THE SPECIAL ORTHOGONAL GROUPS $SO(2n)$ AS FRAMED BOUNDARIES

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1. Introduction

A compact connected Lie group G with its left invariant framing L defines via the Pontrjagin–Thom construction an element $[G, L]$ in the stable homotopy group $\pi_{\dim G}^S$ of spheres. Let C be a finite central subgroup of G . Then the extension $C \rightarrow G \rightarrow G/C$ raises the problem of studying the relation between $[G, L]$ and $[G/C, L]$. In particular, we are interested in the rotation group $SO(n)$ and its universal covering $Spin(n)$. We will identify $SO(n)$ with $Spin(n)/\{\pm 1\}$ below. It is well known [2] that $SO(2)$ and $Spin(3)$ represent the Hopf maps $\eta \in \pi_1^S = \mathbb{Z}/2$ and $\nu \in \pi_3^S = \mathbb{Z}/24$ and $SO(3)$ represents 2ν so that $[SO(3), L] = 2[Spin(3), L]$ holds. Furthermore, we know that $2[SO(2n), L] = 0$ for $n \geq 1$ [3] and in particular $[SO(n), L] = 0$ for $n = 4, 6, 8$ [3, 4]. The purpose of this paper is to strengthen these results as follows.

THEOREM. $[SO(2n), L] = 0$ for $n \geq 2$.

As will be shown later, we have $2[Spin(2n), L] = 0$ for $n \geq 1$. As a consequence we see that $[SO(2n), L] = 2[Spin(2n), L]$ holds for $n \geq 1$. In the final section we also consider the same problem for some of the projective groups of classical groups. Our proof of the theorem is largely based on the method in [7] and the idea used in [8]. In the next section we collect the facts which will be needed.

2. Preliminaries

Let G be as in Section 1 and let us put $d = \dim G$. Let $S \subset G$ be a subgroup isomorphic to S^1 . Then the left invariant framing L on G induces a framing on G/S [5], which we denote by the same symbol L for simplicity.

By a slight modification of the arguments in [8] (see [6] for the explicit proof) we find:

(2.1)

If G has a subgroup isomorphic to either S^3 or $SO(3)$ which contains S as a subgroup, then $[G/S, L] = 0$.

Let $\xi : E \rightarrow G/S$ be a complex line bundle over G/S . Then the total space $S(E)$ of its sphere bundle becomes a framed manifold with the framing, also denoted by L , obtained by pulling the framing L of G/S back along the canonical projection map $G \rightarrow G/S$. We simply write E for ξ .

We denote the framed bordism fundamental class of G/S by $[G/S] \in \pi_{d-1}^S(G/S_+)$ and the reduced J -homomorphism by $\tilde{J} : \tilde{K}^{-1}(S^1 \wedge G/S_+) \rightarrow \pi_S^0(S^1 \wedge G/S_+) = \pi_S^{-1}(G/S_+)$. Here, X_+ denotes the disjoint union of X and a single point $+$ which is taken to be the base point of X_+ . Further, let $\beta \in \tilde{K}(S^2)$ denote the Bott element. By [7] we then know

$$[S(E), L] = -\langle \tilde{J}(\beta E), [G/S] \rangle \quad \text{in } \pi_d^S \quad (2.2)$$

where $\langle , \rangle$ denotes the Kronecker product in the stable homotopy theory.

View $\mathbb{C}$ as a S -module by acting S on the field $\mathbb{C}$ in the standard fashion through the composition $\alpha : S \cong S^1 \subset \mathbb{C}$. We write V_α for this S -module and consider the case where $E = G \times_S V_\alpha$. Then $S(E)$ becomes diffeomorphic to G as framed manifolds, in other words,

$$[G, L] = [S(E), L].$$

In fact, in [7] by using (2.2) and applying the solution of the Adams conjecture to $\tilde{J}(\beta E)$ the order of $[G, L]$ has been estimated at $2^3 \cdot 3^2[G, L] = 0$. In particular, if G is locally isomorphic to a classical group, then it has been shown that

$$2^3 \cdot 3[G, L] = 0 \quad (2.3)$$

which will be used only in the final section.

