ON EQUIVARIANT COBORDISM CLASSES
OF PROJECTIVE SPACES FOR COMPLEX
U(1)-REPRESENTATION SPACES

Masayoshi KAMATA
(Received 18 August 1997)

1. Introduction

Let \( V \) be an \((n + 1)\)-dimensional complex representation space for the unitary group \( U(1) \). The complex projective space \( P(V) \) is the projective space \( CP^n \) with the \( U(1) \)-action induced from the complex representation \( V \). The aim of this paper is to investigate the condition that the projective spaces \( P(V) \) and \( P(W) \) are cobordant as complex \( U(1) \)-manifolds. Let \( L \) be the one-dimensional canonical complex representation space of \( U(1) \). In this paper we shall prove the following theorem.

**Theorem 1.** Let

\[
V = L^{a_0} \oplus L^{a_1} \oplus \cdots \oplus L^{a_n}
\]

and

\[
W = L^{b_0} \oplus L^{b_1} \oplus \cdots \oplus L^{b_n}.
\]

Then \( P(V) \) is cobordant to \( P(W) \) as complex \( U(1) \)-manifolds if and only if there exists a permutation

\[
\sigma : \{0, 1, \ldots, n\} \rightarrow \{0, 1, \ldots, n\}
\]

such that for some integer \( s \)

\[
a_i = b_{\sigma(i)} + s.
\]

2. A proof of Theorem 1

Let \( G \) be a compact Lie group. Denote by \( \pi : EG \rightarrow BG \) the universal principal bundle for \( G \). For a unitary \( G \)-manifold \( M \) we have the associated bundle

\[
\pi_M : EG \times_G M \rightarrow BG
\]

and the Gysin homomorphism

\[
\pi_M^*: H^*(EG \times_G M; Q) \rightarrow H^*(BG; Q).
\]
Let $\xi$ be an $n$-dimensional smooth complex vector bundle over an orientable closed manifold $M$. The tangent bundle $\tau(P(\xi))$ of the projective space bundle associated with $\xi$ is described as

$$\tau(P(\xi)) \oplus \varepsilon = \overline{\eta_\xi} \otimes \pi^*\xi,$$

where $\varepsilon$ is the trivial complex line bundle over $P(\xi)$, $\eta_\xi$ is the tautological line bundle over $P(\xi)$ and $\pi : P(\xi) \rightarrow M$ is the projection. Denote by $c$ the first Chern class of $\overline{\eta_\xi}$. Then the Chern classes of $\xi$ satisfy

$$c^n + \pi^*(c_1(\xi))c^{n-1} + \pi^*(c_2(\xi))c^{n-2} + \cdots + \pi^*(c_n(\xi)) = 0,$$

and

$$c^{n+k-1} = -\pi^*(c_k(\xi)) + \text{decomposable elements of } \pi^*(c_i(\xi)) c^{n-i}$$

+ the lower terms. \hfill (1)

The canonical line bundle $\xi_m$ over the complex projective space $CP^m$ is the conjugate bundle of the line bundle $S^{2m+1} \times_{U(1)} L \rightarrow CP^m$. Let $V = L^{a_0} \oplus L^{a_1} \oplus \cdots \oplus L^{a_n}$ and let $\xi_V$ be the associated bundle $S^{2m+1} \times_{U(1)} V \rightarrow CP^m$. Here we use the following notation for the first Chern classes:

$$x_m = c_1(\xi_m) \quad \text{and} \quad c_V = c_1(\eta_V),$$

where $\eta_V$ is the tautological line bundle over the associated projective space bundle $P(\xi_V)$. We calculate the Gysin homomorphism

$$\pi_{P(V)\downarrow} : H^*(P(\xi_V)) \rightarrow H^*(CP^m)$$

for the projection $\pi_{P(V)} : P(\xi_V) \rightarrow CP^m$. Denote the $i$th elementary symmetric polynomial of $t_1, t_2, \ldots, t_{n+1}$ by $s_i(t_1, t_2, \ldots, t_{n+1})$. Then we have the following proposition.

