CONTINUITY OF COMMUTATORS ON WEIGHTED HERZ-TYPE SPACES

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1. Introduction

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $E_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Let $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$ be the characteristic function of the set E_k .

Definition 1.1. Let $0 < \alpha < \infty$, $0 < p \leq \infty$, $1 < q < \infty$, w_1 and w_2 be non-negative weight functions.

(a) The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1, w_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(w_1, w_2) \equiv \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}, w_2) : \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \equiv \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \|f \chi_k\|_{L^q(w_2)}^p \right\}^{1/p},$$

and the usual modifications are made when $p = \infty$.

(b) The homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ is defined by

$$H\dot{K}_q^{\alpha,p}(w_1, w_2) \equiv \{f \in \mathcal{S}' : Gf \in \dot{K}_q^{\alpha,p}(w_1, w_2)\},$$

where $\mathcal{S}'$ is the space of Schwartz's tempered distributions and Gf is the grand maximal function of f (see [S2, p. 90] and [LY2] for the definition). Moreover, we define

$$\|f\|_{H\dot{K}_q^{\alpha,p}(w_1, w_2)} \equiv \|Gf\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}.$$

Here and in what follows, for any $r \in (1, \infty)$, any non-negative function w and any Lebesgue measurable function f , we write

$$\|f\|_{L^r(w)} \equiv \left\{ \int_{\mathbb{R}^n} |f(x)|^r w(x) dx \right\}^{1/r}.$$

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If $w \equiv w_1 \equiv w_2 \equiv 1$, we simply denote $\dot{K}_q^{\alpha,p}(w_1, w_2)$, $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ and $L^r(w)$ by $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n)$, respectively.

The weighted Herz space and the weighted Herz-type Hardy space are introduced by Lu and Yang, respectively, in [LY1] and [LY2].

Let T be a standard Calderón–Zygmund singular integral operator and $b \in \text{BMO}(\mathbb{R}^n)$. The commutator $[b, T]$ generated by T and b is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for any suitable function f .

A well-known result of Coifman *et al.* [CRW] states that if T is a standard Calderón–Zygmund operator and $b \in \text{BMO}(\mathbb{R}^n)$, then $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$ for $q \in (1, \infty)$. Lu and Yang [LY4] generalized this result to the case of Herz spaces.

On the other hand, Stein in [S1] obtained the celebrated result that if

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} h(x, y)f(y) dy \quad (1.1)$$

is bounded on $L^q(\mathbb{R}^n)$ with some $q \in (1, \infty)$, and if $h(x, y)$ satisfies the standard size condition

$$|h(x, y)| \leq \frac{c}{|x - y|^n} \quad (1.2)$$

for any $x, y \in \mathbb{R}^n$ and $x \neq y$, then T is also bounded on $L^q(|x|^\alpha)$ for $|x|^\alpha \in A_q$ (the Muckenhoupt weight class) or, equivalently, $\alpha \in (-n, n(q - 1))$. Recently, Soria and Weiss [SW] extended Stein's result to more general cases, where the power weight was replaced by more general weight $w \in A_q$ satisfying

$$\sup_{2^{k-2} < |x| \leq 2^{k+1}} w(x) \leq c_1 \inf_{2^{k-2} < |x| \leq 2^{k+1}} w(x), \quad k \in \mathbb{Z}, \quad (1.3)$$

where c_1 is independent of $k \in \mathbb{Z}$.

Note that $L^q(|x|^\alpha) = \dot{K}_q^{\alpha/q, q}(\mathbb{R}^n)$, a special case of the homogeneous Herz space. Lu and Yang generalized Stein's result and the Soria–Weiss result to the case of commutators on homogeneous Herz spaces in [LY4] and to the case of sublinear operators on homogeneous weighted Herz-type spaces in [LY1] and [LY3].

The main purpose of this paper is to generalize the above results to commutators on homogeneous weighted Herz-type spaces. By the well-known reason (see [LY4]), we shall investigate the continuity of the commutator $[b, T]$ on the homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1, w_2)$ in the case of $0 < \alpha < n(1 - 1/q)$, and on the homogeneous weighted atomic Herz-type Hardy space $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ associated with the function b in the case of $\alpha \geq n(1 - 1/q)$. The space $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ is defined as follows.

Definition 1.2. Let $\alpha \in \mathbb{R}$, $s \in \mathbb{N} \cup \{0\}$, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and $b \in L_{\text{loc}}^{q'}(\mathbb{R}^n)$, $w_1 \in A_1$ and w_2 be a non-negative weight function on $\mathbb{R}^n$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q, s; b; w_1, w_2)$ -atom if:

- (i) $\text{supp } a \subset B(0, r)$ for some $r > 0$, where $B(0, r) = \{y \in \mathbb{R}^n : |y| \leq r\}$;
- (ii) $\|a\|_{L^q(w_2)} \leq [w_1(B(0, r))]^{-\alpha/n}$; and
- (iii) $\int_{\mathbb{R}^n} x^\beta a(x) dx = \int_{\mathbb{R}^n} x^\beta a(x) b(x) dx = 0$ for $|\beta| \leq s$,
where $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ and $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$.

Definition 1.3. Let $\alpha \in \mathbb{R}$, $s \in \mathbb{N} \cup \{0\}$, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and $b \in L_{\text{loc}}^{q'}(\mathbb{R}^n)$, $0 < p \leq \infty$, $w_1 \in A_1$ and w_2 be a non-negative weight function on $\mathbb{R}^n$. A tempered distribution f is said to belong to $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ if, in the $\mathcal{S}'(\mathbb{R}^n)$ sense, it can be written as

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j,$$

where a_j is a central $(\alpha, q, s; b; w_1, w_2)$ -atom supported on B_j , $\lambda_j \in \mathbb{C}$, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$.