3. Proof of the theorem

We will use the same notation for S , α and V_α as in Section 2. Take $Spin(n)$ with $n \geq 3$ for G and choose S as the subgroup $Spin(2)$ such that the quotient $Spin(2)/\{\pm 1\}$ is just the subgroup $SO(2) \times I_{n-2}$ of $SO(n)$, where I_{n-2} denotes the identity matrix of degree $n-2$. We denote by E the complex line bundle $E = G \times_S V_\alpha \mapsto G/S$ defined in Section 2.

Let $R^{p,q}$ denote $\mathbb{R}^{p+q}$ with non-trivial $(\mathbb{Z}/2)$ -action on the first p coordinates and let $\Sigma^{p,q}$ be the one-point compactification of $R^{p,q}$. According to [1] the cofibre sequence $S^{1,0} \rightarrow B^{1,0} \rightarrow \Sigma^{1,0}$ then induces the exact sequence

$$\dots \rightarrow KR^{-1+q}(X) \xrightarrow{\chi} KR^{-2+q}(X) \xrightarrow{i^*} K^{-2+q}(X) \xrightarrow{\delta} KR^q(X) \rightarrow \dots, \quad (3.1)$$

where χ is multiplication by $-\eta$, $i^* = c$ is complexification and δ is given by $\delta(\beta x) = r(x)$, the realification of x . Here, in particular if the reality of X is trivial, we have $KR(X) \cong KO(X)$ and so under the identification of them i^* becomes the usual complexification.

Set $G = Spin(n)$, $X = \Sigma^2 \wedge G/S_+$ with trivial reality in (3.1) and consider $\beta(E^{\otimes 2} - 1) \in \tilde{K}(\Sigma^2 \wedge G/S_+)$. Making use of the identity representation of G we then see that $r(E^{\otimes 2})$ is stably trivial and hence $\delta(\beta(E^{\otimes 2} - 1)) = 0$. Together with the exactness of (3.1) this implies that there is

$$\zeta \in \widetilde{KO}(\Sigma^2 \wedge G/S_+)$$

such that

$$c(\zeta) = \beta(E^{\otimes 2} - 1). \quad (3.2)$$

Let J_R be the real J -homomorphism. Since $J = J_R \circ r$ and $r \circ c = 2$, (3.2) shows

$$\tilde{J}(\beta(E^{\otimes 2} - 1)) = 2\tilde{J}_R(\zeta). \quad (3.3)$$

By definition, $\{\pm 1\} \subset S$ so that $S(E^{\otimes 2})$ becomes diffeomorphic to $SO(n)$ as framed manifolds. It is also clear that $S(\underline{\mathbb{C}})$ is diffeomorphic to $G/S \times S$ in the same manner, where $\underline{\mathbb{C}}$ denotes the product bundle over G/S with fibre $\mathbb{C}$. So by applying (3.3) to the formula of (2.2) we get

$$[SO(n), L] + \eta[G/S, L] = -2\langle \tilde{J}_R(\zeta), [G/S] \rangle.$$

Now, since S is contained in a canonical subgroup $Spin(3)$ of G , it follows from (2.1) that $[G/S, L] = 0$. Thus we have

$$[SO(n), L] = -2\langle \tilde{J}_R(\zeta), [G/S] \rangle \quad \text{for } G = Spin(n) \text{ with } n \geq 3. \quad (3.4)$$

