1. Introduction

This is a continuation of Fujiwara and Kunita [8]. In the previous paper, we formulated canonical stochastic differential equations (SDEs) based on semimartingales with spatial parameters and established several fundamental properties of the solutions, in particular, their diffeomorphic property.

In this paper, we will study the dynamics of the flows determined by canonical SDEs toward backward direction in time. In other words, the purpose of this paper is to represent the inverse of the flow established in Section 3 of [8] as a system of solutions of a certain equation. Recall that canonical SDEs are identical to Stratonovich SDEs when the driving processes are continuous in time. In this case, such problems as above have already been studied in Kunita [11] and Arnold and Scheutzow [2].

Now, in order to carry out our investigations, we need to introduce $\mathbb{C}$-valued forward-backward semimartingales and backward integrals based on them, where $\mathbb{C} = \mathbb{C}(\mathbb{R}^d, \mathbb{R}^d)$ is the space of continuous maps from the $d$-dimensional Euclidean space $\mathbb{R}^d$ to itself. In particular, a forward-backward semimartingale $X$ is formulated so that $X$ is a (forward) semimartingale and that the càglàd version $X^-$ is a backward semimartingale. The precise definitions of them will be given in Section 2. In Section 3, we will state our main result, Theorem 3.1, which shows a kind of symmetric property of the representation via canonical SDEs as follows.

Let $X$ be a $\mathbb{C}$-valued forward-backward semimartingale and let $\xi_{s,t}, \ s \leq t$, be a stochastic flow determined by the canonical SDE based on the forward
semimartingale $X_t$:

$$
\xi_{s,t}(x) = x + \int_{(s,t]} X(\xi_{s,r-,}(x), \circ d r),
$$

where $x \in \mathbb{R}^d$. Then, for each $t > 0$, the inverse flow $\xi_{s,t}^{-1}$ is represented as a system of solutions of the canonical backward SDE

$$
\xi_{s,t}^{-1}(y) = y + \int_{(s,t]} (-X^-)(\xi_{r,t}^{-1}(y), \circ d r),
$$

where $y \in \mathbb{R}^d$.

Section 4 will be devoted to the proof of Theorem 3.1.

An assertion similar to Theorem 3.1 of this paper is announced as Theorem 3.2 in Kunita [12] without details of the proof. However, a stronger assumption seems to be needed for its proof. This paper will complete its proof by furnishing necessary conditions.

2. Backward semimartingales and backward integrals

Our purpose in this paper is to study the dynamics of the flows determined by canonical SDEs toward backward direction in time. To this end, we need to formulate backward semimartingales, backward integrals based on them, and finally canonical SDEs based on $\mathbb{C}$-valued backward semimartingales. However, we would like to restrict ourselves to introducing the notion of them briefly because the arguments are parallel to those of forward semimartingales, and so on. Also, we will use the same notation as in Part I [8].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_s:t; 0 \leq s \leq t < \infty\}$ be a family of sub-$\sigma$-fields of $\mathcal{F}$ with the following properties:

1. $\mathcal{F}_{s,t} \subset \mathcal{F}_{u,v}$ if $[s, t] \subset [u, v]$;
2. $\mathcal{F}_{s-,t} = \mathcal{F}_{s,t} = \mathcal{F}_{s,t+}$ for any $s \leq t$, where
   $$\mathcal{F}_{s-,t} := \cap_{\varepsilon > 0} \mathcal{F}_{s-,t+} \quad \text{and} \quad \mathcal{F}_{s,t+} := \cap_{\varepsilon > 0} \mathcal{F}_{s,t+};$$
3. $\mathcal{F}_{s,t}$ contains all the $P$-null sets of $\mathcal{F}$ for any $s \leq t$.

We call such a family of sub-$\sigma$-fields a two parameter filtration on $(\Omega, \mathcal{F}, P)$.

We fix $t > 0$ for a moment. An $\mathbb{R}^d$-value càdlàg process $Y_s$ is called a backward martingale adapted to $\{\mathcal{F}_s\}$, if it is integrable, $Y_s - Y_t$ is $\mathcal{F}_{s,t}$-measurable and satisfies $E[Y_{s_1} - Y_t | \mathcal{F}_{s_2,t}] = Y_{s_2} - Y_t$ for any $s_1 \leq s_2 \leq t$. 

We call a positive random variable $\tau$ a backward stopping time (with respect to $\{F_{s,t}\}_{t}$) if it satisfies $[\tau \geq s] \in F_{s,t}$ for any $s \leq t$.

We call an $\mathbb{R}^{d}$-valued càdlàg process $Y_{s}$ a backward local martingale adapted to $\{F_{s,t}\}_{t}$ if there exists a nonincreasing sequence of backward stopping times $\{\tau_{n}\}_{n}$ such that $\lim_{n \to \infty} \tau_{n} = 0$ a.s. and that the stopped process $Y^{\tau_{n}}_{s}$ is a backward martingale, where

$$Y^{\tau_{n}}_{s} := \begin{cases} Y_{s}, & \tau_{n} \leq s \leq t \\ Y_{\tau_{n}}, & 0 \leq s \leq \tau_{n}. \end{cases}$$

We denote by $\mathcal{P}^{(-)}$ the backward predictable $\sigma$-field (adapted to $\{F_{s,t}\}_{t}$), which is defined as the smallest $\sigma$-field on $[0, \tau] \times \Omega$ making all $\{F_{s,t}\}_{t}$-adapted càdlàg processes measurable.

We call an $\mathbb{R}^{d}$-valued càdlàg process $Y_{s}$ a backward semimartingale adapted to $\{F_{s,t}\}_{t}$ if it is written $Y_{s} = M^{(-)}_{s} + B^{(-)}_{s}$, where $M^{(-)}_{s}$ is a backward local martingale adapted to $\{F_{s,t}\}_{t}$ and $B^{(-)}_{s}$ is a càdlàg bounded variation process such that $B^{(-)}_{t} - B^{(-)}_{s}$ is $\{F_{s,t}\}_{t}$-measurable. In particular, when $B^{(-)}_{s} - B^{(-)}_{t}$ is $\mathcal{P}^{(-)}$-measurable, we call the decomposition canonical.

Next, we introduce the corresponding notion in the case of $\mathbb{C}$-valued processes.

Let $Y_{s}$ be a $\mathbb{C}$-valued càdlàg process. We call it a $\mathbb{C}$-valued backward semimartingale or a $\mathbb{C}$-valued backward local martingale, etc., if, for each $x \in \mathbb{R}^{d}$, $Y_{s}(x)$, which is also denoted by $Y(x, s)$, is an $\mathbb{R}^{d}$-valued backward semimartingale adapted to $\{F_{s,t}\}_{t}$ or an $\mathbb{R}^{d}$-valued backward local martingale, etc., respectively.

Corresponding to stochastic integrals defined in Section 2 of [8], we introduce several kinds of backward stochastic integrals based on $\mathbb{C}$-valued backward semimartingales. Let $Y_{s}$, $s \in [0, t]$, be a $\mathbb{C}$-valued backward semimartingale adapted to $\{F_{s,t}\}_{t}$ and let $\psi_{s}$ be an $\mathbb{R}^{d}$-valued càdlàg process adapted to the filtration. Note that $\psi_{s}$ is a backward predictable (that is, $\mathcal{P}^{(-)}$-measurable) process.

The backward Itô integral of $\psi_{s}$ based on $Y_{s}$ is defined by

$$\int_{[s,t]} Y(\psi_{r}, d\tau) := \lim_{i \to \infty} \sum_{k=0}^{n-1} \{Y(\psi_{k+1} \vee s, t_{k+1} \vee s) - Y(\psi_{k} \vee s, t_{k} \vee s)\}, \quad (2.1)$$

where $\Pi := \{0 = t_{0} < t_{1} < \cdots < t_{n} = t\}$ is a partition of the interval $[0, t]$, if the limit on the right-hand side exists.

Suppose that $Y_{s}$ is a continuous $\mathbb{C}$-valued backward semimartingale. The
backward Stratonovich integral of $\psi_s$ based on $Y_s$ is defined by

$$
\int_{[s,t]} Y(\psi_r, \circ \hat{d}r) := \text{l.i.p.} \sum_{\Pi \mid 0} \frac{1}{2} \sum_{k=0}^{n-1} \left\{ Y(\psi_{t_{k+1} \vee s}, t_{k+1} \vee s) + Y(\psi_{t_k \vee s}, t_k \vee s) - Y(\psi_{t_k \vee s}, t_k \vee s) \right\}
$$

if the limit on the right-hand side exists.

In particular, suppose that $\int_{[s,t]} \nabla Y(\psi_r, \hat{d}r)$ is defined and that $\psi^-$ is a backward semimartingale, where $\psi^-$ denotes the càdlàg version of the process $\psi$, that is, $\psi^- := \psi_{\cdot -}$. Then the Stratonovich integral of (2.2) can be expressed as

$$
\int_{[s,t]} Y(\psi_r, \circ \hat{d}r) = \int_{[s,t]} Y(\psi_r, \hat{d}r) + \frac{1}{2} \left[ \int_{[s,t]} \nabla Y(\psi_r, \hat{d}r), \psi^- \right]_{[s,t]}
$$

Suppose that a $\mathbb{C}$-valued backward semimartingale $Y_s$ has a decomposition $Y = Y_c + Y_d$, where $Y_c$ and $Y_d$ are a continuous $\mathbb{C}$-valued backward semimartingale and a discontinuous one, respectively. Then, using the two kinds of integrals stated above, we define the canonical backward integral of $\psi_s$ based on $Y_s$ by

$$
\int_{[s,t]} Y(\psi_r, \circ \hat{d}r) := \int_{[s,t]} Y_c(\psi_r, \circ \hat{d}r) + \int_{[s,t]} Y_d(\psi_r, \hat{d}r) + \sum_{r \in [s,t)} \{ \text{Exp}(\Delta Y_r)(\psi_r) - \psi_r - \Delta Y_r(\psi_r) \}
$$

Finally, we introduce the notion of forward-backward semimartingales.

Let $X$ be an $\mathbb{R}^d$-valued càdlàg process. We call it a forward-backward semimartingale adapted to a two parameter filtration $\{ \mathcal{F}_{s,t} \}_{s,t}$ if $X_t - X_s$, $t \in [s, \infty)$, is a (forward) semimartingale adapted to the filtration $\mathcal{F}_{s,t}$ for any $s$ and if $X_s^- - X_t^-$, $s \in (0, t]$, is a backward semimartingale adapted to the filtration $\mathcal{F}_{s,t}$ for any $t$.

Next, let $X$ be a $\mathbb{C}$-valued càdlàg process. We call it a forward-backward semimartingale if $X(x, t)$ is an $\mathbb{R}^d$-valued forward-backward semimartingale for any $x \in \mathbb{R}^d$.