Moreover, we define on $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ the quasinorm

$$\|f\|_{H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)} \equiv \inf \left\{ \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all decompositions of f and the usual modifications are made when $p = \infty$.

Obviously, if $1 < q < \infty$, $\alpha \geq n(1 - 1/q)$, $0 < p \leq \infty$, $s \geq [\alpha + n(1/q - 1)]$, and $b = 1$, then $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2) = H\dot{K}_q^{\alpha,p}(w_1, w_2)$, which was studied by Lu and Yang in [LY2] and [LY3].

In Section 2 of this paper, we shall investigate the boundedness on homogeneous weighted Herz spaces for a large class of the commutators related to the Hardy–Littlewood maximal function, the standard Calderón–Zygmund operator and so on. We now state one of our main results.

THEOREM 1.1. Let $0 < p \leq \infty$, $1 < q < \infty$, $0 < \alpha < n(1 - 1/q)$, $w_1, w_2 \in A_1$ and w_2 satisfy (1.3). Let $b \in \text{BMO}(\mathbb{R}^n)$ and the linear operator T satisfy the size condition

$$|Tf(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy \quad (1.4)$$

for any compactly supported function f and any $x \notin \text{supp } f$. Suppose that the commutator $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$. Then, $[b, T]$ is also bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)$.

If we choose $w_1(x) \equiv 1$ and $w_2(x) = |x|^{-\alpha}$ with $\alpha \in [0, n)$ in Theorem 1.1, we then re-obtain Corollary 2.1 in [LY4]. Thus, if $\alpha \geq n(1 - 1/q)$ or $\alpha \leq 0$, Theorem 1.1 may be false; see [LY4]. That is, in some sense, the range of α in our theorems of this section is best possible under the conditions of the theorems.

We point out that the size condition (1.4) is satisfied by many operators in harmonic analysis, such as Calderón–Zygmund operators, C. Fefferman’s singular multiplier, R. Fefferman’s singular integral, Ricci–Stein’s oscillatory singular integral, the Bochner–Riesz operator at the critical index, and so on. For example, as a special case of Theorem 1.1, we have the following corollaries.

COROLLARY 1.1. *Let $0 < p \leq \infty$, $1 < q < \infty$ and $0 < \alpha < n(1 - 1/q)$. Let $b \in \text{BMO}(\mathbb{R}^n)$ and T be the standard Calderón–Zygmund operator. Suppose that $w_1, w_2 \in A_1$ and w_2 satisfies (1.3). Then, $[b, T]$ is also bounded on the Herz spaces $\dot{K}_q^{\alpha,p}(w_1, w_2)$.*

COROLLARY 1.2. *Let $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < n(1 - 1/q)$. Let $b \in \text{BMO}(\mathbb{R}^n)$, h be a standard Calderón–Zygmund kernel, and T be defined by*

$$Tf(x) \equiv \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} h(x-y) f(y) dy,$$

where $P(x, y)$ is a real polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Suppose that $w_1, w_2 \in A_1$ and w_2 satisfies (1.3). Then $[b, T]$ is bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)$ with the operator norm independent of the coefficients of P .

As we have already pointed out, we cannot expect that the commutator $[b, T]$ related to the Calderón–Zygmund operator maps continuously $\dot{K}_q^{\alpha,p}(w_1, w_2)$ into itself in the case when $\alpha \geq n(1 - 1/q)$. However, if we replace $\dot{K}_q^{\alpha,p}(w_1, w_2)$ by the spaces $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ defined as above, then the commutator $[b, T]$ does map continuously $H\dot{K}_{q,b}^{\alpha,p,s}(w_1, w_2)$ into $\dot{K}_q^{\alpha,p}(w_1, w_2)$, which will be proved in Section 3. That is, in Section 3 of this paper, we will discuss the boundedness on the above homogeneous weighted atomic Herz-type Hardy spaces of the commutators related to the local Calderón–Zygmund type operator, the Bochner–Riesz operator and so on.

On the fractional case, we also obtain some theorems parallel to the non-fractional case. We will only state these theorems and omit their proofs which are similar to those of the theorems on the non-fractional case.

For non-homogeneous spaces, we have similar results to all the theorems obtained in this paper for homogeneous spaces; and we omit the details. There are also similar theorems for the maximal commutators associated with a sequence of commutators. To limit the length of this paper, we omit the details; see [SW] and [LY4].

Throughout this paper, c , c_0 and c_1 always mean constants independent of the main parameters involved, but whose values may vary from line to line. For any power exponent $r \in [1, \infty]$, we denote the conjugate one $r/(r-1)$ by r' . Also, the properties of the $BMO(\mathbb{R}^n)$ functions are referred to [S2].

2. $(\dot{K}_q^{\alpha,p}(w_1, w_2), \dot{K}_q^{\alpha,p}(w_1, w_2))$ -type continuity

We begin this section with the boundedness of the Hardy–Littlewood maximal commutators relative to b on weighted Herz spaces $\dot{K}_q^{\alpha,p}(w_1, w_2)$, which will be used in the proofs of our main theorems, and which themselves have independent interest.

