RIGIDITY AND SPHERE THEOREMS FOR SUBMANIFOLDS II

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0. Introduction

This paper is the continuation of our previous work [SX], in which there are typographical errors and mistakes in the statements of theorems. We correct the typographical errors in [SX] and present further developments on this topic. We have improved the results obtained in [SX]. We emphasize that the constants obtained in the Main Theorem and Theorems A and B in [SX] can be replaced by constants independent of the codimension but depending only on the dimension of M . Some examples show that the new constants are sharp. Moreover, we have established the finiteness theorem in Riemannian geometry. The corrections are listed at the end of this paper.

The notation of [SX] is used throughout this paper. Let M be a connected and compact Riemannian n -manifold without boundary and let $\mathbf{F}^{n+p}(c)$ be the complete simply connected $(n + p)$ -dimensional space form of constant sectional curvature $c \geq 0$. Let $\varphi : M \rightarrow \mathbf{F}^{n+p}(c)$ be an isometric immersion let $\mathbf{h}$ be the second fundamental form for φ and let H be the mean curvature. The squared norm S of $\mathbf{h}$ is given by $S = n(n - 1)c - R + n^2H^2$, where R is the scalar curvature of M . Let

$$I(M; \varphi) := \int_M (S - nH^2)^{n/2} dM$$

and

$$I_0(M) := \inf_{\varphi} I(M; \varphi).$$

Then the $I_0(M)$ is independent of the choice of isometric immersion but depends on the Riemannian metric. With this notation the Main Theorem obtained in [SX] is improved as follows.

MAIN THEOREM. *There exists an explicitly given constant $D_n > 0$, depending only on n , such that for every isometric immersion $\varphi : M \rightarrow \mathbf{F}^{n+p}(c)$ of every compact*

Riemannian n -manifold M , we have

$$I(M; \varphi) \geq \begin{cases} D_n \sum_{i=1}^{n-1} \beta_i, & \text{for } n \leq 3 \\ D_n \left(\beta_1 + 3 \sum_{i=2}^{n-2} \beta_i + \beta_{n-1} \right), & \text{for } n \geq 4, \end{cases}$$

where β_i is the i th Betti number of M . In particular, M is homeomorphic to S^n if $I_0(M) < 2D_n$.

Example 1. If $\varphi : M \rightarrow \mathbf{R}^{n+1}$ is a standard n -sphere of constant sectional curvature, then equality holds in the Main Theorem.

Example 2. If M^2 is the squared flat torus minimally embedded into $S^3 \subset \mathbf{R}^4$, then $2D_2 = 8\pi$ and the integral is $4\pi^2$. Here the number of critical points of height functions is greater than three.

Next, Theorems A and B in [SX] are improved as follows.

THEOREM A. Assume that $\varphi : M \rightarrow \mathbf{F}^{n+p}(c)$ admits a parallel mean curvature normal field. Then there exist positive constants A_n, B_n , both depending only on n , such that if $I(M; \varphi) < A_n + B_n \sum_{i=1}^{n-1} \beta_i$, then $S = nH^2$ is constant, φ is totally umbilical and M is isometric to the standard n -sphere of constant sectional curvature $H^2 + c$.

THEOREM B. Assume that $\varphi : M \rightarrow \mathbf{F}^{n+p}(c)$ admits a parallel mean curvature normal field and that φ is not totally umbilical. We then have

$$I(M; \varphi) \geq A_n + B_n \sum_{i=1}^{n-1} \beta_i.$$

The topological finiteness theorem is obtained by relaxing the assumption in our Main Theorem.

THEOREM C. For a given integer $n \geq 2$ and a constant $A > 0$ the class $\mathcal{M}(n, A) := \{M : I_0(M) < A\}$ of all compact Riemannian n -manifolds has at most finitely many homeomorphism types.

1. Proof of the Main Theorem

We now prove our Main Theorem and obtain the new constants. They are independent of codimension but depend on n . The proofs of Theorems A and B follow from that of the Main Theorem and the discussion developed in [SX], and are omitted here.

Proof of the Main Theorem. The proof is divided into two cases.