3. A representation theorem of inverse flows via backward SDEs

In this section, we will give a precise statement of the main theorem in this paper. In fact, we will represent the inverse of the flow determined by some canonical SDE as a system of solutions of a canonical backward SDE corresponding to the given forward equation.
Let us begin with introducing several conditions which will be necessary to accomplish our aim. Throughout this and the next sections, we will consider a \( \mathbb{C} \)-valued forward-backward semimartingale \( X \) adapted to a two-parameter filtration \( \{ \mathcal{F}_{s,t} \}_{s,t} \). Moreover, we will assume that there exists an increasing sequence \( \{ U_M ; M \in \mathbb{N} \} \) of Borel sets of \( \mathbb{C} \) such that \( \hat{N}_p((0, t], (U_M)^c) < \infty \) a.s. for each \( t > 0 \) and \( M \), where \( \hat{N}_p(dr \, dv) \) denotes the compensator of the counting measure \( N_p(dr \, dv) \) defined by (2.1) in [8]. Note that for each \( M \in \mathbb{N} \), \( X_t^{U_M} := X_t - \sum_{r \in (0, t]} \Delta X_r I_r(U_M) \Delta X_r \) is also a \( \mathbb{C} \)-valued forward-backward semimartingale.

Now, let us denote by \( (a(x, y, t), b(x, t), v_t(dv), A_t; U_M) \) the characteristics of the forward semimartingale \( X_t \) associated with \( U_M \). Recall that this concept was introduced in Section 2 of [8]. We will assume that the following condition is satisfied.

**Condition C.** Let \( \delta \) be some number in \( (0, 1] \).

1. \( a(t) \) is a continuous \( \mathcal{C}^{2+\delta}_b \)-valued process satisfying \( \|a(t)\|_{2+\delta} \leq K_t \).
2. \( c(t) \) of (2.9) in [8] is a \( \mathcal{C}^{0+1}_b \)-valued process satisfying \( \|c(t)\|_{0+1} \leq K_t \).
3. \( b(t) \) is a \( \mathcal{C}^{0+1}_b \)-valued process satisfying \( \|b(t)\|_{0+1} \leq K_t \).
4. The measures \( v_t(dv) \) are supported by \( S := \{ v \in \mathbb{C} ; \|v\|_{2+\delta} \vee \|\nabla v \cdot v\|_{\text{Lip}} < \infty \} \).

\[
U_M \uparrow S \quad \text{as} \quad M \uparrow \infty .
\]

\[
\sup_{v \in U_M} \|v\|_{2+\delta} < \infty .
\]

\[
\int_{U_M} \left\{ \|v\|_{2+\delta} \vee \|\nabla v \cdot v\|_{\text{Lip}} \right\} v_t(dv) \leq K_t .
\]

\[
v_t((U_M)^c) \leq K_t .
\]

Here, \( K_t \) is a nonnegative predictable process satisfying \( \int_{(0, t]} K_r \, dA_r < \infty \) for each \( t > 0 \).

Next, we will introduce the characteristics of the backward semimartingale \( X_t^- \) associated with \( U_M \). We fix \( t > 0 \). Then since \( (X_t^{U_M})^- - (X_t^{U_M})^- \) is a backward semimartingale for each \( x \in \mathbb{R}^d \), we have the canonical decomposition \( (X_t^{U_M})^- - (X_t^{U_M})^- = M^{(-)}(x, s) + B^{(-)}(x, s) \), where \( M^{(-)}(x, s) \) is a backward local martingale and \( B^{(-)}(x, s) \) is a \( \mathcal{F}^{(-)} \)-measurable, bounded variation process. Furthermore, we denote by \( M^{(-)}_c(x, s) \) and \( M^{(-)}_d(x, s) \) the continuous and the discontinuous parts of \( M^{(-)}(x, s) \), respectively. We also denote by \( B^{(-)}_c(x, s) \) and \( B^{(-)}_d(x, s) \) the continuous and the discontinuous parts of \( B^{(-)}(x, s) \), respectively. We will use the notation that \( X^{(-)}_c(x, s) := M^{(-)}_c(x, s) + B^{(-)}_c(x, s) \) and \( X^{(-)}_d(x, s) := \)
\[ \{X^{-}(x, s) - X^{-}(x, t)\} - X^{-}_{\epsilon}(x, s) \text{ and call them the continuous and the discontinuous parts of the backward semimartingale } X^{-}_{\epsilon}, \text{ respectively.} \]

On the other hand, note that \( N_{p}(dr \, dv) \) of (2.1) in [8] is also the counting measure of the process \( X^{-}_{\epsilon} \). Indeed, since \( \Delta X^{-}_{\epsilon} = X^{-}_{\epsilon} - X^{-}_{\epsilon} = \Delta X_{\epsilon} \), we have \( \mathbb{I}_{\{r \in [s, t]; \Delta X^{-}_{\epsilon} \in A \setminus \{0\}\}} = N_{p}((s, t), A) \) for any \( 0 < s < t \) and \( A \in \mathcal{B}(\mathbb{R}) \). We denote by \( \hat{N}_{p}(-)(dr \, dv) \) the (backward) compensator of \( N_{p}(dr \, dv) \), which is the unique \( \mathcal{P}(-) \)-measurable random measure on \( (0, t) \times \mathbb{C} \) such that \( N_{p}((s, t), A) = \hat{N}_{p}(-)((s, t), A) \) is a backward local martingale for any \( A \in \mathcal{B}(\mathbb{C}) \) that makes \( N_{p}((s, t), A) \) locally integrable.

Now, we assume that there exist a \( \mathcal{P}(-) \)-measurable process \( (a(-)(x, y, s), b(-)(x, s), v_{s}(-)(dv)) \) and a nondecreasing \( \mathcal{P}(-) \)-measurable càdlàg process \( A_{s}(-) \) such that

\[
[M_{\epsilon}^{-}(x, \cdot), M_{\epsilon}^{-}(y, \cdot)]_{[s, t]} = \int_{[s, t]} a(-)(x, y, r) \, dA_{r}^{-},
\]

\[
B^{-}(x, t) - B^{-}(x, s) = \int_{[s, t]} b(-)(x, r) \, dA_{r}^{-},
\]

\[
\hat{N}_{p}(-)((s, t), dv) = \int_{[s, t]} v_{r}^{-}(dv) \, dA_{r}^{-}.
\]

We call the above \( (a(-)(x, y, s), b(-)(x, s), v_{s}(-)(dv), A_{s}(-); U_{M}) \) the (backward) characteristics of \( X^{-}_{s} \) associated with \( U_{M} \). We will assume that the following condition is satisfied.

**Condition D.**

1. \( a(-)(s) \) is a \( \mathbb{C}_{b}^{2+\delta} \)-valued \( \mathcal{P}(-) \)-measurable process satisfying \( \|a(-)(s)\|_{2+\delta} \leq K_{s}(-) \) for some \( \delta \in (0, 1) \).

2. \( b(-)(s) \) is a \( \mathbb{C}_{b}^{0+1} \)-valued \( \mathcal{P}(-) \)-measurable process satisfying \( \|b(-)(s)\|_{0+1} \leq K_{s}(-) \).

3. \( \int_{U_{M}} \|v\|_{0+1}^{2} + \|\nabla v \cdot v\|_{\text{Lip}} v_{s}^{-}(dv) \leq K_{s}^{-} \).

Here, \( K_{s}(-) \) is a nonnegative \( \mathcal{P}(-) \)-measurable process satisfying \( \int_{[s, t]} K_{r}^{-}(s) \, dA_{r}^{-} < \infty \) for each \( s \in (0, t) \).

Condition D will only be used to imply the existence of the canonical backward integral based on \( X^{-} \) and the backward semimartingale property. Therefore, the condition can be replaced by any other one which ensures these properties.

We are now in a position to state our main result precisely.
THEOREM 3.1. Let \( X \) be a \( \mathbb{C} \)-valued forward-backward semimartingale satisfying Conditions C and D. Let \( \xi_{s,t}, s \leq t \), be a stochastic flow of homeomorphisms determined by the canonical SDE based on the forward semimartingale \( X_t \):

\[
\xi_{s,t}(x) = x + \int_{(s,t]} X(x, r) \, \circ d\tau.
\]

(3.1)

Then, for each \( t > 0 \), the inverse flow \( \xi_{s,t}^{-1}, 0 < s \leq t \), is represented as a system of solutions of the canonical backward SDE based on \( -X^- \):

\[
\xi_{s,t}^{-1}(y) = y + \int_{(s,t]} (-X^-)(\xi_{r,t}^{-1}(y), \circ d\tau).
\]

(3.2)

where \( y \in \mathbb{R}^d \). Moreover, for each \( t > 0 \) and \( y \in \mathbb{R}^d \), \( \xi_{s,t}^{-1}(y), 0 < s \leq t \), is a backward semimartingale.

We will specialize Theorem 3.1 to two cases. The first one is the case where the driving process \( X \) is linear in the sense described in Section 2 of [8].

COROLLARY 3.1. Let \( X \) be a \( \mathbb{C} \)-valued process of the form \( X(x, r) = f(x)Z_r \), where \( f \in \mathcal{C}_B^2(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d) \) with \( \| \nabla f \cdot f \|_{\text{Lip}} < \infty \) and \( Z_r \) is an \( \mathbb{R}^d \)-valued forward-backward semimartingale. Then the conclusion of Theorem 3.1 holds.

Remark. If, in addition, the coefficient \( f \) has the property that \( \| \nabla^2 f \|_{0,\delta} < \infty \) for some \( \delta \in (0, 1] \), then Conditions C and D are fully satisfied. However, owing to the linearity of \( X \), we can follow the proof of Theorem 3.1 which will be given in the next section without using the Hölder continuity.

The second case is where \( X \) belongs to the class of \( \mathbb{C} \)-valued Lévy processes. In this case, note that \( X \) can be regarded as a \( \mathbb{C} \)-valued forward-backward semimartingale in the sense stated in Section 2 and that the backward characteristics completely coincide with the forward one because they are deterministic. Thus, we have the following result.

COROLLARY 3.2. Let \( X \) be a \( \mathbb{C} \)-valued Lévy process and suppose that the characteristics \((a, b, \nu, \Lambda, M)\) satisfy the following conditions. Let \( \delta \) be some number in \((0, 1] \).

1. \( a \in \mathcal{C}_B^{2+\delta} \) and \( c \in \mathcal{C}_B^{0+1} \), where \( c \) is the function defined by (2.15) in [8].
2. \( b \in \mathcal{C}_B^{0+1} \).
3. The measure \( \nu(d\nu) \) is supported by the function space \( S \) introduced in Condition C.

\( U_M \uparrow S \) as \( M \uparrow \infty \).
\[ \sup_{v \in U_M} \|v\|_{2+\delta} < \infty. \]
\[ \int_{U_M} \{ \|v\|_{2+\delta}^2 \vee \|\nabla v \cdot v\|_{Lip} \} (du) < \infty. \]
\[ v((U_M)^c) < \infty. \]

Then the conclusion of Theorem 3.1 holds.

4. Proof of Theorem 3.1

Throughout this section, the process \( X \) given in Theorem 3.1 is assumed to satisfy Conditions C and D, and we will use the same notation as in Sections 2 and 3.