Definition 2.1. Let $b \in BMO(\mathbb{R}^n)$ and $l \in (0, n)$. The maximal commutator M_b is defined by

$$M_b(f)(x) \equiv \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |b(x) - b(y)| \cdot |f(y)| dy,$$

where $B(x, r) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$, and the fractional maximal commutator M_b^l is defined by

$$M_b^l(f)(x) \equiv \sup_{r>0} \frac{1}{r^{n-l}} \int_{B(x,r)} |b(x) - b(y)| \cdot |f(y)| dy.$$

We need the following decompositional characterization of $\dot{K}_q^{\alpha,p}(w_1, w_2)$ in [LY1].

Definition 2.2. Let $0 < \alpha < \infty$, $0 < p \leq \infty$, $1 \leq q < \infty$, w_1 , and w_2 be non-negative weight functions on $\mathbb{R}^n$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q; w_1, w_2)$ -block if:

- (i) $\text{supp } a \subset B(0, r)$, $r > 0$;
- (ii) $\|a\|_{L^q(w_2)} \leq [w_1(B(0, r))]^{-\alpha/n}$.

LEMMA 2.1. Let $0 < \alpha < \infty$, $0 < p \leq \infty$, $1 \leq q < \infty$, $w_1 \in A_1$, and w_2 be a non-negative weight function on $\mathbb{R}^n$. Then the following two facts are equivalent:

- (i) $f \in \dot{K}_q^{\alpha,p}(w_1, w_2)$;

(ii) f can be represented by

$$f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x),$$

where a_j is a central $(\alpha, q; w_1, w_2)$ -block with the support B_j , and $\sum_{j \in \mathbb{Z}} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \sim \inf \left\{ \left(\sum_{j \in \mathbb{Z}} |\lambda_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all the above decompositions of f .

THEOREM 2.1. Let $1 < q < \infty$, $0 < \alpha < n(1 - 1/q)$, and $0 < p \leq \infty$. Let $b \in \text{BMO}(\mathbb{R}^n)$ and M_b be defined as in Definition 2.1. Suppose that $w_1, w_2 \in A_1$ and w_2 satisfies (1.3). Then M_b is bounded on the Herz space $\dot{K}_q^{\alpha,p}(w_1, w_2)$.

Proof. We consider two cases of p . When $0 < p \leq 1$, let a_j be a central $(\alpha, q; w_1, w_2)$ -block supported on B_j . In this case, it suffices to show $\|M_b(a_j)\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}^p \leq c$, where c is independent of j . Write

$$\begin{aligned} \|M_b(a_j)\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}^p &= \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|\chi_k M_b(a_j)\|_{L^q(w_2)}^p \\ &\quad + \sum_{k=j+3}^{\infty} [w_1(B_k)]^{\alpha p/n} \|\chi_k M_b(a_j)\|_{L^q(w_2)}^p \\ &\equiv D_1 + D_2. \end{aligned}$$

For D_1 , using the $L^q(w_2)$ -boundedness of M_b (see [GHST]) and $w_1 \in A_1$, we then obtain

$$\begin{aligned} D_1 &\leq c \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|a_j\|_{L^q(w_2)}^p \\ &\leq c \sum_{k=-\infty}^{j+2} \left[\frac{w_1(B_k)}{w_1(B_j)} \right]^{\alpha p/n} \\ &\leq c + c \sum_{k=-\infty}^{j-1} 2^{(k-j)\alpha p \delta/n} \\ &\leq c, \end{aligned}$$

where $\delta > 0$ such that $w_1(B_k)/w_1(B_j) < c[|B_k|/|B_j|]^\delta$ with c and δ independent of k and j ; see [GR, p. 401].

To estimate D_2 , we first compute $\|\chi_k M_b(a_j)\|_{L^q(w_2)}$ for $k \geq j + 3$. Let b_j be the mean value of b on the ball B_j . By $b \in \text{BMO}(\mathbb{R}^n)$, $w_2 \in A_1$ and (1.3), for $x \in E_k$, $y \in B_j$ and $k \geq j + 3$, we have

$$\begin{aligned}
& \|\chi_k M_b(a_j)\|_{L^q(w_2)} \\
& \leq c2^{-kn} \left\{ \int_{E_k} \left[\int_{B_j} |b(x) - b(y)| \cdot |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
& \leq c2^{-kn} \left\{ \int_{E_k} \left[|b(x) - b_j| \int_{B_j} |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
& \quad + c2^{-kn} \left\{ \int_{E_k} \left[\int_{B_j} |b_j - b(y)| \cdot |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
& \leq c2^{-kn} \left[\sup_{x \in E_k} w_2(x) \right]^{1/q} \left[\int_{B_j} |a_j(y)| dy \right] \left[\int_{E_k} |b(x) - b_j|^q dx \right]^{1/q} \\
& \quad + c2^{-kn} \left[\sup_{x \in E_k} w_2(x) \right]^{1/q} |E_k|^{1/q} \int_{B_j} |b(y) - b_j| \cdot |a_j(y)| dy \\
& \leq c2^{-kn} \left[\inf_{x \in E_k} w_2(x) \right]^{1/q} |B_j|^{1-1/q} \|a_j\|_{L^q(\mathbb{R}^n)} (k-j) |E_k|^{1/q} \|b\|_{\text{BMO}(\mathbb{R}^n)} \\
& \quad + c2^{-kn} \left[\inf_{x \in E_k} w_2(x) \right]^{1/q} |E_k|^{1/q} \left(\int_{B_j} |b(y) - b_j|^{q'} dy \right)^{1/q'} \|a_j\|_{L^q(\mathbb{R}^n)} \\
& \leq c2^{(j-k)n(1-1/q)} \left[\frac{w_2(B_k)}{|B_k|} \right]^{1/q} \|a_j\|_{L^q(\mathbb{R}^n)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \\
& \quad + c2^{-kn} \left[\frac{w_2(B_k)}{|B_k|} \right]^{1/q} |E_k|^{1/q} |B_j|^{1/q'} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(\mathbb{R}^n)} \\
& \leq c2^{(j-k)n(1-1/q)} (k-j) \left[\frac{w_2(B_j)}{|B_j|} \right]^{1/q} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(\mathbb{R}^n)} \\
& \quad + c2^{(j-k)n(1-1/q)} \left[\frac{w_2(B_j)}{|B_j|} \right]^{1/q} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(\mathbb{R}^n)} \\
& \leq c2^{(j-k)n(1-1/q)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)} \\
& \quad + c2^{(j-k)n(1-1/q)} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)}.
\end{aligned}$$