Proof of the theorem. Let $G = Spin(2n)$ with $n \geq 2$. Observe the involutive automorphism τ of $SO(2n)$ given by $A \mapsto (I_{2n-1} \times (-1))A(I_{2n-1} \times (-1))$. This can be obviously lifted onto G . We denote this involutive automorphism by the same symbol. Viewing G as a $(\mathbb{Z}/2)$ -manifold with action by τ , the $(\mathbb{Z}/2)$ -trivialization of the tangent vector bundle $T(G)$ of G canonically induced by left translations by the elements of G becomes

$$T(G) \cong \underline{R}^{2n-1, d-2n+1}.$$

From this bundle isomorphism we see that τ induces a diffeomorphism from G to itself with the opposite framing. So immediately we have

$$2[Spin(2n), L] = 0 \quad (\text{similarly } 2[SO(2n), L] = 0) \quad (3.5)$$

which is what we propose to prove following the theorem in Section 1. Now because τ leaves S fixed, we can consider G/S as a $(\mathbb{Z}/2)$ -manifold. Then the usual decomposition of the tangent vector bundle $T(G/S)$ of G/S ,

$$T(G) \cong \pi^* T(G/S) \oplus F_\pi,$$

gives rise to the stable $(\mathbb{Z}/2)$ -trivialization such that

$$\underline{R}^{2n-1, d-2n+1} \cong T(G/S) \oplus \underline{R}^{0,1},$$

where $\pi : G \rightarrow G/S$ is the projection and F_π the vector bundle along the fibres of the principal S -bundle $S \rightarrow G \rightarrow G/S$. This bundle equation defines a stable $(\mathbb{Z}/2)$ -map

$$\Sigma^{2n-1, d-2n+1} \rightarrow \Sigma^{0,1} \wedge G/S_+$$

via the Pontrjagin–Thom construction.

We let the realities of both $\mathbb{Z}/2$ and X be trivial and consider the following $(\mathbb{Z}/2)$ -equivariant version of (3.1) with $X = \Sigma^2 \wedge G/S_+$:

$$\cdots \rightarrow K R_{\mathbb{Z}/2}^{-1+q}(X) \xrightarrow{\chi} K R_{\mathbb{Z}/2}^{-2+q}(X) \xrightarrow{i^*} K_{\mathbb{Z}/2}^{-2+q}(X) \xrightarrow{\delta} K R_{\mathbb{Z}/2}^q(X) \rightarrow \cdots.$$

Since E has a canonical $(\mathbb{Z}/2)$ -linear action induced from τ and $r(E^{\otimes 2}) \oplus \underline{R}^{1, 2n-3} \cong \underline{R}^{1, 2n-1}$ follows using the identity representation of G again, we then see that $\delta(\beta(E^{\otimes 2} - 1)) = 0$ and therefore we can take ζ in $\widetilde{K\mathcal{O}}_{\mathbb{Z}/2}(\Sigma^2 \wedge G/S_+)$. Namely, we may assume that the ζ given preceding (3.2) is provided with a $(\mathbb{Z}/2)$ -linear action and so we see that the $\tilde{J}_R$ -image of this stable $(\mathbb{Z}/2)$ -vector bundle ζ is represented by the stable $(\mathbb{Z}/2)$ -map of the form

$$\Sigma^{0,1} \wedge G/S_+ \rightarrow \Sigma^{0,0}.$$

Denote the composite of the two stable $(\mathbb{Z}/2)$ -maps described above by

$$\tilde{f} : \Sigma^{2n-1, d-2n+1} \rightarrow \Sigma^{0,0}$$

and by f its forgetful map. Then by construction the stable homotopy class of f is obviously equal to $\langle \tilde{J}_R(\zeta), [G/S] \rangle$. Let the stable maps $\tau_1 : \Sigma^{2n-1, d-2n+1} \rightarrow \Sigma^{2n-1, d-2n+1}$ and $\tau_2 : \Sigma^{0,1} \rightarrow \Sigma^{0,1}$ denote the $(\mathbb{Z}/2)$ -action on each space. Since $\tilde{f} \circ \tau_1 = \tau_2 \circ \tilde{f}$, considering the degrees of τ_1 and τ_2 we see that $f \simeq -f$, which implies $2\langle \tilde{J}_R(\zeta), [G/S] \rangle = 0$. Hence by (3.4) we have $[SO(2n), L] = 0, n \geq 2$. This completes the proof of the theorem. $\square$

4. Remarks

We use the notation from the preceding sections. Here, let C denote the whole centre of G and write PG for G/C , usually called the projective group of G . Then we obtain the following.