We first make a reduction of the proof of Theorem 3.1 as follows. Suppose that Theorem 3.1 holds for \( X^{UM} \) for each \( M \in \mathbb{N} \). Let \( \xi_{s,t} \) and \( \xi_{s,t}^M \) be the stochastic flows determined by the canonical SDEs based on \( X \) and \( X^{UM} \), respectively. Then by the hypothesis, the inverse flow \( \chi_{s,t}^M := (\xi_{s,t}^M)^{-1} \) satisfies the canonical backward SDE based on \(- (X^{UM})^{-}\), which means that for each \( y \in \mathbb{R}^d \) and \( t > 0 \)

\[
\chi_{s,t}^M(y) = y - \int_{(s,t]} (X^{UM})^{-}(\chi_{s,t}^M(y), \hat{a}r) + \frac{1}{2} \int_{(s,t]} [\nabla X^{-}(\xi^{-1}, \chi_{s,t}^M(y), \hat{a}r)
+ \sum_{r \in (s,t]} \{\text{Exp}(-\Delta X_r)(\chi_{s,t}^M(y)) - \chi_{s,t}^M(y) + \Delta X_r(\chi_{s,t}^M(y))\}] I_{UM}(\Delta X_r).
\]  

(4.1)

Here, \( X^{-} \) is the continuous part of the backward semimartingale \( X^{-} \) and \( U^M := (U_M)^c \). For each \( M \in \mathbb{N} \), let \( S_M := \inf \{ r > 0 ; \Delta X_r \in U^M \} \). Obviously, if \( t < S_M \), then \( X^{UM} = X_r \) for all \( r \in (0, t] \). Hence, it holds that, on \( t < S_M \), \( \xi_{s,t}^M = \xi_{s,t} \) and so \( \chi_{s,t}^M = \chi_{s,t}^{-1} \) for all \( r \in (0, t] \). Therefore, approximating \( \chi_{s,t}^M(y) \) and \( \chi_{s,t}^{-1}(y) \) by simple processes as in (2.1), we have

\[
\int_{(s,t]} (X^{UM})^{-}(\chi_{s,t}^M(y), \hat{a}r) = \int_{(s,t]} X^{-} (\chi_{s,t}^{-1}(y), \hat{a}r) \quad \text{on} \ t < S_M.
\]

Observing these facts, we see from (4.1) that on \( t < S_M \)

\[
\xi_{s,t}^{-1}(y) = y - \int_{(s,t]} X^{-} (\chi_{s,t}^{-1}(y), \hat{a}r) + \frac{1}{2} \int_{(s,t]} [\nabla X^{-}(\xi^{-1}, \chi_{s,t}^{-1}(y), \hat{a}r)
+ \sum_{r \in (s,t]} \{\text{Exp}(-\Delta X_r)(\chi_{s,t}^{-1}(y)) - \chi_{s,t}^{-1}(y) + \Delta X_r(\chi_{s,t}^{-1}(y))\}].
\]  

(4.2)
Since \( \lim_{M \to \infty} P[t < S_M] = 1 \), (4.2) holds a.s. and hence so does (3.2). Moreover, since 
\[
\xi_{s-t}^{-1}(y) - \xi_{s-t}^{-1}(y) = \text{Exp}(-\Delta X_s)\{\xi_{s-t}^{-1}(y) - \xi_{s-t}^{-1}(y)\},
\]
adding the quantity to both sides of (4.2), we see that
\[
\xi_{s-t}^{-1}(y) = y + \int_{[s,t]} (-X^-(\xi_{r-t}^{-1}(y), \omega) \, dr),
\]
and therefore we see that \( \xi_{s-t}^{-1}(y) \) is a backward semimartingale. Consequently, we
have shown that Theorem 3.1 holds for \( X \).

Thus, by the reduction argument, it is sufficient to prove Theorem 3.1 for \( X^{U_M} \)
for each \( M \in \mathbb{N} \). Therefore, from here to the end of this section, we will fix \( M \in \mathbb{N} \)
and use \( X \) and \( \xi_{s,t} \) to denote \( X^{U_M} \) and \( \xi_{s,t}^{M} \), respectively.

Now let us recall that the canonical SDE (3.1) is equivalent to the Itô SDE
\[
\xi_{s,t}(x) = x + \int_{(s,t]} X(\xi_{s,r-}(x), dr) + \frac{1}{2} \int_{(s,t]} [\nabla X_c, X_c](\xi_{s,r-}(x), dr)
+ \sum_{r \in (s,t]} \{\text{Exp}(\Delta X_r)(\xi_{s,r-}(x)) - \xi_{s,r-}(x) - \Delta X_r(\xi_{s,r-}(x))\}. \quad (4.3)
\]

We will prove Theorem 3.1 by substituting \( x \) with \( \xi_{s,t}^{-1}(y) \). Therefore, the first and the
most important step in our proof is to establish the following proposition.

**Proposition 4.1.**
\[
\int_{(s,t]} X(\xi_{s,r-}(x), dr) \bigg|_{x=\xi_{s,t}^{-1}(y)} = \int_{(s,t]} X^{-}(\xi_{r-t}^{-1}(y), dr)
- \int_{(s,t]} [\nabla X_c^{(-)}, X_c^{(-)}](\xi_{r-t}^{-1}(y), dr)
- \sum_{r \in (s,t]} \{\Delta X_r(\xi_{r-t}^{-1}(y)) - \Delta X_r(\xi_{r-t}^{-1}(y))\}. \quad (4.4)
\]

We will prove this proposition after preparing several lemmas. The first is a
general result concerning the compact-uniform convergence of a sequence of random
fields.

**Lemma 4.1.** Let \( Z^m, m \in \mathbb{N} \), and \( Z \) be \( C \)-valued random variables, and assume that
for any \( x \in \mathbb{R}^d \),
\[
\lim_{m \to \infty} |Z^m(x) - Z(x)| = 0. \quad (4.5)
\]
Moreover, assume that for any $K > 0$ there exist $\beta > 0$, $\gamma > d$ and $C_K > 0$ such that

$$\sup_m E[[Z^m(x) - Z^m(y)]^{\beta}] \leq C_K |x - y|^{\gamma}, \quad |x|, |y| \leq K,$$

$$\sup_m E[[Z^m(x)]^{\beta}] \leq C_K, \quad |x| \leq K.$$ 

Then it holds that

$$\limsup_{m \to \infty} \sup_{|x| \leq K} |Z^m(x) - Z(x)| = 0. \tag{4.6}$$

Proof of Lemma 4.1. Let $K > 0$ be arbitrary. We denote by $P^m$ the law of $(Z^m - Z)$ as a $\mathbb{C}(|x| \leq K), \mathbb{R}^d$-valued random variable. Since it is easy to see from our assumption that for any $|x|, |y| \leq K$,

$$E[|Z(x) - Z(y)]^{\beta}] \leq C_K |x - y|^{\gamma},$$

$$E[|Z(x)]^{\beta}] \leq C_K,$$ 

we have

$$E[[Z^m(x) - Z(x) - (Z^m(y) - Z(y))^{\beta}] \leq C_K |x - y|^{\gamma},$$

$$E[[Z^m(x) - Z(x)]^{\beta}] \leq C_K.$$ 

Therefore, it follows from Theorem 1.4.7 in [11] (p. 38) that the family of the laws $(P^m)_m$ is tight. Furthermore, combining this fact with assumption (4.5), we see from Theorem 1.4.5 in [11] (p. 37) that $(P^m)_m$ converges weakly to a probability $P^\infty$ on $\mathbb{C}(|x| \leq K), \mathbb{R}^d$ as $m$ tends to infinity and that $P^\infty$ is actually the Dirac measure concentrated at the single point $0 \in \mathbb{C}(|x| \leq K), \mathbb{R}^d$. Set $F^\delta := \{f \in \mathbb{C}(|x| \leq K), \mathbb{R}^d; \sup_{|x| \leq K} |f(x)| \geq \delta\}$ for any $\delta > 0$. It is a closed subset of $\mathbb{C}(|x| \leq K), \mathbb{R}^d)$. Therefore we have

$$\limsup_{m \to \infty} P^m(F^\delta) \leq P^\infty(F^\delta) = 0.$$ 

This means (4.6). \qed

The next lemma is also a general result which gives us $L^p$-estimates of stochastic integrals based on $\mathbb{C}$-valued semimartingales.

**Lemma 4.2.** Let $Z$ be a $\mathbb{C}$-valued semimartingale with the canonical decomposition $Z = M^Z + B^Z$. We denote by $\nu^Z(dr df)$ the compensator of the counting measure of
the jumps $\Delta Z_r$. Assume that for all $x, y \in \mathbb{R}^d$, $u \leq v$ and for some $p \geq 2$,

$$
\langle M^Z(x) - M^Z(y) \rangle_{(u,v)} \leq |x - y|^2 (D_v - D_u),
$$

(4.7)

$$
\langle B^Z(x) - B^Z(y) \rangle_{(u,v)} \leq |x - y|(D_v - D_u),
$$

(4.8)

$$
\int_C |f(x) - f(x)|^p v^Z((u, v), df) \leq |x - y|^p (D_v - D_u),
$$

(4.9)

where $D_r$ is a predictable nondecreasing process and $|B|_{(u,v)}$ denotes the total variation of the process $B$ on $(u,v)$. We also assume that $D_t$ is bounded by a positive constant $N$. Then there exists a positive constant $C_N$ such that

$$
E \left[ \sup_{u \in (s,t]} \left| \int_{(s,u]} Z(\xi_{t-r}, dr) - \int_{(s,u]} Z(\eta_{t-r}, dr) \right|^p \right] \leq C_N E \left[ \sup_{r \in (s,t]} |\xi_{t-r} - \eta_{t-r}|^p \right]
$$

for any adapted càdlàg processes $\xi_r$ and $\eta_r$. (The constant $C_N$ may depend on $p$.)

Moreover, assume that

$$
\langle M^Z(x) \rangle_{(u,v)} \leq (1 + |x|)^2 (D_v - D_u),
$$

(4.10)

$$
\langle B^Z(x) \rangle_{(u,v)} \leq (1 + |x|)(D_v - D_u),
$$

(4.11)

$$
\int_C |f(x)|^p v^Z((u, v), df) \leq (1 + |x|)^p (D_v - D_u).
$$

(4.12)

Then it holds that

$$
E \left[ \sup_{u \in (s,t]} \left| \int_{(s,u]} Z(\xi_{t-r}, dr) \right|^p \right] \leq C_N \left\{ 1 + E \left[ \sup_{r \in (s,t]} |\xi_{t-r}|^p \right] \right\}.
$$

Proof of Lemma 4.2. The first assertion is nothing but a restatement of Lemma V.2.1 in [3] (p. 117). The second assertion can be proved in the same manner. 