Thus, we obtain

$$\|\chi_k M_b(a_j)\|_{L^q(w_2)} \leq c 2^{(j-k)n(1-1/q)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} [w_1(B_j)]^{-\alpha/n}. \quad (2.1)$$

By substituting (2.1) into D_2 and $\alpha < n(1-1/q)$, we obtain

$$D_2 \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)} \sum_{k=j+3}^{\infty} (k-j)^p 2^{(k-j)(\alpha-n(1-1/q))p} \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)}.$$

Now let $1 < p < \infty$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a central $(\alpha, q; w_1, w_2)$ -block supported on B_j . By Minkowski's inequality, we obtain

$$\begin{aligned} & \|M_b(f)\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \\ & \leq \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|\chi_k M_b(a_j)\|_{L^q(w_2)} \right)^p \right\}^{1/p} \\ & \quad + \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\chi_k M_b(a_j)\|_{L^q(w_2)} \right)^p \right\}^{1/p} \\ & \equiv D_3 + D_4. \end{aligned}$$

For D_3 , by the $L^q(w_2)$ -boundedness of M_b and $w_1 \in A_1$, using Hölder's inequality and $\alpha > 0$, we have

$$\begin{aligned} D_3 & \leq \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=k-2}^{\infty} |\lambda_j|^p \|\chi_k M_b(a_j)\|_{L^q(w_2)}^{p/2} \right) \right. \\ & \quad \times \left. \left(\sum_{j=k-2}^{\infty} \|\chi_k M_b(a_j)\|_{L^q(w_2)}^{p'/2} \right)^{p/p'} \right\}^{1/p} \\ & \leq c \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{j=k-2}^{\infty} |\lambda_j|^p \left(\frac{w_1(B_k)}{w_1(B_j)} \right)^{\alpha p/(2n)} \right] \left[\sum_{j=k-2}^{\infty} \left(\frac{w_1(B_k)}{w_1(B_j)} \right)^{\alpha p'/(2n)} \right]^{p/p'} \right\}^{1/p} \\ & \leq c \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{j=k-2}^{\infty} |\lambda_j|^p \left(\frac{w_1(B_k)}{w_1(B_j)} \right)^{\alpha p/(2n)} \right] \left[c + \sum_{j=k+1}^{\infty} 2^{(k-j)\alpha \delta p'/2} \right]^{p/p'} \right\}^{1/p} \\ & \leq c \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(c + \sum_{k=-\infty}^{j-1} 2^{(k-j)\alpha \delta p/2} \right) \right\}^{1/p} \\ & \leq c \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \right\}^{1/p}, \end{aligned}$$

where $\delta > 0$ depends only on w_1 ; see [GR, p. 401].

For D_4 , by $w_1 \in A_1$ and (1.3), using (2.1) and $\alpha < n(1 - 1/q)$, we have

$$\begin{aligned} D_4 &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{j=-\infty}^{k-3} |\lambda_j| (k-j) 2^{(k-j)[\alpha-n(1-1/q)]} \right]^p \right\}^{1/p} \\ &\leq c \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{(k-j)[\alpha-n(1-1/q)]p/2} \right] \right. \\ &\quad \times \left. \left[\sum_{j=-\infty}^{k-3} (k-j)^{p'} 2^{(k-j)[\alpha-n(1-1/q)]p'/2} \right]^{p/p'} \right\}^{1/p} \\ &\leq c \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \right\}^{1/p}. \end{aligned}$$

The case $p = \infty$ is easy and we omit the details.

This finishes the proof of Theorem 2.1. $\square$

Now let us state one of our main theorems.

THEOREM 2.2. *Let p, q, α and w_1, w_2 be as in Theorem 2.1. Let $b \in \text{BMO}(\mathbb{R}^n)$ and let the linear operator T satisfy the local size condition*

$$|Tf(x)| \leq \frac{c_0}{|x|^n} \int_{\mathbb{R}^n} |f(y)| dy, \quad \text{if } \text{dist}(x, \text{supp } f) > |x|/2 \quad (2.2)$$

for any compactly supported function $f \in L^1(\mathbb{R}^n)$. Suppose that the commutator $[b, T]$ is bounded on $L^q(w_2)$. Then, $[b, T]$ is also bounded on the Herz space $\dot{K}_q^{\alpha, p}(w_1, w_2)$.