First we assume that $c = 0$. The same computations as in [SX, p. 303, (i)] are employed. We have in the case $n \leq 3$,

$$\begin{aligned} \int_M (S - nH^2)^{n/2} dM &\geq \sum_{k=1}^{n-1} \frac{D_{n,p}}{\omega_{n+p-1}} \int_{S^{n+p-1}} c_k(\langle \varphi, z \rangle) dv_z \\ &\geq D_{n,p} \sum_{k=1}^{n-1} \beta_k \geq D_n \sum_{k=1}^{n-1} \beta_k. \end{aligned} \quad (1)$$

In the case where $n \geq 4$, we have

$$\begin{aligned} &\int_M (S - nH^2)^{n/2} dM \\ &\geq \frac{D_{n,p}}{(n-1)^{(2-n)/2} \omega_{n+p-1}} \sum_{k=1}^{n-1} k^{(n-2k)/2} (n-k)^{(2k-n)/2} \int_{S^{n+p-1}} c_k(\langle \varphi, z \rangle) dv_z \\ &\geq D_{n,p} \left[c_1 + c_{n-1} + \sum_{k=2}^{n-2} k^{(n-2k)/2} (n-k)^{(2k-n)/2} (n-1)^{(n-2)/2} c_k \right] \\ &\geq D_{n,p} \left(\beta_1 + 3 \sum_{k=2}^{n-2} \beta_k + \beta_{n-1} \right) \geq D_n \left(\beta_1 + 3 \sum_{k=2}^{n-2} \beta_k + \beta_{n-1} \right). \end{aligned} \quad (2)$$

Here we set

$$D_{n,p} := \max\{pn^{n/2}(n-1)^{(2-n)/2}\omega_{n+p-1}\omega_{p-1}^{-1}, \frac{1}{2}n^{n/2}(n-1)^{(2-n)/2}\omega_n\}$$

and

$$D_n = \min_{p \geq 1} D_{n,p}.$$

By the assumption that $I_0(M) < 2D_n$, we find an isometric immersion $\varphi : M \rightarrow F^{n+p}(c)$ such that $I(M; \varphi) < 2D_n$. Then the above inequality implies that there is a

Morse function $f := \langle \varphi, z \rangle$ on M such that

$$\begin{aligned} \sum_{k=1}^{n-1} c_k(f) &< 2, \quad \text{for } n \leq 3, \\ c_1(f) + 3 \sum_{k=2}^{n-2} c_k(f) + c_{n-1}(f) &< 2, \quad \text{for } n \geq 4. \end{aligned} \quad (3)$$

Therefore there exists at most one critical point with index k for some $1 \leq k \leq n-1$. Assume that f has a critical point with such an index k . Then the inequality (3) implies that $k = 1$ or $k = n-1$. We see from Morse theory (see [M1, M2]) that f has exactly two critical points with index 0 and one critical point with index n . Note that $-f$ has exactly one critical point with index 0 and two critical points with index n . By the cancellation of critical points (see [M2]), we see that one pair of critical points with index 0 and 1 (or $n-1$ and n) can be eliminated. Therefore there is a Morse function on M which has only two critical points. Then the classical Reeb theorem implies that M is homeomorphic to S^n .

We finally assume that $c > 0$. The proof in this case is done in the same manner as stated in [SX, p. 303, (ii)]. $\square$

The constants A_n, B_n used in Theorems A and B are given as follows:

$$A_n := \begin{cases} 4\pi, & \text{for } n = 2 \\ \frac{1}{2}\gamma_n, & \text{for } n \geq 3, \end{cases}$$

and

$$B_n := \begin{cases} 2\pi, & \text{for } n = 2 \\ \frac{1}{2}D_n, & \text{for } n \geq 3. \end{cases}$$

They are obtained in the same manner as in [SX]. The proof of Theorems A and B has essentially been established in the previous paper.

Proof of Theorem C. It follows from (1) that every $M \in \mathcal{M}(n, A)$ admits an isometric immersion $\mathbf{x} : M \rightarrow \mathbf{R}^{n+p}$ and a height function ψ which is a Morse function with at most $2A/D_n + 2$ critical points. From the standard Morse theory we may consider that all the critical values of ψ are distinct. Let $a_0 < \cdots < a_k$ be the set of all the critical values of ψ . If $a_i < t_1 < t_2 < a_{i+1}$, then each component of the $\psi^{-1}([t_1, t_2])$ is homeomorphic to a cylinder $S^{n-1} \times [t_1, t_2]$. Thus there exists a constant $N > 0$ depending only on n, p and A such that the number of n -disks needed to cover M is at most N . $\square$

2. Corrections to the previous work

All the corrections to our previous paper [SX] are listed below.

On page 291:

- (1) ↑ 1 and ↑ 2: $D_{n,p}$ should read D_n
- (2) ↑ 2: ‘and p ’ should be deleted

On page 292:

- (1) ↓ 4: $D_{n,p}$ should read D_n
- (2) ↓ 9, ↓ 11 and ↓ 17: $B_{n,p}$ should read B_n
- (3) ↓ 10: a comma should be deleted
- (4) ↑ 2: ‘closed’ should be replaced by ‘complete’
- (5) ↑ 1: $S < C(n, p, H, c)$ should read $\sup_M S < C(n, p, H, c)$

On page 293:

- (1) ↑ 3, ↑ 4: $h_{i\ell}^{(\cdot)}$ and $h_{ik}^{(\cdot)}$ should read $h_{i\ell}^\alpha$ and h_{ik}^α
- (2) ↑ 1: c_α should read e_α

On page 294:

- (1) ↓ 9: S_ℓ should read S_I . The same errors are on page 295: ↓ 1, ↓ 2 and ↓ 7, on page 299: ↓ 3, ↓ 4, ↓ 5 and ↓ 8, and on page 300: ↓ 4.