**Proposition 2.**

$$\pi_{P(V)\downarrow}(c^{n+k}_V) = [(-1)^{k+1} s_k(a_0, a_1, \ldots, a_n)$$

$$+ \text{decomposable elements of } s_j(a_0, a_1, \ldots, a_n)] x_m^k.$$  

**Proof:** We denote the fundamental class of an orientable closed manifold $M$ by $[M]$. Put

$$\pi_{P(V)\downarrow}(c^{n+k}_V) = \lambda_k x_m^k.$$
Then we use formula (\(*\)) to calculate $\lambda_k$.

\[
\lambda_k = \langle \pi_{P(V)!}(c^{n+k}_V), x^{m-k}_m \cap [CP^m] \rangle \\
= \langle 1, x^{m-k}_m \cap (\pi_{P(V)!}(c^{n+k}_V) \cap [P(\xi_V)]) \rangle \\
= \langle (\pi_{P(V)}^*(x_m))^m_{m-k} c^{n+k}_V, [P(\xi_V)] \rangle \\
= \langle (-1)^{k+1} s_k(a_0, a_1, \ldots, a_n) \rangle \\
\quad + \text{decomposable elements of } s_j \} (\pi_{P(V)}^*(x_m))^m_{m-k} c^{n+k}_V, [P(\xi_V)] \rangle \\
= (-1)^{k+1} s_k(a_0, a_1, \ldots, a_n) + \text{decomposable elements of } s_j(a_0, a_1, \ldots, a_n).
\]

\[\square\]

For a complex $U(1)$-representation space $V$ we consider the associated bundle

\[
\pi_{P(V)}: EU(1) \times_{U(1)} P(V) \rightarrow BU(1)
\]

and the Gysin homomorphism

\[
\pi_{P(V)!}: \lim_{m \rightarrow \infty} H^*(S^{2m+1}) \times_{U(1)} P(V) \cong H^*(EU(1) \times_{U(1)} P(V)) \\
\rightarrow H^*(BU(1)) \cong \lim_{m \rightarrow \infty} H^*(CP^m).
\]

Let us use the same notation as the bundle associated with the principal bundle $S^{2m+1} \rightarrow CP^m$:

\[
c_V = c_1(\eta_V) \quad \text{and} \quad x = c_1(\xi),
\]

where $\eta_V$ is the tautological line bundle over the associated projective space bundle $EU(1) \times_{U(1)} P(V)$ and $\xi$ is the canonical line bundle over $BU(1)$. Then we have the following corollary.

**COROLLARY 3.** Let $V = L^{a_0} \oplus L^{a_1} \oplus \cdots \oplus L^{a_n}$. Then

\[
\pi_{P(V)!}(c^{n+k}_V) = \langle (-1)^{k+1} s_k(a_0, a_1, \ldots, a_n) + \text{decomposable elements of } s_j \rangle x^k.
\]

We take a characteristic formal power series (cf. [1])

\[
f(z) = 1 + q_1 z + q_2 z^2 + \cdots + q_n z^n + \cdots \in \mathbb{Q}{[z]}.
\]

Let

\[
P_f(s_1, s_2, \ldots, s_n) = f(z_1) f(z_2) \cdots f(z_n).
\]

where $s_i$ is the $i$th elementary symmetric polynomial of $z_j$'s. For a complex vector bundle $\xi$ over a space $X$ we put

\[
\Phi_f(\xi) = P_f(c_1(\xi), c_2(\xi), \ldots, c_n(\xi)).
\]
For a unitary $G$-manifold $M$ we have the $G$-equivariant stable tangent bundle $\tau'_M$, and the quotient bundle of the product $EG \times \tau'_M$:

$$EG \times_G \tau'_M : EG \times_G E(\tau'_M) \rightarrow EG \times_G M.$$ 

Let us consider the associated bundle

$$\pi_M : EG \times_G M \rightarrow BG.$$ 

We denote the characteristic class $\Phi_f(EG \times_G \tau'_M)$ by $\Phi_f(M)$. If unitary $G$-manifolds $M$ and $N$ are cobordant, then

$$\pi_{M!}(\Phi_f(M)) = \pi_{N!}(\Phi_f(N)) \in H^*(BG; Q) \quad \text{(**) \quad (cf. [3]).}$$