Proof. We consider two cases of p . When $0 < p \leq 1$, let $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a central $(\alpha, q; w_1, w_2)$ -block supported on B_j . Then,

$$\begin{aligned} \|[b, T](f)\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)}^p &\leq \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \|\chi_k [b, T](a_j)\|_{L^q(w_2)}^p \right) \\ &= \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|\chi_k [b, T](a_j)\|_{L^q(w_2)}^p \right) \\ &\quad + \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=j+3}^{\infty} [w_1(B_k)]^{\alpha p/n} \|\chi_k [b, T](a_j)\|_{L^q(w_2)}^p \right) \\ &\equiv J_1 + J_2. \end{aligned}$$

For J_1 , using the $L^q(w_2)$ -boundedness of $[b, T]$ and $w_1 \in A_1$, we then obtain

$$\begin{aligned}
 J_1 &\leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|a_j\|_{L^q(w_2)}^p \right) \\
 &\leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} \left[\frac{w_1(B_k)}{w_1(B_j)} \right]^{\alpha p/n} \right) \\
 &\leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(c + c \sum_{k=-\infty}^{j-1} 2^{(k-j)\alpha p\delta/n} \right) \\
 &\leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p,
 \end{aligned}$$

where $\delta > 0$ depends only on w_1 ; see [GR, p. 401].

For J_2 , noting that $x \in E_k$ and $k \geq j + 3$ and using (2.2), we then obtain

$$|[b, T](a_j)(x)| \leq \frac{c}{|x|^n} \int_{|x-y| \leq 5|x|/4} |b(x) - b(y)| \cdot |a_j(y)| dy \leq c M_b(a_j)(x).$$

Thus, for $x \in E_k$ and $k \geq j + 3$, we have

$$\|\chi_k[b, T](a_j)\|_{L^q(w_2)} \leq c \|\chi_k M_b(a_j)\|_{L^q(w_2)}. \quad (2.3)$$

Therefore,

$$J_2 \leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p D_2 \leq c \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

When $1 < p < \infty$, by Minkowski's inequality, we have

$$\begin{aligned}
 &\|[b, T](f)\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \\
 &\leq \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \cdot \|\chi_k[b, T](a_j)\|_{L^q(w_2)} \right)^p \right\}^{1/p} \\
 &\quad + \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \cdot \|\chi_k[b, T](a_j)\|_{L^q(w_2)} \right)^p \right\}^{1/p} \\
 &\equiv J_3 + J_4.
 \end{aligned}$$

Applying the $L^q(w_2)$ -boundedness of $[b, T]$ and (2.3), respectively, to the estimates for J_3 and J_4 and by some computational techniques similar to the estimates for D_3

and D_4 , we can obtain

$$J_3 + J_4 \leq c \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \right\}^{1/p}.$$

We omit the proof of the case $p = \infty$.

This finishes the proof of Theorem 2.2. $\square$

Remark 2.1. By the Alvertz–Babgy–Kurtz–Pérez theorem (see Theorem 2.13 in [ABKP]), in Theorem 2.2 the condition that the commutator $[b, T]$ is bounded on $L^q(w_2)$ can be replaced by the condition that T is bounded on $L^q(w)$ for all $w \in A_1$.

From Theorem 1.4 in [LY4] and Theorem 2.2, we can easily deduce Theorem 1.1, which is convenient in applications.

There are some theorems relative to the fractional cases which are parallel to the above theorems. To state them, let us first introduce the weight function class $A(q_1, q_2)$; see [ST].

Definition 2.3. A non-negative and locally integrable function w on $\mathbb{R}^n$ is said to belong to $A(q_1, q_2)$ with $1 \leq q_1, q_2 \leq \infty$ if

$$\left(\frac{1}{|B|} \int_B [w(x)]^{-q'_1} dx \right)^{1/q'_1} \left(\frac{1}{|B|} \int_B [w(x)]^{q_2} dx \right)^{1/q_2} \leq c_0 \quad (2.4)$$

holds for any ball B with some positive constant c_0 . The smallest constant c satisfying (2.4) will be called the constant of w in the class $A(q_1, q_2)$.

On the boundedness of M_b^l we now have the following.

THEOREM 2.3. Let $b \in \text{BMO}(\mathbb{R}^n)$, $l \in (0, n)$, and M_b^l be defined as in Definition 2.1. Let $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $0 < \alpha < n(1 - 1/q_1)$ and $0 < p \leq \infty$. If $w_1, w_2 \in A_1$, and $w_2 \in A(q_1, q_2)$ satisfies (1.3), then M_b^l is bounded from $\dot{K}_{q_1}^{\alpha, p}(w_1, w_2^{q_1})$ to $\dot{K}_{q_2}^{\alpha, p}(w_1, w_2^{q_2})$.

The proof of this theorem is parallel to that of Theorem 2.1 by noting the continuity of M_b^l from $L^{q_1}(w_2^{q_1})$ to $L^{q_2}(w_2^{q_2})$ (see [ST]); we omit the details.

On the fractional commutator, we have the following theorem which is parallel to Theorem 1.1.

THEOREM 2.4. Let $b \in \text{BMO}(\mathbb{R}^n)$, $l \in (0, n)$, and T^l satisfy the size condition

$$|T^l f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-l}} dy$$

for any compactly supported function f and any $x \notin \text{supp } f$. Let q_1, q_2, α, p, w_1 and w_2 be as in Theorem 2.3. If $[b, T^l]$ is bounded from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, then $[b, T^l]$ is also bounded from $\dot{K}_{q_1}^{\alpha, p}(w_1, w_2^{q_1})$ to $\dot{K}_{q_2}^{\alpha, p}(w_1, w_2^{q_2})$.

By Theorem 1.4 in [LY4], we know that $[b, T^l]$ is bounded from $L^{q_1}(w_2^{q_1})$ to $L^{q_2}(w_2^{q_2})$. Using this fact, we can prove Theorem 2.4 by a similar procedure to that of Theorem 2.2. We omit the details here.