Before starting the proof of Proposition 4.1, we give a remark and introduce some notation. In order to simplify notation, we will consider only the one-dimensional case: $d = 1$. For fixed $s$ and $t$, $\Pi$ will denote a finite partition $\{s = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[s, t]$. Associated with a càdlàg process $\varphi_r$ and the partition $\Pi$, define a simple process $\varphi_r^\Pi$ by

$$
\varphi_r^\Pi := \sum_{k=0}^{n-1} \varphi_{t_k} I_{[t_k, t_{k+1})}(r).
$$

Now let us begin the proof of Proposition 4.1. The first step is to prove the following lemma, which states that the Itô integral $\int_{(s,t]} X(\xi_{t-r}(x), dr)$ can be uniformly approximated on any compact set in $\mathbb{R}^d$ by a sequence of simple processes with parameter $x$. 

LEMMA 4.3. For any $K > 0$,
\[ \text{l.i.p. sup}_{|\Pi| \leq K} \left| \int_{[s,t]} X(\xi^{\Pi}_{s,r-}(x), dr) - \int_{[s,t]} X(\xi_{s,r-}(x), dr) \right| = 0. \]

Proof of Lemma 4.3. For each $N > 0$, let $T_N$ be the stopping time defined by (3.10) in [8]. Note that, on the set $\{t < T_N\}$, $(\xi_{s,r}^{\Pi}(x))_{r \leq t} = (\xi_{s,r}^{T_N-}(x))_{r \leq t} = \xi_{s,r}^{T_N-}(x)$ for all $r \leq t$, and so we use $\xi_{s,r}^{T_N-}(x)$ to denote these processes in a unified manner. Furthermore, since $X^{T_N-}_r = X_r$ and $\xi_{s,r}^{T_N-}(x) = \xi_{s,r}(x)$ for all $r \leq t$ on $\{t < T_N\}$, we see from the local property of stochastic integrals (Property (8) in [3] (p. 44)) that for any $\delta > 0$,
\[ P\left[ \text{sup}_{|x| \leq K} \left| \int_{[s,t]} X(\xi^{\Pi}_{s,r-}(x), dr) - \int_{[s,t]} X(\xi_{s,r-}(x), dr) \right| > \delta \right] \leq P[T_N \leq t] \]
\[ + P\left[ \text{sup}_{|x| \leq K} \left| \int_{[s,t]} X^{T_N-}(\xi^{T_N-\Pi}_{s,r-}(x), dr) - \int_{[s,t]} X^{T_N-}(\xi_{s,r-}^{T_N-}(x), dr) \right| > \delta \right]. \]

Since $\lim_{N \to \infty} P[T_N \leq t] = 0$, our proof of this lemma is reduced to showing that the second term converges to 0 as $|\Pi| \downarrow 0$ for any $N > 0$. Because of Lemma 4.1, this will be done once we establish the following:
\[ \text{l.i.p.} \int_{[s,t]} X^{T_N-}(\xi^{T_N-\Pi}_{s,r-}(x), dr) = \int_{[s,t]} X^{T_N-}(\xi_{s,r-}^{T_N-}(x), dr), \quad \text{for each } x \in \mathbb{R}^d, \]
(4.13)

\[ \sup_{\Pi} E\left[ \left| \int_{[s,t]} X^{T_N-}(\xi^{T_N-\Pi}_{s,r-}(x), dr) - \int_{[s,t]} X^{T_N-}(\xi_{s,r-}^{T_N-}(y), dr) \right|^p \right] \leq C|x-y|^p, \quad x, y \in \mathbb{R}^d, \]
(4.14)

\[ \sup_{\Pi} E\left[ \left| \int_{[s,t]} X^{T_N-}(\xi^{T_N-\Pi}_{s,r-}(x), dr) \right|^p \right] \leq C(1 + |x|)^p, \quad x \in \mathbb{R}^d, \]
(4.15)

where we take $p > d \vee 2$.

We will first prove (4.13). Let $X = M + B$ be the canonical decomposition of the
forward semimartingale $X$. Observe that

$$
E \left[ \int_{(s,t]} M^{TN^-}(\xi^{TN^-}_{s,r}, (x), dr) - \int_{(s,r]} M^{TN^-}(\xi^{TN^-}_{s,r}, (x), dr) \right] \\
= E \left[ \int_{(s,t]} \left[ \tilde{a}(\xi^{TN^-}_{s,r}, (x), \xi^{TN^-}_{s,r}, (x), r) - 2\tilde{a}(\xi^{TN^-}_{s,r}, (x), \xi^{TN^-}_{s,r}, (x), r) \\
+ \tilde{a}(\xi^{TN^-}_{s,r}, (x), \xi^{TN^-}_{s,r}, (x), r) \right] d\tilde{A}^{TN^-}_r \right] \\
\leq E \left[ \int_{(s,t]} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (x)|^2 d\tilde{A}^{TN^-}_r \right],
$$

(4.16)

where we set $\tilde{a}(x, y, r) := a(x, y, r) + \int_{C} v(x)u(y)v_{r}(dv)$. Fix $\omega \in \Omega$ for a moment. Then, $\lim_{|\Pi| \downarrow 0} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (x)| = 0$ for each $r$. Also, $\sup_{x} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (x)|^2 \leq 2 \sup_{x \in (s,t]} |\xi^{TN^-}_{s,r}, (x)|^2$. Since the right-hand side is a bounded function (for each $\omega$), it is also integrable with respect to the Stieltjes measure $d\tilde{A}^{TN^-}_r$. Hence, we see from Lebesgue’s dominated convergence theorem that $\int_{(s,t]} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (x)|^2 d\tilde{A}^{TN^-}_r$ converges to 0 as $|\Pi| \downarrow 0$ a.s. for each $x$. Moreover, since $2 \sup_{x \in (s,t]} |\xi^{TN^-}_{s,r}, (x)|^2$ is in $L^1(dP)$, it follows from Lebesgue’s dominated convergence theorem again that the right-hand side of (4.16) converges to 0 as $|\Pi| \downarrow 0$. Similarly, we can show that

$$
\lim_{|\Pi| \downarrow 0} E \left[ \int_{(s,t]} B^{TN^-}(\xi^{TN^-}_{s,r}, (x), dr) - \int_{(s,t]} B^{TN^-}(\xi^{TN^-}_{s,r}, (x), dr) \right] = 0,
$$

and therefore we get (4.13).

Next, note that, under Condition C, the $\mathbb{C}$-valued semimartingale $X^{TN^-}$ satisfies all of the assumptions in Lemma 4.2 for $p$ taken above. Hence, by the first assertion of the lemma, we see that

$$
\sup_{\Pi} E \left[ \left| \int_{(s,t]} X^{TN^-}(\xi^{TN^-}_{s,r}, (x), dr) - \int_{(s,t]} X^{TN^-}(\xi^{TN^-}_{s,r}, (y), dr) \right|^p \right] \\
\leq C \sup_{\Pi} E \left[ \sup_{r \in (s,t]} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (y)|^p \right] \\
\leq CE \left[ \sup_{r \in (s,t]} |\xi^{TN^-}_{s,r}, (x) - \xi^{TN^-}_{s,r}, (y)|^p \right] \\
\leq C|x - y|^p,
$$

(4.17)

where we have used Lemma 3.2 of [8] to get the last inequality. Thus, we have verified (4.14).
Finally, we will prove (4.15). Applying Lemma 4.2 as above, we see from the second assertion that
\[
\sup_{\Pi} E \left[ \left| \int_{(s,t)} X_{T_N}^{\infty, \Pi} \left( \xi_{s,r}^{T_N, \Pi} (x), dr \right) \right|^p \right] \leq C \left\{ 1 + E \left[ \sup_{r \in (s,t)} |\xi_{s,r}^{T_N, \Pi} (x)|^p \right] \right\}.
\]

On the other hand, in the same manner as in the proof of Lemma 3.2 of [8], we can obtain the following estimate. For each \( N > 0 \) and \( p \geq 2 \),
\[
E \left[ \sup_{r \in (s,t)} |\xi_{s,r}^{T_N, \Pi} (x)|^p \right] \leq C_{M,N,p} (1 + |x|)^p,
\]
where \( C_{M,N,p} \) is a constant depending only on \( M \) in Condition C, \( N \) and \( p \). By these facts, it is now obvious that (4.15) holds. Thus, we have proved Lemma 4.3. \( \square \)

In view of Lemma 4.3, we see that
\[
\int_{(s,t)} X(\xi_{s,r}^{-}(x), dr) \bigg|_{x = \xi_{s,t}^{-1}(y)} = \text{l.i.p.} \int_{(s,t)} X(\xi_{s,r}^{1}(x), dr) \bigg|_{x = \xi_{s,t}^{-1}(y)} = \text{l.i.p.} \sum_{k=0}^{n-1} \{ X(\xi_{s,t}^{-1}(y), t_{k+1}) - X(\xi_{s,t}^{-1}(y), t_k) \},
\]
where we have used the flow property that \( \xi_{s,t} \circ \xi_{s,t}^{-1} = \xi_{s,t}^{-1} \). Now, we set
\[
I_1^{\Pi} (X) := \sum_{k=0}^{n-1} \{ X(\xi_{s,t}^{-1}(y), t_{k+1}) - X(\xi_{s,t}^{-1}(y), t_k) \},
\]
and further decompose it into
\[
I_1^{\Pi} (X) = I_2^{\Pi} (X),
\]
where
\[
I_2^{\Pi} (X) := \sum_{k=0}^{n-1} \{ \{ X(\xi_{s,t}^{-1}(y), t_{k+1}) - X(\xi_{s,t}^{-1}(y), t_k) \} - \{ X(\xi_{s,t}^{-1}(y), t_{k+1}) - X(\xi_{s,t}^{-1}(y), t_k) \} \}.
\]

By these considerations, the next step should be to show that \( I_1^{\Pi} (X) \) converges in probability as \( |\Pi| \downarrow 0 \) and, furthermore, to identify the limit for each \( i = 1, 2 \). In fact, the proof of Proposition 4.1 will be complete once we can prove the following two lemmas.

**Lemma 4.4.**
\[
\text{l.i.p.} I_1^{\Pi} (X) = \int_{(s,t)} X^{-}(\xi_{s,t}^{-1}(y), dr).
\]
Lemma 4.5.

\[ l.i.p. \int_\Pi^\Pi_2 (X) = \int_{(s,t]} [\nabla X_c^{(-)}, X_c^{(-)}](\xi_{r,t}^{-1}(y), \hat{dr}) \]
\[ + \sum_{r \in (s,t]} \{\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,t}^{-1}(y))\}. \]

Before we proceed to prove Lemma 4.4, we prepare a lemma which will be useful in the proofs of Lemmas 4.4 and 4.5. To state it, we need some notation. For given semimartingales \( X \) and \( Y \), and a partition \( \Pi : s = t_0 < \cdots < t_n = t \) of \([s, t] \), set

\[ [X, Y]_{(s, t]}^\Pi := \sum_{k=0}^{n-1} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k}). \]

We will also use the notation \([X]_{(s, t]}^\Pi := [X, X]_{(s, t]}^\Pi\).