On page 299:

- (1) ↓ 9: $\int_\epsilon$ should read $\int_M$
- (2) ↓ 11: $\|g_{n/(n-2)}\|$ should read $\|g^2\|_{n/(n-2)}$

On page 301:

- (1) ↓ 8: ‘the well known Calabi theorem’ should be deleted.
- (2) ↓ 12 to ↓ 13: ‘and’ should be inserted between (4.10) and ‘Calabi’s theorem’
- (3) ↑ 6: ‘sharp’ should read ‘optimal’

On page 302:

- (1) ↓ 1: ‘Theorem C.’ should read ‘Main Theorem I.’

On page 303:

- (1) ↓ 9: ‘is’ should be inserted between ‘ $c_k(f)$ ’ and ‘the’
- (2) ↓ 13: This sentence should be rewritten as follows.
This together with (4.13), (4.14) and the proof of the Chen-Willmore theorem ([Ch]) yields

- (3) ↑ 9 and ↑ 10: The two lines should read as follows.

$$\begin{aligned} \int_M (S - nH^2)^{n/2} &\geq \sum_{k=1}^{n-1} \frac{D_{n,p}}{\omega_{n+p-1}} \int_{S^{n+p-1}(1)} c_k(\langle \varphi, z \rangle) \\ &\geq D_{n,p} \left(\sum_{k=1}^{n-1} c_k \right) \geq D_n \left(\sum_{k=1}^{n-1} \beta_k \right), \end{aligned} \quad (4.16)$$

where

$$D_{n,p} := \max\{pn^{n/2}(n-1)^{(2-n)/2}\omega_{n+p-1}\omega_{p-1}^{-1}, \frac{1}{2}n^{n/2}(n-1)^{(2-n)/2}\omega_n\}$$

and

$$D_n := \min_{p \geq 1} D_{n,p}.$$

On page 304:

- (1) ↓ 1: ‘Proof of Theorem D.’ should read ‘Proof of Main Theorem II.’
 (2) ↓ 1: The following sentence should be inserted in between the above phrase and (i).

Now we assume that

$$\int_M (S - nH^2)^{n/2} < D_n.$$

- (3) ↓ 1: ‘(4.13), (4.14), (4.15)’ should read ‘(4.16)’
 (4) ↓ 3: z_0 in the integrand should read z
 (5) ↓ 4: $D_{n,p}$ should read D_n
 (6) ↓ 7: $c_0(\langle \varphi, z_0 \rangle)$ should read $c_0(\langle \varphi, z_0 \rangle)$
 (7) ↓ 8: ‘Theorem D’ should read ‘The proof’
 (8) ↓ 9: ‘proof of theorem C’ should read ‘case I(ii)’
 (9) ↑ 10: ‘ $B_{n,p}$ and $D_{n,p}$ ’ should read ‘ B_n and D_n ’
 (10) ↑ 9: ‘C’ should read ‘the main theorem’
 (11) ↑ 7, ↑ 2: $I(S)$ should read $I(M)$
 (12) ↑ 4: ‘E’ should read ‘4’

On page 305:

- (1) ↓ 1: $\delta_{n,p}$ should read δ_n
 (2) ↓ 3: ‘Theorem E’ should read ‘Theorem 4’
 (3) ↓ 3, ↓ 4: $J(H)$ should read $J(M)$
 (4) ↓ 7: ‘Theorem F’ should read ‘Theorem 5’
 (5) ↓ 1, ↓ 8, ↓ 11: $I(S)$ should read $I(M)$
 (6) ↓ 8, ↓ 9: $\varepsilon_{n,p}$ should read ε_n

- (7) ↓ 9: The sentence should be rewritten as follows.
where ε_n is an explicit positive constant depending only on n .
- (8) ↓ 11, ↓ 12: $\tau_{n,p}$ should read τ_n
- (9) ↓ 12: The sentence should be rewritten as follows.
where δ_n and τ_n are explicit positive constants depending only on n .

REFERENCES

- [CE] J. Cheeger and D. Ebin. Comparison Theorems in Riemannian Geometry (Math Library, 9). North-Holland, 1975.
- [M1] J. Milnor. Morse Theory. Princeton University Press, Princeton, 1963.
- [M2] J. Milnor. Lectures on the h-cobordism Theorem. Princeton University Press, Princeton, 1965.
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