Finally, we point out that if we choose $w_1 \equiv 1$ and $w_2(x) = |x|^\alpha$, respectively, in Theorems 2.1, 2.3 and 2.4, then we re-obtain Theorems 2.1, 2.2 and Corollary 2.4 in [LY4]. Therefore, as we pointed out in the introduction, the range of α in Theorems 2.3 and 2.4 is best possible under the conditions of the theorems.

3. $(H\dot{K}_{q,b}^{\alpha, p, s}(w_1, w_2), \dot{K}_q^{\alpha, p}(w_1, w_2))$ -type continuity

Let us now consider the continuity of commutators from the weighted atomic Herz-type Hardy spaces to weighted Herz spaces.

THEOREM 3.1. *Let $b \in \text{BMO}(\mathbb{R}^n)$ and w_1, w_2 be as in Theorem 2.1. Let T be a standard Calderón–Zygmund operator, that is,*

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} h(x, y) f(y) dy$$

with the kernel $h(x, y)$ satisfying

$$|h(x, y) - h(x, z)| \leq c \frac{|y - z|^\varepsilon}{|x - z|^{n+\varepsilon}} \quad (3.1)$$

for some $\varepsilon \in (0, 1]$ and $2|y - z| < |x - z|$. If $1 < q < \infty, n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \varepsilon$, and $0 < p \leq \infty$, then the commutator $[b, T]$ maps $H\dot{K}_{q,b}^{\alpha, p, 0}(w_1, w_2)$ continuously into $\dot{K}_q^{\alpha, p}(w_1, w_2)$.

Proof. Similar to the proof of Theorem 2.1, we consider two cases of p . When $0 < p \leq 1$, let a_j be a central $(\alpha, q, 0; b; w_1, w_2)$ -atom supported on B_j . In this case, it suffices to show $\|[b, T](a_j)\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)}^p \leq c$, where c is independent of j .

Write

$$\begin{aligned} \|[b, T](a_j)\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)}^p &= \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|\chi_k [b, T](a_j)\|_{L^q(w_2)}^p \\ &\quad + \sum_{k=j+3}^{\infty} [w_1(B_k)]^{\alpha p/n} \|\chi_k [b, T](a_j)\|_{L^q(w_2)}^p \\ &\equiv I_1 + I_2. \end{aligned}$$

For I_1 , using the $L^q(\mathbb{R}^n)$ -boundedness of $[b, T]$ (see [CRW] and Theorem 1.4 in [LY4]), we know that $[b, T]$ is bounded on $L^q(w_2)$. By this and $w_1 \in A_1$, we then obtain

$$\begin{aligned} I_1 &\leq c \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|a_j\|_{L^q(w_2)}^p \\ &\leq c \sum_{k=-\infty}^{j+2} \left[\frac{w_1(B_k)}{w_1(B_j)} \right]^{\alpha p/n} \\ &\leq c + c \sum_{k=-\infty}^{j-1} 2^{(k-j)\alpha p \delta/n} \\ &\leq c, \end{aligned}$$

where $\delta > 0$ is as in the proof of Theorem 2.1.

To estimate I_2 , we first compute $\|\chi_k [b, T](a_j)\|_{L^q(w_2)}$ for $k \geq j+3$. Let b_j be the mean value of b on the ball B_j . Note that if $x \in E_k$, $y \in B_j$ and $k \geq j+3$, then $2|y| < |x|$. By the vanishing moments of a_j , $b \in \text{BMO}(\mathbb{R}^n)$, $w_2 \in A_1$ satisfying (1.3), the Hölder inequality and (3.1), we have

$$\begin{aligned} &\|\chi_k [b, T](a_j)\|_{L^q(w_2)} \\ &\leq \|\chi_k (b - b_j) T(a_j)\|_{L^q(w_2)} + \|\chi_k T((b - b_j) a_j)\|_{L^q(w_2)} \\ &\leq \left\{ \int_{E_k} \left[|b(x) - b_j| \int_{B_j} |h(x, y) - h(x, 0)| \cdot |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\ &\quad + \left\{ \int_{E_k} \left[\int_{B_j} |b_j - b(y)| \cdot |h(x, y) - h(x, 0)| \cdot |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \end{aligned}$$

$$\begin{aligned}
&\leq c \left\{ \int_{E_k} \left[|b(x) - b_j| \int_{B_j} \frac{|y|^\varepsilon}{|x|^{n+\varepsilon}} |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
&\quad + c \left\{ \int_{E_k} \left[\int_{B_j} |b_j - b(y)| \frac{|y|^\varepsilon}{|x|^{n+\varepsilon}} |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
&\leq c 2^{(j-k)\varepsilon - kn} \left\{ \int_{E_k} \left[|b(x) - b_j| \int_{B_j} |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
&\quad + c 2^{(j-k)\varepsilon - kn} \left\{ \int_{E_k} \left[\int_{B_j} |b_j - b(y)| \cdot |a_j(y)| dy \right]^q w_2(x) dx \right\}^{1/q} \\
&\leq c 2^{(j-k)[n(1-1/q)+\varepsilon]} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)} \\
&\quad + c 2^{(j-k)[n(1-1/q)+\varepsilon]} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)},
\end{aligned}$$

where we have used some computational techniques as we did in the proof of (2.1).