Lemma 4.6. Let \( Z \) be a \( \mathbb{C} \)-valued semimartingale satisfying (4.7)–(4.12). Then, for any \( K > 0 \),

\[ l.i.p. \sup_{|\Pi| \leq K} \left\| \left[ \int_{(s, \cdot]} Z(\xi_{r, r-}^{\Pi}(x), dr), \xi_{s, \cdot}(x) \right]_{(s, t]}^\Pi \right\| = 0. \]

Proof of Lemma 4.6. For each \( N > 0 \), set \( T_N' := T_N \wedge \inf\{t > 0 : D_t \geq N\} \). By the localization argument used before, it is sufficient to show that for each \( N > 0 \),

\[ l.i.p. \sup_{|\Pi| \leq K} \left\| \left[ \int_{(s, \cdot]} Z^{T_N'}(\xi_{r, r-}^{T_N'}, \xi_{r, r-}^{T_N'}, (x), dr), \xi_{s, \cdot}^{T_N'}(x) \right]_{(s, t]}^\Pi \right\| = 0. \]

Moreover, this is reduced to showing that for \( i = 1, 2, \)

\[ l.i.p. \sup_{|\Pi| \leq K} \int_t^\Pi (x) = 0, \quad (4.19) \]
where we set
\[
J_1^\Pi(x) := \left[ \int_{(s, r]} Z_{(s, r]}^N (\xi_{(s, r]}^N, \Pi) (x), dr \right]_{(s, t]}
\]
\[
- \left[ \int_{(s, r]} Z_{(s, r]}^N (\xi_{(s, r]}^N, \Pi) (x), dr \right]_{(s, t]}
\]
\[
J_2^\Pi(x) := \left[ \int_{(s, r]} Z_{(s, r]}^N (\xi_{(s, r]}^N, \Pi) (x), dr \right]_{(s, t]}
\]
\[
- \left[ \int_{(s, r]} Z_{(s, r]}^N (\xi_{(s, r]}^N, \Pi) (x), dr \right]_{(s, t]}
\]
\[
(4.20)
\]

Then, by Lemma 4.1, we will obtain (4.19) once we prove the following:
\[
\text{i.p. } J_1^\Pi(x) = 0, \quad \text{for each } x \in \mathbb{R}^d,
\]
\[
(4.21)
\]
\[
\sup_{\Pi} E[|J_1^\Pi(x) - J_1^\Pi(y)|^p] \leq C_K |x - y|^p, \quad |x|, |y| \leq K,
\]
\[
(4.22)
\]
\[
\sup_{\Pi} E[|J_1^\Pi(x)|^p] \leq C(1 + |x|^p), \quad x \in \mathbb{R}^d,
\]
\[
(4.23)
\]

where we take \( K > 0 \) arbitrarily and \( p > d \lor 2 \). To this end, the following estimates are fundamental:
\[
\sup_{\Pi} E[|\xi_{(s, t]}^N - \xi_{(s, t]}^N(y)|^{\Pi}] \leq C|y - x|^{2p},
\]
\[
(4.24)
\]
\[
\sup_{\Pi} E[|\xi_{(s, t]}^N(x)|^{\Pi}] \leq C(1 + |x|)^{2p},
\]
\[
(4.25)
\]

Obviously, these results are also valid for \( T_N^r \). We first show (4.25). Recall that, using the same notation as in the proof of Lemma 3.2 in [8],
\[
\xi_{(s, u]}^N(x) = \xi_{(s, u]}(x) + \int_{(s, u]} \bar{M}(\xi_{(s, r]}^N, \bar{d}) (x), dr,
\]
for \( s \leq u \), where we set \( M(x, r) := \bar{M}_c(x, r) + \bar{M}_d(x, r) \). Hence,
\[
[\xi_{(s, t]}^N(x)]^{\Pi}_{(s, t]} \leq 2 \left\{ \int_{(s, t]} \bar{M} (\xi_{(s, r]}^N, (x), dr) \right\}^{\Pi}_{(s, t]} + \left\{ \int_{(s, t]} \bar{B} (\xi_{(s, r]}^N, (x), dr) \right\}^{\Pi}_{(s, t]}
\]

Since \( \int_{(s, t]} \bar{M} (\xi_{(s, r]}^N, (x), dr); k = 0, 1, \ldots, n \) is a martingale with the discrete parameter \( t_k \), it follows from the discrete version of Burkholder’s inequality (Stroock [15] (p. 341)) that
\[
E \left[ \left| \int_{(s, t]} \bar{M} (\xi_{(s, r]}^N, (x), dr) \right|^{\Pi}_{(s, t]} \right] \leq CE \left[ \sup_{k} \left| \int_{(s, t_k]} \bar{M} (\xi_{(s, r]}^N, (x), dr) \right|^{2p} \right]
\]
\[
\leq CE \left[ \sup_{u \in (s, t]} \left| \int_{(s, u]} \bar{M} (\xi_{(s, r]}^N, (x), dr) \right|^{2p} \right],
\]
where $C$ is a constant not depending on $\Pi$. Moreover, since $\tilde{M}$ satisfies all of the assumptions in Lemma 4.2, it follows from the second assertion of the lemma that

$$E\left[ \sup_{u \in (s, t]} \left| \int_{(s, u]} \tilde{M}(\xi^{T_N^-, (x)}_{s, r}, dr) \right|^{2p} \right] \leq C \left\{ 1 + E \left[ \sup_{r \in (s, t]} |\xi^{T_N^-, (x)}_{s, r}|^{2p} \right] \right\}.$$

Therefore, we see from (4.18) that

$$\sup_{\Pi} E\left[ \left( \int_{(s, t]} \tilde{M}(\xi^{T_N^-, (x)}_{s, r}, dr) \right)^{\Pi} \right] \leq C(1 + |x|)^{2p}.$$

On the other hand, since $\tilde{B}(x, u) = \int_{(0, u]} \tilde{b}(x, r) \, dA^{T_N^-}_r$, we have

$$\left( \int_{(s, t]} \tilde{B}(\xi^{T_N^-, (x)}_{s, r}, dr) \right)^{\Pi}_{(s, t]} \leq \sum_{k=0}^{n-1} \int_{(t_k, t_{k+1}]} \tilde{b}(\xi^{T_N^-, (x)}_{s, r}, r) \, dA^{T_N^-}_r \right)^2 \leq \sum_{k=0}^{n-1} \left\{ \int_{(t_k, t_{k+1}]} \tilde{b}(r) \, dA^{T_N^-}_r \right\}^2 \times \left( 1 + \sup_{r \in (s, t]} |\xi^{T_N^-, (x)}_{s, r}| \right)^2 \leq C_{M, N} \left( 1 + \sup_{r \in (s, t]} |\xi^{T_N^-, (x)}_{s, r}| \right)^2,$$

where $C_{M, N}$ is a constant depending only on $M$ and $N$. Therefore, combining this with (4.18), we have

$$\sup_{\Pi} E\left[ \left( \int_{(s, t]} \tilde{B}(\xi^{T_N^-, (x)}_{s, r}, dr) \right)^{\Pi} \right] \leq C(1 + |x|)^{2p}.$$

Thus, we get (4.25). On the other hand, using (3.9) of [8] instead of (4.18), we can obtain (4.24) in the same way as we just obtained (4.25).

Now we return to the proof of (4.21)–(4.23). However, we will only check (4.21) and (4.22) for $i = 1$, because we can similarly show the others.

We first check (4.21) for $i = 1$. By combining the arguments used to prove (4.13)
and (4.25), we have
\[
E[|J_1^\Pi(x)|] \leq E \left[ \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(x), dr \right| \right]^{1/2} \left[ \left| E[|\xi_{s, r}^{T_N^-}(x)|] \right| \right]^{1/2} \\
\leq C E \left[ \int_{(s, t]} |\xi_{s, r}^{T_N^-}(x) - \xi_{s, r}^{T_N^-}(x)|^2 dD_r^{T_N} \right]^{1/2} \\
\times E[|\xi_{s, r}^{T_N^-}(x)|]^{1/2},
\]
where C is a constant not depending on Π. As we have shown in the proof of Lemma 4.3, the first term on the right-hand side converges to 0 as |Π| ↓ 0. Moreover, because of (4.25), we see that (4.21) holds for i = 1.

Next, in order to check (4.22) for i = 1, observe that
\[
|J_1^\Pi(x) - J_1^\Pi(y)| \leq \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(x), dr \right| - \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(y), dr \right| \xi_{s, r}^{T_N^-}(x) \left|_{(s, t]} \right| \\
+ \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(x), dr \right| - \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(x), dr \right| \xi_{s, r}^{T_N^-}(y) \left|_{(s, t]} \right| \\
+ \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(y), dr \right| \xi_{s, r}^{T_N^-}(y) \left|_{(s, t]} \right| \\
+ \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(y), dr \right| \xi_{s, r}^{T_N^-}(y) \left|_{(s, t]} \right| \\
=: K_1^\Pi + K_2^\Pi + K_3^\Pi + K_4^\Pi,
\]
so that our problem is reduced to showing that
\[
\sup_{\Pi} E[|K_j^\Pi|^p] \leq C_K |x - y|^p \quad (4.26)
\]
for j = 1, \ldots, 4. Using the same argument as we used to prove (4.24) and (4.25), we see that
\[
\sup_{\Pi} E \left[ \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(x), dr \right| - \left| \int_{(s, t]} Z_{T_N}^r - (\xi_{s, r}^{T_N^-}, \Pi)(y), dr \right| \right]^p \left|_{(s, t]} \right| \\
\leq C |x - y|^{2p}.
\]
Combining this estimate with (4.25), we see that (4.26) holds for \( j = 1 \). Furthermore, by a similar argument, it is easy to see that (4.26) holds for \( j = 2, 3, 4 \). Thus, we have checked (4.22) for \( i = 1 \), and therefore we have proved Lemma 4.6.

We are now in a position to prove Lemma 4.4.