A proof of Theorem 1

Suppose that $a_0 + a_1 + \cdots + a_n \equiv b_0 + b_1 + \cdots + b_n \mod n + 1$. Then there exists an integer $s$ such that

$$a_0 + a_1 + \cdots + a_n = b_0 + b_1 + \cdots + b_n + (n + 1)s.$$ 

Here we consider the characteristic formal power series

$$f(z) = \exp(z).$$

$P(W')$, $W' = L^{b_0 + s} \oplus L^{b_1 + s} \oplus \cdots \oplus L^{b_n + s}$, is $U(1)$-equivariant diffeomorphic to $P(W)$. Suppose that $P(V)$ and $P(W)$ are $G$-cobordant. Then by (**),

$$\pi_{P(V)!}(\Phi_f(P(V))) = \pi_{P(V)!}(\exp((n + 1)c_V)\pi_{P(V)!}(\exp(-(a_0 + a_1 + \cdots + a_n)x)))$$

coincides with

$$\pi_{P(W')!}(\Phi_f(P(W')))$$

$$= \pi_{P(W')!}(\exp((n + 1)c_{W'})\pi_{P(W')!}(\exp(-(b_0 + b_1 + \cdots + b_n + (n + 1)s)x))).$$

Therefore it follows from Corollary 3 that for any $k$,

$$(-1)^{k+1}s_k(a_0, a_1, \ldots, a_n) + \text{decomposable elements of } s_j(a_0, \ldots, a_n)$$

$$= (-1)^{k+1}s_k(b_0 + s, b_1 + s, \ldots, b_n + s)$$

$$+ \text{decomposable elements of } s_j(b_0 + s, \ldots, b_n + s)$$

and

$$s_k(a_0, a_1, \ldots, a_n) = s_k(b_0 + s, b_1 + s, \ldots, b_n + s).$$
Equivariant cobordism classes of projective spaces

Thus under the assumption that \( a_0 + a_1 + \cdots + a_n \equiv b_0 + b_1 + \cdots + b_n \) modulo \( n + 1 \), if \( P(V) \) is cobordant to \( P(W) \) as complex \( U(1) \)-manifolds then there exists a permutation

\[ \sigma : \{0, 1, \ldots, n\} \rightarrow \{0, 1, \ldots, n\} \]

such that for some integer \( s \),

\[ a_i = b_{\sigma(i)} + s. \]

In the general case, for \( P(V), V = L^{a_0} \oplus L^{a_1} \oplus \cdots \oplus L^{a_n} \), and \( P(W), W = L^{b_0} \oplus L^{b_1} \oplus \cdots \oplus L^{b_n} \), consider complex representations

\[ V' = L^{(n+1)a_0} \oplus L^{(n+1)a_1} \oplus \cdots \oplus L^{(n+1)a_n} \]

and

\[ W' = L^{(n+1)b_0} \oplus L^{(n+1)b_1} \oplus \cdots \oplus L^{(n+1)b_n}. \]

If \( P(V) \) is cobordant to \( P(W) \) as complex \( U(1) \)-manifolds, then \( P(V') \) is cobordant to \( P(W') \) as complex \( U(1) \)-manifolds. Therefore using the above result we complete the proof of Theorem 1.

Acknowledgement. I am grateful to N. Iwase for many discussions about this problem.

REFERENCES


Masayoshi Kamata
Graduate School of Mathematics
Kyushu University 33
Hakozaki Fukuoka 812
Japan