Thus, we obtain

$$\|\chi_k[b, T](a_j)\|_{L^q(w_2)} \leq c 2^{(j-k)[n(1-1/q)+\varepsilon]} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} [w_1(B_j)]^{-\alpha/n}. \quad (3.2)$$

By substituting (3.2) into I_2 and $\alpha < n(1 - 1/q) + \varepsilon$, we obtain

$$I_2 \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)} \sum_{k=j+3}^{\infty} (k-j)^p 2^{(k-j)[\alpha - n(1-1/q) - \varepsilon]p} \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)}.$$

In the case when $1 < p < \infty$, let $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where a_j is a central $(\alpha, q, 0; b; w_1, w_2)$ -atom supported on B_j . By Minkowski's inequality, we obtain

$$\begin{aligned}
&\|[b, T](f)\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)} \\
&\leq \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left[\sum_{j=k-2}^{\infty} |\lambda_j| \|\chi_k[b, T](a_j)\|_{L^q(w_2)} \right]^p \right\}^{1/p} \\
&\quad + \left\{ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{\alpha p/n} \left[\sum_{j=-\infty}^{k-3} |\lambda_j| \|\chi_k[b, T](a_j)\|_{L^q(w_2)} \right]^p \right\}^{1/p} \\
&\equiv I_3 + I_4.
\end{aligned}$$

By (3.1) and $0 < n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \varepsilon$, using the $L^q(w_2)$ -boundedness of $[b, T]$ and (3.2), respectively, for I_3 and I_4 , we can obtain

$$I_3 + I_4 \leq c \left\{ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \right\}^{1/p};$$

see the estimates of D_3 and D_4 for some details.

The proof of the case $p = \infty$ is easy and we omit the details.

This finishes the proof of Theorem 3.1. $\square$

Remark 3.1. In the proof of Theorem 3.1, if we replace (3.1) by the local condition

$$|h(x, y) - h(x, 0)| \leq c \frac{|y|^\varepsilon}{|x|^{n+\varepsilon}}$$

for $\varepsilon \in (0, 1]$ and $2|y| < |x|$, then we can easily see that Theorem 3.1 still holds if we suppose the corresponding operator T is bounded on $L^q(w_2)$.

If we assume more regularity on the kernel $h(x, y)$ in Theorem 3.1, we can extend Theorem 3.1 to a larger range of α and s ; see the following theorem.

THEOREM 3.2. *Let $l \in (0, n)$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $\alpha \geq n(1 - 1/q_1)$ and $0 < p \leq \infty$. Let*

$$I_l f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-l}} dy$$

and $w_1, w_2 \in A_1$ be as in Theorem 2.3. If $b \in \text{BMO}(\mathbb{R}^n)$ and $s+1 > \alpha + n(1/q_1 - 1)$, then $[b, I_l]$ maps $H\dot{K}_{q_1, b}^{\alpha, p, s}(w_1, w_2^{q_1})$ continuously into $\dot{K}_{q_2, b}^{\alpha, p, s}(w_1, w_2^{q_2})$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1; we omit the details (also see Theorem 3.2 in [LY4]).

Finally, we study the continuity of the maximal commutators of the Bochner–Riesz operators with BMO functions. Let $b \in \text{BMO}(\mathbb{R}^n)$. We define the maximal operator $B_{*, b}^\delta$ associated with the commutators of the Bochner–Riesz operators by

$$B_{*, b}^\delta f(x) = \sup_{r>0} |[b, B_r^\delta]f(x)|,$$

where $(\widehat{B_r^\delta f})(\xi) = (1 - r^2|\xi|^2)_+^\delta \widehat{f}(\xi)$ and $\widehat{f}$ means the Fourier transform of f . We also set

$$B_*^\delta f(x) = \sup_{r>0} |B_r^\delta f(x)|.$$

THEOREM 3.3. *Let $1 < q < \infty$, $\alpha \geq n(1 - 1/q)$, $\delta > \alpha + n/q - (n+1)/2$ and $0 < p \leq \infty$. Let $b \in \text{BMO}(\mathbb{R}^n)$ and w_1, w_2 be as in Theorem 2.1. If $s+1 > \alpha + n(1/q - 1)$, then $B_{*, b}^\delta$ maps $H\dot{K}_{q, b}^{\alpha, p, s}(w_1, w_2)$ continuously into $\dot{K}_q^{\alpha, p}(w_1, w_2)$.*

Proof. We consider two cases of p and omit the case $p = \infty$. When $0 < p \leq 1$, let a_j be a central $(\alpha, q, s; b; w_1, w_2)$ -atom supported on B_j . In this case, it suffices to show

$$\|B_{*,b}^\delta(a_j)\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p \leq c,$$

where c is independent of j . To do so, we write

$$\begin{aligned} \|B_{*,b}^\delta(a_j)\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}^p &= \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|\chi_k B_{*,b}^\delta(a_j)\|_{L^q(w_2)}^p \\ &\quad + \sum_{k=j+3}^{\infty} [w_1(B_k)]^{\alpha p/n} \|\chi_k B_{*,b}^\delta(a_j)\|_{L^q(w_2)}^p \\ &\equiv G_1 + G_2. \end{aligned}$$

For G_1 , when $\alpha \geq n(1-1/q)$ and $\delta > \alpha + n/q - (n+1)/2 \geq (n-1)/2$, using the $L^q(w_2)$ -boundedness of $B_{*,b}^\delta$, which is deduced from [SS] and [ABKP], and $w_1 \in A_1$, we then obtain

$$\begin{aligned} G_1 &\leq c \sum_{k=-\infty}^{j+2} [w_1(B_k)]^{\alpha p/n} \|a_j\|_{L^q(w_2)}^p \\ &\leq c \sum_{k=-\infty}^{j+2} \left[\frac{w_1(B_k)}{w_1(B_j)} \right]^{\alpha p/n} \\ &\leq c + c \sum_{k=-\infty}^{j-1} 2^{(k-j)\alpha p\delta/n} \\ &\leq c, \end{aligned}$$

where $\delta > 0$ is as in the proof of Theorem 2.1.