Proof of Lemma 4.4. First, observe that \( I_1(\{X\}) = I_1(\{X^-\}) + I_1(\{\Delta X\}) \). Then, owing to Condition D, it follows from an argument parallel to the one which we follow to construct stochastic integrals based on forward semimartingales that

\[
\text{l.i.p. } I_1(\{X^-\}) = \int_{[t,\infty)} X^-(\xi_{r,t}^{-1}(y), \, dr).
\]

Therefore, it is clear that the proof of this lemma will be done if we can show that

\[
\text{l.i.p. } I_1(\{\Delta X\}) = \Delta X_t(y) - \Delta X_s(\xi_{s,t}^{-1}(y)).
\]

Furthermore, since

\[
I_1(\{\Delta X\}) = \sum_{k=0}^{n-1} \{ \Delta X_{t_{k+1}}(\xi_{t_{k+1},t}^{-1}(y)) - \Delta X_{t_k}(\xi_{t_k,t}^{-1}(y)) \}
+ \sum_{k=0}^{n-1} \{ \Delta X_{t_k}(\xi_{t_k,t}^{-1}(y)) - \Delta X_{t_{k+1}}(\xi_{t_{k+1},t}^{-1}(y)) \}
= \Delta X_t(\xi_{t,t}^{-1}(y)) - \Delta X_s(\xi_{s,t}^{-1}(y))
+ \sum_{k=0}^{n-1} \{ \Delta X_{t_k}(\xi_{t_k,t}^{-1}(y)) - \Delta X_{t_{k+1}}(\xi_{t_{k+1},t}^{-1}(y)) \},
\]

our problem is reduced to proving that the last term on the right-hand side converges to 0 in probability as \( |\Pi| \downarrow 0 \). To this end, it is sufficient to show that

\[
\lim_{\epsilon \downarrow 0} \limsup_{|\Pi| \downarrow 0} P \left[ \sum_{k=0}^{n-1} | \Delta X_{t_k}(\xi_{t_k,t}^{-1}(y)) - \Delta X_{t_{k+1}}(\xi_{t_{k+1},t}^{-1}(y)) | I_{V_{t_k}}(\Delta X_{t_k}) > \delta \right] = 0 \quad (4.27)
\]

for any \( \delta > 0 \) and that

\[
\text{l.i.p. } \sum_{k=0}^{n-1} | \Delta X_{t_k}(\xi_{t_k,t}^{-1}(y)) - \Delta X_{t_{k+1}}(\xi_{t_{k+1},t}^{-1}(y)) | I_{V^c}(\Delta X_{t_k}) = 0 \quad (4.28)
\]

for any \( \epsilon > 0 \). Here, \( V_{\epsilon} := \{ v \in \mathbb{C}; \| v \|_{0+1} \leq \epsilon \} \) and \( V^c \) is the complement of the set \( V_{\epsilon} \).
To prove (4.27), first observe that for any $N > 0$

$$P\left[\sum_{k=0}^{n-1} |\Delta X_k(x_{k+1}^{-1}(y)) - \Delta X_k(x_{k+1}^{-1}(y))| I_{\mathcal{V}_c}(\Delta X_k) > \delta \right]$$

$$\leq P\left[\sum_{k=0}^{n-1} |\Delta X_k(x_{k+1}^{-1}(y)) - \Delta X_k(x_{k+1}^{-1}(y))| I_{\mathcal{V}_c}(\Delta X_k) > \delta; t < T_N \right]$$

$$+ P[T_N \leq t].$$

Furthermore, for any $K > 0$,

$$P\left[\sum_{k=0}^{n-1} |\Delta X_k(x_{k+1}^{-1}(y)) - \Delta X_k(x_{k+1}^{-1}(y))| I_{\mathcal{V}_c}(\Delta X_k) > \delta; t < T_N \right]$$

$$\leq P\left[\left\{\sum_{k=0}^{n-1} \|\Delta X_k\|_{\text{Lip}}^2 I_{\mathcal{V}_c}(\Delta X_k)\right\}^{1/2} \times \left\{\sum_{k=0}^{n-1} |x_{k+1}^{-1}(y) - x_{k+1}^{-1}(y)|^2 \right\}^{1/2} > \delta; t < T_N \right]$$

$$\leq P\left[\sum_{k=0}^{n-1} \|\Delta X_k\|_{\text{Lip}}^2 I_{\mathcal{V}_c}(\Delta X_k) > \delta^2/K; t < T_N \right]$$

$$+ P\left[\sum_{k=0}^{n-1} |x_{k+1}^{-1}(y) - x_{k+1}^{-1}(y)|^2 > K \right]$$

$$\leq (K/\delta^2) E\left[\int_{(0,t]} \int_{\mathcal{V}_c} \|\mathbf{v}\|_{\text{Lip}}^2 I_{(0,T_N)}(r) \tilde{N}_p(dr dv) \right]$$

$$+ P\left[\sum_{k=0}^{n-1} |x_{k+1}^{-1}(y) - x_{k+1}^{-1}(y)|^2 > K \right].$$

Here, note that for any $K > 0$,

$$\text{l.i.p. sup}_{|\eta| \leq K} \|\xi_{\eta} . (x)(t) - \xi_{\eta} . (x)(s, t)\| = 0. \quad (4.29)$$

Indeed, in (4.20), take $Z = \tilde{X}$ of (3.3) of [8] and $T' = T_N$. Then (4.19) immediately
implies (4.29). Therefore, it follows that

\[
\begin{align*}
\text{l.i.p.} & \left| \sum_{k=0}^{n-1} \left[ \xi_{i_{k+1}, t}^{-1}(y) - \xi_{i_{k}, t}^{-1}(y) \right]^2 \right| \\
& = \text{l.i.p.} \left| \sum_{k=0}^{n-1} \left[ \xi_{i_{k}, t}^{-1}(x) - \xi_{i_{k}, t}^{-1}(x) \right] \right|_{x=\xi_{i_{k}, t}^{-1}(y)} \\
& = \text{l.i.p.} \left[ \xi_{i_{k}, t}(x) \right]_{(0, t)} \left| x=\xi_{i_{k}, t}^{-1}(y) \right. \\
& = \left[ \xi_{i_{k}, t}(x) \right]_{(0, t)} \left| x=\xi_{i_{k}, t}^{-1}(y) \right. ,
\end{align*}
\]  

and so we see that

\[
\limsup_{n \to \infty} P \left[ \sum_{k=0}^{n-1} \left| \xi_{i_{k+1}, t}^{-1}(y) - \xi_{i_{k}, t}^{-1}(y) \right|^2 > K \right] \leq P \left[ \left[ x=\xi_{i_{k}, t}^{-1}(y) \right] > K/2 \right].
\]

Thus, we have

\[
\limsup_{n \to \infty} P \left[ \sum_{k=0}^{n-1} \left| \Delta X_{i_k} \left( \xi_{i_{k+1}, t}^{-1}(y) \right) - \Delta X_{i_k} \left( \xi_{i_{k}, t}^{-1}(y) \right) \right| I_v (\Delta X_{i_k}) > \delta; t < T_N \right] \\
\leq (K/\delta^2) E \left[ \int_{(0, t)} \int_{V_v} \|v\|_{Lip}^2 I(0, T_N)(r) \tilde{N}_p (dr, dv) \right] \\
+ P \left[ \left[ x=\xi_{i_{k}, t}^{-1}(y) \right] > K/2 \right].
\]

Since the sequence of random variables \( \{\int_{(0, t)} \int_{V_v} \|v\|_{Lip}^2 I(0, T_N)(r) \tilde{N}_p (dr, dv)\}_{e} \) converges to 0 a.s. as \( e \to 0 \) and it is uniformly bounded by \( N \), it converges to 0 in \( L^1 (dP) \). Furthermore, since \( \lim_{K \to \infty} P \left[ \left[ x=\xi_{i_{k}, t}^{-1}(y) \right] > K/2 \right] = 0 \), we have

\[
\liminf_{n \to \infty} P \left[ \sum_{k=0}^{n-1} \left| \Delta X_{i_k} \left( \xi_{i_{k+1}, t}^{-1}(y) \right) - \Delta X_{i_k} \left( \xi_{i_{k}, t}^{-1}(y) \right) \right| I_v (\Delta X_{i_k}) > \delta; t < T_N \right] = 0
\]

for any \( N > 0 \). Therefore, it is now clear that (4.27) holds.

Next, we prove (4.28). To this end, for each \( \varepsilon > 0 \), set \( J^\varepsilon := \{ r \in (0, t]; \Delta X_r \in V^\varepsilon \} \). Then, note that \( J^\varepsilon \) is finite almost surely on the set \( \{ t < T_N \} \). Indeed, it is easy to see from Condition C that \( E[N_p(s, t), V^\varepsilon]; t < T_N \) < \( \infty \). Hence, we fix \( \omega \in \{ t < T_N \} \) and let \( 0 < r_1 < \cdots < r_m < t \) be the elements of \( J^\varepsilon \). Then,

\[
\sum_{k=0}^{n-1} \left| \Delta X_{i_k} \left( \xi_{i_{k+1}, t}^{-1}(y) \right) - \Delta X_{i_k} \left( \xi_{i_{k}, t}^{-1}(y) \right) \right| I_v (\Delta X_{i_k}) \\
\leq \sum_{j; i_k = r_j < i_{k+1}} \left| \Delta X_{r_j} \left( \xi_{i_{k+1}, t}^{-1}(y) \right) - \Delta X_{r_j} \left( \xi_{i_{k}, t}^{-1}(y) \right) \right| I_v (\Delta X_{r_j})
\]
\[ \leq \sum_{j=1}^{m} |\Delta X_{r_j}(\xi_j^{-1}(y)) - \Delta X_{r_j}(\xi_k^{-1}(y))| I_{r_k=r_j}(k) I_{\nu}(\Delta X_{r_j}) \]

\[ \xrightarrow{|\nu| \downarrow 0, \text{m}} \sum_{j=1}^{m} |\Delta X_{r_j}(\xi_j^{-1}(y)) - \Delta X_{r_j}(\xi_k^{-1}(y))| I_{\nu}(\Delta X_{r_j}) \]

\[ = 0, \]

and so, for any \( \varepsilon > 0 \), \( N > 0 \) and \( \delta > 0 \),

\[ \lim sup_{|\nu| \downarrow 0} P \left[ \sum_{k=0}^{n-1} |\Delta X_{k}(\xi_k^{-1}(y)) - \Delta X_{k}(\xi_{k+1}^{-1}(y))| I_{\nu}(\Delta X_{k}) > \delta; t < T_N \right] = 0. \]

Therefore, it is now clear that (4.28) holds. Thus, we have completed the proof of Lemma 4.4.

\[ \Box \]

We next turn to the proof of Lemma 4.5.

