrow{\overline{\pi}} \overline{X} \xrightarrow{\text{inclusion}} \mathbb{C}^3 \]
\[ X \xrightarrow{\pi} \sigma |_{\overline{X}} \xrightarrow{\sigma} \mathbb{C}^3 \]

**Figure 16.**

$A_1$:
- $C_1$
- $1$
- $\overline{C}_1$

$A_n$:
- $C_1$
- $\cdots$
- $C_2$
- $1$
- $1$
- $1$
- $\overline{C}_1$
- $\overline{C}_2$
- $A_{n-2}$

$n \geq 2$

$D_4$:
- $1$
- $C_1$
- $1$
- $2$
- $1$
- $\overline{C}_1$
- $A_1 A_1 A_1$

$D_n$:
- $C_1$
- $1$
- $2$
- $\cdots$
- $2$
- $1$
- $\overline{C}_1$
- $D_{n-2} A_1$

$n \geq 5$

$E_6$:
- $C_1$
- $1$
- $2$
- $3$
- $2$
- $1$
- $\overline{C}_1$
- $A_5$

$E_7$:
- $C_1$
- $2$
- $3$
- $4$
- $3$
- $2$
- $1$
- $\overline{C}_1$
- $D_6$

$E_8$:
- $C_1$
- $3$
- $2$
- $4$
- $6$
- $5$
- $4$
- $3$
- $2$
- $\overline{C}_1$
- $E_7$

**Figure 17.**
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