To estimate G_2 , we shall compute $\|\chi_k B_{*,b}^\delta(a_j)\|_{L^q(w_2)}$ for $k \geq j+3$. Let b_j be the mean value of b on the ball B_j . Then,

$$\begin{aligned} \|\chi_k B_{*,b}^\delta(a_j)\|_{L^q(w_2)} &\leq \|\chi_k(b - b_j)B_{*,b}^\delta(a_j)\|_{L^q(w_2)} \\ &\quad + \|\chi_k B_{*,b}^\delta((b - b_j)a_j)\|_{L^q(w_2)} \\ &\equiv L_1 + L_2. \end{aligned}$$

Let $\max\{(n-1)/2, \alpha + n/q - (n+1)/2\} < \delta_0 < \min\{\delta, (n+1)/2 + s\}$. We consider the following two cases.

Case 1. $0 < r \leq 2^j$. In this case, note that $|x| > 2|y|$ and $|B^\delta(z)| \leq c(1 + |z|)^{-(\delta+(n+1)/2)}$. From Hölder's inequality, $w_2 \in A_1$ and (1.3), it follows that

$$\begin{aligned} L_1 &\equiv \left[\int_{E_k} |b(x) - b_j|^q \left| \int_{B_j} a_j(y) B^\delta \left(\frac{x-y}{r} \right) dy \right|^q w_2(x) dx \right]^{1/q} \\ &\leq cr^n \left\{ \int_{E_k} |b(x) - b_j|^q \left[\int_{B_j} |a_j(y)|^q dy \right] \right. \\ &\quad \times \left. \left[\int_{B_j} \left| B^\delta \left(\frac{x-y}{r} \right) \right|^{q'} dy \right]^{q/q'} w_2(x) dx \right\}^{1/q} \\ &\leq c2^{jn-k(\delta_0+(n+1)/2)} r^{\delta_0+(n+1)/2} (k-j) \left[\frac{w_2(B_k)}{w_2(B_j)} \right]^{1/q} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)} \\ &\leq c2^{(j-k)(\delta_0+(n+1)/2-n/q)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)}, \end{aligned}$$

and

$$\begin{aligned} L_2 &\equiv \left\{ \int_{E_k} \left| \int_{B_j} (b_j - b(y)) a_j(y) B^\delta \left(\frac{x-y}{r} \right) dy \right|^q w_2(x) dx \right\}^{1/q} \\ &\leq c2^{jn-k(\delta_0+(n+1)/2)} r^{\delta_0+(n+1)/2} \left[\frac{w_2(B_k)}{w_2(B_j)} \right]^{1/q} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)} \\ &\leq c2^{(j-k)(\delta_0+(n+1)/2-n/q)} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)}, \end{aligned}$$

where c is independent of j and r .

Case 2. $r > 2^j$. In this case, by the vanishing moments of a_j and the s -order Taylor expansion of $B^\delta[(x-y)/2]$ at x/r , we obtain

$$\begin{aligned} L_1 &\leq cr^{-n-s-1} \left\{ \int_{E_k} |b(x) - b_j|^q \left[\int_{B_j} |a_j(y)|^q dy \right] \right. \\ &\quad \times \left. \left[\int_{B_j} \left(\sum_{|\beta|=s+1} \left| \nabla^\beta B^\delta \left(\frac{x-\theta y}{r} \right) \right|^{q'} \right) |y|^{(s+1)q'} dy \right]^{q/q'} w_2(x) dx \right\}^{1/q}, \end{aligned}$$

where $\theta \in (0, 1)$, $\beta = (\beta_1, \dots, \beta_n)$, $|\beta| \equiv \beta_1 + \dots + \beta_n = s+1$, $\beta_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, n$, and

$$\nabla^\beta = \left(\frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\beta_n}.$$

Using $|\nabla^\beta B^\delta(z)| \leq c(1 + |z|)^{-(\delta+(n+1)/2)}$ for any $\beta \in (\mathbb{N} \cup \{0\})^n$ and the Hölder inequality, we have

$$L_1 \leq c2^{(j-k)(\delta_0+(n+1)/2-n/q)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)},$$

where c is independent of j and r . Similarly, we can prove that

$$L_2 \leq c 2^{(j-k)(\delta_0+(n+1)/2-n/q)} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)}$$

with c independent of j and r .

Combining Case 1 and Case 2, we obtain

$$\|\chi_k B_{*,b}^\delta(a_j)\|_{L^q(w_2)} \leq c 2^{(j-k)(\delta_0+(n+1)/2-n/q)} (k-j) \|b\|_{\text{BMO}(\mathbb{R}^n)} \|a_j\|_{L^q(w_2)}. \quad (3.3)$$

Therefore, for $k \geq j+3$, we have

$$G_2 \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)}^p \sum_{k=j+3}^{\infty} (k-j)^p 2^{(j-k)(\delta_0+(n+1)/2-n/q-\alpha)p} \leq c \|b\|_{\text{BMO}(\mathbb{R}^n)}^p,$$

where c is independent of j .

This finishes the proof of the case $0 < p \leq 1$. Using (3.3) and Minkowski's inequality, we can finish the proof of the case $1 < p < \infty$ in a similar way to the proof of Theorem 2.1. We omit the details.

This finishes the proof of Theorem 3.3. $\square$

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