**Proof of Lemma 4.5.** For each \( \varepsilon > 0 \), we decompose \( X \) as \( X = M_\varepsilon + B_\varepsilon + X^\varepsilon \), where

\[ M_\varepsilon(x, t) := M_\varepsilon(x, t) + \int_{(0, t]} \int_{V_\varepsilon} v(x) \tilde{N}_p(dr dv), \]

\[ B_\varepsilon(x, t) := B(x, t) - \int_{(0, t]} \int_{V_\varepsilon} v(x) \tilde{N}_p(dr dv), \]

\[ X^\varepsilon(x, t) := \int_{(0, t]} \int_{V_\varepsilon} v(x) N_p(dr dv). \]

Then, since it holds that \( I_2^\Pi(X) = I_2^\Pi(M_\varepsilon) + I_2^\Pi(B_\varepsilon) + I_2^\Pi(X^\varepsilon) \), our proof of this lemma will be reduced to showing that for any \( \delta > 0 \),

\[ \lim_{\varepsilon \downarrow 0} \lim_{|\nu| \downarrow 0} P \left[ \left| I_2^\Pi(M_\varepsilon) - \int_{(s, t]} [\nabla X_\varepsilon^{(-)}, X_\varepsilon^{(-)}](\xi^{-1}_r(y), \hat{a}r) \right| > \delta \right] = 0, \quad (4.31) \]

\[ \lim_{\varepsilon \downarrow 0} \lim_{|\nu| \downarrow 0} P[|I_2^\Pi(B_\varepsilon)| > \delta] = 0, \quad (4.32) \]

\[ \lim_{\varepsilon \downarrow 0} \lim_{|\nu| \downarrow 0} P \left[ \left| I_2^\Pi(X^\varepsilon) - \sum_{r \in (s, t]} |\Delta X_r(\xi_{r-1}^{-1}(y)) - \Delta X_r(\xi_{r-1}^{-1}(y))| \right| > \delta \right] = 0. \quad (4.33) \]
We will first prove (4.31). By the mean value theorem, it holds that

\[ M_\epsilon(x_{k+1}, t)(y, t) - M_\epsilon(x_k, t)(y, t) = \nabla M_\epsilon(x_k, t)(y, t) \cdot (x_{k+1} - x_k) \]

for \( k = 0, \ldots, n - 1 \), where \( x_k := x_k \) and \( y = x_k \). Hence,

\[ \limsup_{\epsilon \downarrow 0} \mathbb{P} \left( \max_{|\Pi| = 0} \left| I_{2,1}^{\Pi}(M_\epsilon)(x) - \int_{[s,t]} \nabla \xi_{t,r}(x) \cdot d\epsilon \right| > \delta \right) = 0, \]  

(4.34)

and

\[ \limsup_{\epsilon \downarrow 0} \mathbb{P} \left( \max_{|\Pi| = 0} \left| I_{2,2}^{\Pi}(M_\epsilon) \right| > \delta \right) = 0. \]  

(4.35)

To show (4.34), observe that

\[ I_{2,1}^{\Pi}(M_\epsilon)(x) = \sum_{k=0}^{n-1} \nabla M_\epsilon(x_k, t_k) \cdot (x_{k+1} - x_k) \]

for \( k = 0, \ldots, n - 1 \), where \( x_k := x_k \) and \( y = x_k \). Hence, taking \( Z = \nabla M_\epsilon \) in Lemma 4.6, we see that for any \( K > 0 \),

\[ \limsup_{\epsilon \downarrow 0} \mathbb{P} \left( \max_{|\Pi| = 0, |x| \leq K} \left| I_{2,1}^{\Pi}(M_\epsilon)(x) - \int_{[s,t]} \nabla M_\epsilon(x, r) \cdot d\epsilon \right| > \delta \right) = 0. \]  

(4.36)
Set $M_{\epsilon,d}(x,t) := \int_{(0,t]} \int_{\mathcal{V}_e} v(x) \tilde{N}_p(dr \, dv)$. Then note that
\[
\begin{align*}
\left[ \int_{(s,t]} \nabla M_\epsilon(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]} &= \left[ \int_{(s,t]} \nabla M_c(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]} \\
&\quad + \left[ \int_{(s,t]} \nabla M_{\epsilon,d}(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]} \\
&= \int_{(s,t]} \left[ \nabla X_c, X_c \right] \xi_{s,r-}(x), dr \\
&\quad + \left[ \int_{(s,t]} \nabla M_{\epsilon,d}(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]} \\
&= \int_{(s,t]} \left[ \nabla X_c, X_c \right] \xi_{s,r-}(x), dr \\
&\quad + \left[ \int_{(s,t]} \nabla M_{\epsilon,d}(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]}
\end{align*}
\] (4.37)

Moreover, it holds that
\[
\text{l.i.p. } \sup_{\epsilon \downarrow 0} \left[ \int_{(s,t]} \nabla M_{\epsilon,d}(\xi_{s,r-}(x), dr), \xi_{s,}(x) \right]_{(s,t]} = 0. \tag{4.38}
\]

To see this, by the localization argument and by the fact that $\sup_{|x| \leq K} \xi_{s,}(x)_{(s,t]} < \infty$ a.s., it is sufficient to show that for each $N > 0$,
\[
\text{l.i.p. } \sup_{\epsilon \downarrow 0} \left[ \int_{(s,t]} \nabla M^{T_N}_{\epsilon,d}(\xi_{s,r-}(x), dr) \right]_{(s,t]} = 0.
\]

Indeed, we see this from the observation that
\[
\begin{align*}
E \left[ \sup_{|x| \leq K} \left[ \int_{(s,t]} \nabla M^{T_N}_{\epsilon,d}(\xi_{s,r-}(x), dr) \right]_{(s,t]} \right] \\
&\leq 2E \left[ \sup_{|x| \leq K} \left\{ \int_{(s,t]} \int_{\mathcal{V}_e} |\nabla v(\xi_{s,r-}^N(x))| I_{(0,T_N)}(r) N_p(dr \, dv) \\
&\quad + \int_{(s,t]} \int_{\mathcal{V}_e} |\nabla v(\xi_{s,r-}^N(x))| I_{(0,T_N)}(r) \tilde{N}_p(dr \, dv) \right\} \right] \\
&\leq 4E \left[ \int_{(s,t]} \int_{\mathcal{V}_e} \|\nabla v\|_{\infty} I_{(0,T_N)}(r) \tilde{N}_p(dr \, dv) \right],
\end{align*}
\]

which converges to 0 as $\epsilon \downarrow 0$.

Consequently, it follows from (4.36)–(4.38) that for any $\delta > 0$,
\[
\lim_{\epsilon \downarrow 0} \lim_{|\Pi| \downarrow 0} P \left[ \sup_{|x| \leq K} \left| L_{2,1}^{\Pi}(M_\epsilon)(\xi_{s,}(x)) - \int_{(s,t]} \left[ \nabla X_c, X_c \right] \xi_{s,r-}(x), dr \right| > \delta \right] = 0.
\]
In addition to this fact, observe that
\[
\int_{(s,t]} [\nabla X_c, X_c](\xi_{s,r,-}(x), dr) \bigg|_{x = \xi_{s,r,-}^{-1}(y)} = \int_{(s,t]} [\nabla X_c, X_c](\xi_{s,r}^{-1}(y), \hat{dr})
\]
\[
= \int_{(s,t]} [\nabla X_c, X_c](\xi_{s,r}^{-1}(y), \hat{dr}), \tag{4.39}
\]
because \([\nabla X_c, X_c](x, r) = [\nabla X_c(x), X_c(-)](x, r)\) for all \(x \in \mathbb{R}^d\) and \(r \geq 0\). Thus, we obtain (4.34).

To show (4.35), observe that
\[
\|I_{2,2}^1(M_\varepsilon)\| \leq \sup_{k,u} |\nabla^2 M_\varepsilon(\Xi_{k,u}, t_{k+1}) - \nabla^2 M_\varepsilon(\Xi_{k,u}, t_k)|
\]
\[
\times \sum_{k=0}^{n-1} |\xi_{s+r,k+1}^{-1}(y) - \xi_{s+r,k}^{-1}(y)|^2,
\]
and so, in view of (4.30), it is sufficient to prove that for any \(\delta > 0\)
\[
\lim_{\varepsilon \downarrow 0} \lim_{|\Pi| \downarrow 0} \sup_{k,u} \left[ \sup |\nabla^2 M_\varepsilon(\Xi_{k,u}, t_{k+1}) - \nabla^2 M_\varepsilon(\Xi_{k,u}, t_k)| > \delta \right] = 0. \tag{4.40}
\]
To see this, note that for any \(K > 0\),
\[
\left\{ \sup_{k,u} |\nabla^2 M_\varepsilon(\Xi_{k,u}, t_{k+1}) - \nabla^2 M_\varepsilon(\Xi_{k,u}, t_k)| > \delta \right\}
\]
\[
\subset \left\{ \sup_k \sup_{|x| \leq K} |\nabla^2 M_c(x, t_{k+1}) - \nabla^2 M_c(x, t_k)| > \delta/2 \right\}
\]
\[
\cup \left\{ \sup_{|x| \leq K} \sup_{r \in [s,t]} |\nabla^2 M_{c,d}(x, r)| > \delta/2 \right\} \cup \left\{ \sup_{r \in [s,t]} |\xi_{r,-}^{-1}(y)| > K \right\}.
\]
Here, since \(\nabla^2 M_c\) is a continuous \(C\)-valued process, we have
\[
\text{l.i.p.} \sup_{|\Pi| \downarrow 0} \sup_{k \leq K} |\nabla^2 M_c(x, t_{k+1}) - \nabla^2 M_c(x, t_k)| = 0.
\]
On the other hand, applying Lemma 4.1 as in the proof of Lemma 4.3, we can show that
\[
\text{l.i.p.} \sup_{|\Pi| \downarrow 0} \sup_{|x| \leq K} |\nabla^2 M_{c,d}(x, r)| = 0.
\]
Therefore, from these observations we have

$$\lim_{\varepsilon \downarrow 0} \lim_{|\Pi| \downarrow 0} \sup_{k,u} \left[ \sup_{k,u} |\nabla^2 M_\varepsilon (\mathbb{E}_{k,u}, t_k + 1) - \nabla^2 M_\varepsilon (\mathbb{E}_{k,u}, t_k)| > \delta \right]$$

\[ \leq P \left[ 3 \sup_{r \in [s,t]} |\xi_{t_0}^{-1} (y)| > K \right]. \]

Since \( K \) is arbitrary, we get (4.40). Thus, we have shown (4.31).

Next, we will prove (4.32). Note that

$$B_\varepsilon (x, t) = \int_{(0,t]} b(x, r) dA_r - \int_{(0,t]} \int_{V_\varepsilon} v(x) v_r (dv) dA_r = \int_{(0,t]} b_\varepsilon (x, r) dA_r,$$

where \( b_\varepsilon (x, r) := b(x, r) - \int_{V_\varepsilon} v(x) v_r (dv) \). Hence,

$$|I_2^\Pi (B_\varepsilon)| \leq \sum_{k=0}^{n-1} \int_{(t_{k}, t_{k+1}]} |b_\varepsilon (\xi_{t_{k+1}}^{-1} (y), r) - b_\varepsilon (\xi_{t_k}^{-1} (y), r)| dA_r$$

\[ \leq \sum_{k=0}^{n-1} \int_{(t_{k}, t_{k+1}]} \|b_\varepsilon (r)\|_{0+1} dA_r \times |\xi_{t_{k+1}}^{-1} (y) - \xi_{t_k}^{-1} (y)| \]

\[ \leq \|D_\varepsilon (t_s, t)\|^{1/2} \times \left\{ \sum_{k=0}^{n-1} |\xi_{t_{k+1}}^{-1} (y) - \xi_{t_k}^{-1} (y)|^2 \right\}^{1/2}, \]

where we set \( D_\varepsilon (t) := \int_{(0,t]} \|b_\varepsilon (r)\|_{0+1} dA_r \). Moreover, observe that

$$\Delta D_\varepsilon (r) = \|b(\cdot, r) \Delta A_r - \int_{V_\varepsilon} v(\cdot) v_r (dv) \Delta A_r\|_{0+1}$$

\[ = \left\| \int_{V_\varepsilon} v(\cdot) \tilde{N}_p (\{r\}, dA_r) - \int_{V_\varepsilon} v(\cdot) \tilde{N}_p (\{r\}, dv) \right\|_{0+1} \]

\[ \leq \int_{V_\varepsilon} \|v\|_{0+1} \tilde{N}_p (\{r\}, dv), \]

and hence, we have

$$\text{l.i.p. } [D_\varepsilon]_{(s,t]} = [D_\varepsilon]_{(s,t]} = \sum_{r \in (s,t]} |\Delta D_\varepsilon (r)|^2 \leq \int_{(s,t]} \int_{V_\varepsilon} \|v\|^2_{0+1} \tilde{N}_p (dr, dv).$$

