QUOTIENTS OF EXCEPTIONAL LIE GROUPS AS FRAMED BOUNDARIES

Haruo MINAMI

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1. Introduction

Let G be a compact connected simply-connected Lie group and let T be a maximal torus of G . By explicitly constructing a framed manifold W with boundary ∂W that is G/T , Pittie and Smith in [4] have shown that G/T is framed null-cobordant. In a similar way we also showed [3] that the same kind of thing is true for the real, complex and quaternionic Stiefel manifolds. The first purpose of this paper is to generalize such a method, as follows.

Let H and L be closed connected subgroups of G such that $H \subset L$, and let ρ_H and ρ_L be real representations of H and L respectively. Assume that there exist real G -modules R and S satisfying the equalities

$$\rho_L + R|_L = \text{Ad}_L + S|_L, \quad \rho_L|_H = \rho_H + \ell + 1 \quad \text{and} \quad \text{Ad}_L|_H = \text{Ad}_H + \rho_H + \ell, \quad (1)$$

where Ad_H and Ad_L denote the adjoint representations of H and L respectively and n denotes the trivial n -dimensional real representation. Furthermore, let L act on L/H by a natural left translation and let us assume that there is an L -equivariant diffeomorphism

$$L/H \approx S(V), \quad (2)$$

where V denotes the representation space of ρ_L and $S(V)$ the unit sphere of V . With assumptions (1) and (2) we see that G/H becomes canonically framed and with respect to this framing we have the following.

THEOREM 1. *G/H is framed null-cobordant.*

All the framed manifolds of the form G/H are classified in [5]. Theorem 1 applies to the quotients of exceptional Lie groups listed there except the case when H is a toral subgroup of G . Then we obtain the following.

THEOREM 2. *Let (G, H) be the pair $(G_2, SU(2))$, $(G_2, SU(3))$, $(F_4, SU(3))$, $(F_4, Spin(8))$, $(E_6, Spin(8))$ or $(E_7, Spin(8))$ with the natural embeddings. Then G/H is framed null-cobordant.*

Remark. Let G be as in Theorem 1 and let S^1 be a circle subgroup of G . If G contains a subgroup $S^3 = Sp(1)$ with S^1 as a maximal torus, then G/S^1 is seen to be framed null-cobordant from Theorem 1 and the fact that Ad_{S^3} coincides with the canonical non-trivial three-dimensional real representation of S^3 . This fact is exactly one of the essentials in the proof of [4].

2. Framing of G/H

In this section we recall a canonical construction of the framing of G/H . We denote the tangent bundle of a manifold M by $\tau(M)$. Consider the principal H -bundle

$$H \rightarrow G \xrightarrow{\pi} G/H.$$

We then have a decomposition of $\tau(G)$:

$$\tau(G) \cong \pi^* \tau(G/H) \oplus \tau_H(G),$$

where $\tau_H(G)$ is the bundle of tangents along the fibres. This isomorphism is compatible with the natural right action of H , so that we obtain an isomorphism of vector bundles over G/H :

$$\tau(G)/H \cong \tau(G/H) \oplus \tau_H(G)/H.$$

We denote the tangent space at the identity e of G by $\tau_e(G)$ and also denote the adjoint representation of G on $\tau_e(G)$ by Ad_G as in Section 1. Further, let $L_g : \tau_e(G) \rightarrow \tau(G)$ denote the differential of left multiplication by $g \in G$. Then the map $(g, v) \mapsto L_g(v)$ gives rise to a framing of G :

$$G \times \tau_e(G) \cong \tau(G)$$

which is also compatible with the action of H , where H acts on $\tau_e(G)$ via the restriction of Ad_G to H , so that we see

$$G/H \times \tau_e(G) \cong \tau(G)/H$$

as vector bundles over G/H . Analogously, observing the linear maps from $\tau_e(H)$ to $\tau_H(G)$ induced by the left transformations of G we have

$$G \times_{Ad_H} \tau_e(H) \cong \tau_H(G)/H$$

as vector bundles over G/H . Combining the above three bundle equations yields an isomorphism of vector bundles over G/H :

$$\tau(G/H) \oplus G \times_{\text{Ad}_H} \tau_e(H) \cong G/H \times \tau_e(G). \quad (*)$$

Suppose that there exist real G -modules P and Q such that $P|_H = \text{Ad}_H + Q|_H$. (According to [5] this is a necessary and sufficient condition for G/H to become framed.) Then equation $(*)$ induces a stable isomorphism of vector bundles over G/H :

$$\tau(G/H) \oplus \varepsilon^p \cong G/H \times \tau_e(G) \oplus \varepsilon^q \quad (**)$$

which is the canonical framing of G/H , where $p = \dim P$, $q = \dim Q$ and ε^n denotes a trivial n -dimensional real vector bundle over G/H .

It is clear that there is a real H -module such that the direct sum of this and ρ_H becomes a real G -module. So we name this G -module T . Then by the equalities of (1) we may take $T + R + \ell + 1$ and $T + S + \ell$ as P and Q stated after $(*)$ respectively, so that G/H of Theorem 1 becomes framed.

3. Proof of Theorem 1

Let ξ denote the vector bundle

$$G \times_L V \rightarrow G/L$$

associated with the principal bundle $G \rightarrow G/L$ with fibre V acted on by L via ρ_L . The equivariant diffeomorphism of (2) implies that the total space of the sphere bundle $S(\xi)$ of ξ is diffeomorphic to G/H . Hence we have

$$\partial W \approx G/H,$$

where W denotes the total space of the disc bundle $D(\xi)$ of ξ . To prove the theorem it suffices to show that the framing of G/H extends over W .

Identifying G/H with the total space of $S(\xi)$ via the above diffeomorphism, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H & \xrightarrow{p} & G/L \\ & & \cap & \nearrow q & \\ & & W & & \end{array}$$

where the arrows p and q denote the bundle projections and π is as above. Then it is easily verified that $p^*\xi$ has a natural non-zero cross-section, so that there is a decomposition

$$p^*\xi \cong \eta \oplus \varepsilon^1.$$

In fact, it follows from the second equality of (1) that η agrees with the vector bundle associated with the principal bundle $G \rightarrow G/H$ with the representation space of $\rho_H + \ell$ as fibre. Furthermore, we see that there exist decompositions

$$\tau(G/H) \cong p^* \tau(G/L) \oplus \eta \quad \text{and} \quad \tau(W) \cong q^*(\tau(G/L) \oplus \xi). \quad (3)$$

Apply the equalities of (1) to equation (*) with L in place of H , we then have

$$\tau(G/L) \oplus \xi \oplus G/L \times R \cong G/L \times (\tau_e(G) \oplus S),$$

where R and S denote themselves with G -actions that are forgotten. Pulling back this equation along p and q and using the formulas of (1) and (3), we get the framings of G/H and W :

$$\tau(G/H) \oplus \varepsilon^1 \oplus G/H \times R \cong G/H \times (\tau_e(G) \oplus S)$$

and

$$\tau(W) \oplus W \times R \cong W \times (\tau_e(G) \oplus S),$$

where R and S are as above. The second equation defines a framing of W which has restriction to G/H that becomes the first one. Clearly, this is equivalent to the framing given by (*), which completes the proof of Theorem 1.

4. Proof of Theorem 2

We know [1, 6] that there is a natural sequence of inclusions

$$E_7 \supset E_6 \supset F_4 \supset Spin(9) \supset Spin(8) \supset G_2 \supset SU(3) \supset SU(2)$$

and four embeddings of them induce $G_2/SU(2) \approx V_{7,2}$, $G_2/SU(3) \approx S^6$, $Spin(9)/Spin(8) \approx S^8$ and $SU(3)/SU(2) \approx S^5$. In order to prove Theorem 2, by Theorem 1 it suffices to find the subgroup L and the representations ρ_L and ρ_H satisfying the assumptions (1) and (2) for the pairs (G, H) mentioned in Theorem 2. We check this case by case.

In the case $(G, H) = (E_7, Spin(8))$, take $L = Spin(9)$ and choose for ρ_L and ρ_H the standard representations of L and H respectively. Then obviously $\rho_L|_H = \rho_H + 1$, $Ad_L|_H = Ad_H + \rho_H$ and $L/H \approx S(V)$, where V denotes the representation space of ρ_L as before.

Let Δ denote the real spin representation of L and denote by c and r the complexification and realification functors respectively. By information given in [1, p. 52] we have $Ad_G|_L = Ad_L + 3\rho_L + 4\Delta + 3$ and by Corollary 8.2 of [1] there is a

56-dimensional complex representation ρ of G such that $\rho|L = 2c(\rho_L) + 2c(\Delta) + 6$, so that $r(\rho)|L = 4\rho_L + 4\Delta + 12$. Hence we see $\rho_L + \text{Ad}_G|L + 9 = \text{Ad}_L + r(\rho)|L$.

In the case $(G, H) = (E_6, \text{Spin}(8)), (F_4, \text{Spin}(8))$, let L and ρ_L be as in the above case. By Corollaries 8.1 and 8.3 of [1] there are a 26-dimensional real representation ρ_{F_4} of F_4 and a 27-dimensional complex representation ρ_{E_6} of E_6 such that $\rho_{F_4}|L = \rho_L + \Delta + 1$ and $\rho_{E_6}|F_4 = c(\rho_{F_4}) + 1$ respectively. We also have $\text{Ad}_{E_6}|F_4 = \text{Ad}_{F_4} + \rho_{F_4}$ by Lemma 14.4 of [1]. Combining this with the equality preceding the Note of [1, p. 53] gives $\text{Ad}_{F_4}|L = \text{Ad}_L + \Delta$ (see [7] for the direct reference). From these formulas it is easily deduced that $\rho_L + \text{Ad}_{F_4}|L + 1 = \text{Ad}_L + \rho_{F_4}|L$ and $\rho_L + \text{Ad}_{E_6}|L + 3 = \text{Ad}_L + r(\rho_{E_6})|L$.

In the case $(G, H) = (F_4, \text{SU}(3))$, take $L = G_2$ which has a seven-dimensional real representation ρ_L , its representation space denoted also by V . Then we have $L/H \approx S(V)$ by Theorem 5.5 of [1]. Further, by [6] we have $\text{Ad}_L|H = \text{Ad}_H + \rho_H$ and $\rho_L|H = \rho_H + 1$, where ρ_H denotes the realification of the identity representation of H and, moreover, we have $\rho_G|L = 3\rho_L + 5$ and $\text{Ad}_G|L = \text{Ad}_L + 5\rho_L + 3$, so that $\rho_L + \text{Ad}_G|L + 7 = \text{Ad}_L + 2\rho_G|L$.

The assertion for the remaining cases follows immediately by using the diffeomorphism and the first two equalities stated in the final case together with the equalities $\text{Ad}_{\text{SU}(3)}| \text{SU}(2) = \text{Ad}_{\text{SU}(2)} + \rho_{\text{SU}(2)} + 1$ and $\rho_{\text{SU}(3)}| \text{SU}(2) = \rho_{\text{SU}(2)} + 2$, where $\rho_{\text{SU}(n)}$ denotes the realification of the identity representation of $\text{SU}(n)$. This completes the proof of Theorem 2.

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Haruo Minami
 Department of Mathematics
 Nara University of Education
 Takabatake-cho, Nara 630-8528
 Japan