Combining the fact that the last term converges to 0 in probability as \( \varepsilon \downarrow 0 \) with (4.30), it is easy to see that (4.32) holds.
Finally, we will prove (4.33). To this end, observe that for any $\delta > 0$ and $N > 0$, we have
\[
P\left[ I_2^n (X^e) - \sum_{r \in (s, t)} \{\Delta X_r (\xi^{-1}_{r,t} (y)) - \Delta X_r (\xi^{-1}_{r-t} (y))\} I_{V^e} (\Delta X_r) > \delta \right]
\leq P\left[ I_2^n (X^e) - \sum_{r \in (s, t)} \{\Delta X_r (\xi^{-1}_{r,t} (y)) - \Delta X_r (\xi^{-1}_{r-t} (y))\} \times I_{V^e} (\Delta X_r) > \delta; t < T_N \right] + P[T_N \leq t].
\]

Here, note that, as we have already shown in the proof of Lemma 4.4, the set $J^e = \{r \in (s, t); \Delta X_r \in V^e\}$ is finite almost surely on the set $\{t < T_N\}$, and so we fix $\omega \in \{t < T_N\}$ and let $s < r_1 < \cdots < r_m \leq t$ be the elements of $J^e$. Since we can take $|\Pi|$ sufficiently small so that each subinterval $(t_k, t_{k+1})$ includes at most one jumping time $r_j$, we see that
\[
I_2^n (X^e) = \sum_{k; t_k < r_j \leq t_{k+1}} \{[X^e (\xi^{-1}_{k+1,t} (y), t_{k+1}) - X^e (\xi^{-1}_{k,t} (y), t_k)]
- [X^e (\xi^{-1}_{k,t} (y), t_{k+1}) - X^e (\xi^{-1}_{k+1,t} (y), t_k)]\}
= \sum_{j; t_k < r_j \leq t_{k+1}} \{(\Delta X^e)_{r_j} (\xi^{-1}_{k,t} (y)) - (\Delta X^e)_{r_j} (\xi^{-1}_{k+1,t} (y))\}
\xrightarrow{|\Pi| \downarrow 0} \sum_{j=1}^m \{(\Delta X^e)_{r_j} (\xi^{-1}_{r_j,t} (y)) - (\Delta X^e)_{r_j} (\xi^{-1}_{r_j-t} (y))\}
\]
\[
= \sum_{r \in (s, t)} \{\Delta X_r (\xi^{-1}_{r,t} (y)) - \Delta X_r (\xi^{-1}_{r-t} (y))\} I_{V^e} (\Delta X_r).
\]
Hence,
\[
\lim_{|\Pi| \downarrow 0} P\left[ I_2^n (X^e) - \sum_{r \in (s, t)} \{\Delta X_r (\xi^{-1}_{r,t} (y)) - \Delta X_r (\xi^{-1}_{r-t} (y))\} \times I_{V^e} (\Delta X_r) > \delta; t < T_N \right] = 0.
\]
Combining this with the fact that $\lim_{N \uparrow \infty} P[T_N \leq t] = 0$, we see that
\[
\limsup_{|\Pi| \downarrow 0} P\left[ I_2^n (X^e) - \sum_{r \in (s, t)} \{\Delta X_r (\xi^{-1}_{r,t} (y)) - \Delta X_r (\xi^{-1}_{r-t} (y))\} I_{V^e} (\Delta X_r) > \delta \right] = 0.
\]
Therefore, it is now clear that (4.33) will follow once we show that
\[
\sum_{r \in (s,t)} |\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,-t}^{-1}(y))| < \infty \quad \text{a.s.} \quad (4.41)
\]

Observe that for any \( r \in (s,t) \),
\[
|\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,-t}^{-1}(y))| \leq \|\Delta X_r\|_{\text{lip}}|\xi_{r,t}^{-1}(y) - \text{Exp}(\Delta X_r)(\xi_{r,t}^{-1}(y))| \\
\leq \|\Delta X_r\|_{\text{lip}}\|\Delta X_r\|_{\text{lip}}e^{\|\Delta X_r\|_{\text{lip}}(1 + |\xi_{r,t}^{-1}(y)|)} \\
\leq C_M\|\Delta X_r\|_{0+1}^2 \left(1 + \sup_{r \in (s,t)} |\xi_{r,t}^{-1}(y)|\right),
\]

where \( C_M \) is a constant depending only on \( M \). Since \( \sup_{r \in (s,t)} |\xi_{r,t}^{-1}(y)| < \infty \) a.s. and \( \sum_{r \in (s,t)} \|\Delta X_r\|_{0+1}^2 < \infty \) a.s., we obtain (4.41).

Thus, we have completed the proof of Lemma 4.5. \( \Box \)

Though we have had much to prove, we can now say that Proposition 4.1 has been established.

We are now in a position to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** In addition to (4.4) and (4.39), we need to establish the fact that
\[
\left( \sum_{r \in (s,t)} [\text{Exp}(\Delta X_r)(\xi_{s,r}^{-}(x)) - \xi_{s,r}^{-}(x) - \Delta X_r(\xi_{s,r}^{-}(x))] \right)_{x=\xi_{r,t}^{-1}(y)} \\
= -\sum_{r \in (s,t)} [\text{Exp}(\Delta X_r)(\xi_{r,t}^{-1}(y)) - \xi_{r,t}^{-1}(y) - (-\Delta X_r)(\xi_{r,t}^{-1}(y))] \\
+ \sum_{r \in (s,t)} [\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,-t}^{-1}(y))], \quad (4.42)
\]

which will be proved from the following observation. Since
\[
|\text{Exp}(\Delta X_r)(x) - x - \Delta X_r(x)| \leq \|\Delta X_r\|_{0+1}^2 e^{\|\Delta X_r\|_{0+1}(1 + |x|)},
\]
it holds that for any \( K > 0 \) and \( N > 0 \),
\[
E \left[ \sup_{|x| \leq K} \sum_{r \in (s,t)} |\text{Exp}(\Delta X_r)(\xi_{s,r}^{-}(x)) - \xi_{s,r}^{-}(x) - \Delta X_r(\xi_{s,r}^{-}(x))|; t < T_N \right] \\
\leq C_M E \left[ \int_{(s,t)} \int_{\mathcal{C}} \|v\|_{0+1}^2 N_p(dr \, dv) \times \sup_{|x| \leq K} \sup_{r \in (s,t)} (1 + |\xi_{s,r}^{-}(x)|); t < T_N \right] 
\]
\[ \leq C_M E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv) \right)^{1/2} \]
\[ \times E \left( \sup_{|x| \leq K} \sup_{r \in (s,t]} (1 + |\xi_{s,s,r}^{T_N^+}(x)|^2) \right)^{1/2}, \]

where \( C_M \) is a constant depending only on \( M \). Moreover,

\[ E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv)^2 \right) \]
\[ \leq 2 \left( E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv)^2 \right) \right)^{1/2} \]
\[ + E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv)^2 \right) \]
\[ \leq 4E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^4 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv) \right) \]
\[ + 2E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv)^2 \right) \]
\[ \leq C_M \left( E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv) \right) \right)^{1/2} \]
\[ + E \left( \int_{(s,t]} \int_C \|v\|_{0+1}^2 I_{(0,T_N)}(t) \hat{N}_p(\,dr\,dv)^2 \right) \]
\[ \leq C_M \left( E[\bar{A}_t^{T_N^-}] + E[\bar{A}_t^{T_N^-}]^2 \right) \]
\[ < \infty. \]

On the other hand, it is now clear that

\[ E \left( \sup_{|x| \leq K} \sup_{r \in (s,t]} |\xi_{s,s,r}^{T_N^+}(x)|^2 \right) < \infty. \]

Therefore, there exists a \( P \)-null set \( N_0 \) (independent of \( x \)) such that on \( (N_0)^c \),

\[ \sum_{r \in (s,t]} |\text{Exp}(\Delta X_r)(\xi_{s,r^-}(x)) - \xi_{s,r^-}(x) - \Delta X_r(\xi_{s,r^-}(x))| \times I_{[\xi_{s,r^-}(x) \leq T_N]} < \infty. \]

Hence, on \( (N_0)^c \cap \{ t < T_N \} \), we have

\[ \left( \sum_{r \in (s,t]} \{\text{Exp}(\Delta X_r)(\xi_{s,r^-}(x)) - \xi_{s,r^-}(x) - \Delta X_r(\xi_{s,r^-}(x))\} \right)_{x=\xi_{s,r}^{-1}(y)} \]
\[ = \sum_{r \in (s,t]} \{\text{Exp}(\Delta X_r)(\xi_{r^-}(y)) - \xi_{r^-}(y) - \Delta X_r(\xi_{r^-}(y))\} \]
= \sum_{r \in (s, t]} (\xi_{r,t}^{-1}(y) - \text{Exp}(-\Delta X_r)(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,t}^{-1}(y))) \\
+ \sum_{r \in (s, t]} \{\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,t}^{-1}(y))\} \\
= -\sum_{r \in (s, t]} \{\text{Exp}(-\Delta X_r)(\xi_{r,t}^{-1}(y)) - \xi_{r,t}^{-1}(y) - (-\Delta X_r)(\xi_{r,t}^{-1}(y))\} \\
+ \sum_{r \in (s, t]} \{\Delta X_r(\xi_{r,t}^{-1}(y)) - \Delta X_r(\xi_{r,t}^{-1}(y))\}.

Since \(\lim_{N \to \infty} P[t < T_N] = 1\), we see that (4.42) holds.

Now substitute \(x\) in (3.1) (or equivalently in (4.3)) with \(\xi_{s,t}^{-1}(y)\). Then it follows from (4.4), (4.39) and (4.42) that

\[
y - \xi_{s,t}^{-1}(y) = (\xi_{s,t}(x) - x)_{|x=\xi_{s,t}^{-1}(y)}
\]

\[
= \int_{(s, t]} X^- (\xi_{r,t}^{-1}(y), \hat{d}r) - \frac{1}{2} \int_{(s, t]} [\nabla X^{-,-}_c, X^{-,-}_c](\xi_{r,t}^{-1}(y), \hat{d}r) \\
- \sum_{r \in (s, t]} \{\text{Exp}(-\Delta X_r)(\xi_{r,t}^{-1}(y)) - \xi_{r,t}^{-1}(y) - (-\Delta X_r)(\xi_{r,t}^{-1}(y))\}.
\]

It is now obvious that, for each \(y \in \mathbb{R}^d\), \(\xi_{s,t}^{-1}(y)\) satisfies the canonical backward SDE (3.2) and that \(\xi_{s,t}^{-1}(y)\) is a backward semimartingale. \(\square\)

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