PROPOSITION. *Let G be $SU(2n+1)$ with $n \geq 1$, $SU(4n+2)$ and $Sp(2n+1)$ with $n \geq 0$. Then*

$$[PG, L] = |C|[G, L] \quad (|C| = \text{order of } C).$$

Further, let G be $SO(4n+2)$ with $n \geq 1$. Then

$$[PG, L] = 2[G, L] + \eta[G/S, L] \quad \text{so that} \quad [PG, L] = \eta[G/S, L],$$

where S denotes the standard diagonal circle subgroup of G .

Proof. Denote by $\text{diag}(A_1, \dots, A_k)$ the matrix with one-by-one or two-by-two diagonal boxes A_i .

Case $G = Sp(2n+1)$. Let S be the circle subgroup consisting of all matrices $\text{diag}(z, \dots, z)$ where $z \in S^1 \subset \mathbb{C}$. Define α by $\text{diag}(z, \dots, z) \mapsto z$. Considering the complexification $G \subset U(4n+2)$ of the identity representation G into itself, we see that

$$(2n+1)(E \oplus \bar{E}) \cong (4n+2)\underline{\mathbb{C}}, \quad \text{so that} \quad (2n+1)(E^{\otimes 2} \oplus \underline{\mathbb{C}}) \cong (4n+2)E.$$

The centre C is of order 2 and generated by $\text{diag}(-1, \dots, -1)$. So $C \subset G$ as a subgroup and hence $S(E^{\otimes 2})$ becomes diffeomorphic to PG as framed manifolds. By (2.2) we therefore have

$$(2n+1)([PG, L] + \eta[G/S, L]) = (4n+2)[G, L].$$

Now by (2.1), $[G/S, L] = 0$ since S is contained in the diagonal subgroup of the product group of $2n+1$ copies of $Sp(1)$. Thus we have

$$[PG, L]_{(2)} = 2[G, L]_{(2)},$$

where the index (2) indicates the 2-component of the element added to it. Because $\psi^2(\beta E) = 2\beta E^{\otimes 2}$, we have $\tilde{J}(2\beta E^{\otimes 2})_{(3)} = \tilde{J}(\beta E)_{(3)}$ by the solution of the Adams conjecture. So using (2.3) the formula (2.2) means that $[PG, L]_{(3)} = 2[G, L]_{(3)}$. Therefore we conclude that $[PG, L] = 2[G, L]$.

Case $G = SU(2n+1)$. Take S as the circle subgroup consisting of all matrices $\text{diag}(\bar{z}^{2n}, z, \dots, z)$, where $z \in S^1$ and define α to be $\text{diag}(\bar{z}^{2n}, z, \dots, z) \mapsto z$. Since

C is of order $2n + 1$ and generated by $\text{diag}(\omega, \dots, \omega)$, where ω is a $(2n + 1)$ th root of unity, we see that $C \subset S$ as a subgroup and hence $S(E^{\otimes(2n+1)})$ is diffeomorphic to PG as framed manifolds. Observing the restriction to S of the identity representation of G into itself then gives an isomorphism of vector bundles

$$E^{\otimes(2n+1)} \oplus 2n\mathbb{C} \cong (2n + 1)E.$$

Hence we have $[PG, L] = (2n + 1)[G, L]$ analogously.

Case $G = SU(4n + 2)$. We define S and α in a similar fashion to $SU(2n + 1)$. Then from the same reasoning as for the above case we find

$$E^{\otimes(4n+2)} \oplus (4n + 1)\mathbb{C} \cong (4n + 2)E.$$

Since C is of order $4n + 2$, $C \subset S$ as a subgroup and so $S(E^{\otimes(4n+2)})$ is diffeomorphic to PG as framed manifolds, this isomorphism produces

$$[PG, L] + \eta[G/S, L] = (4n + 2)[G, L].$$

To prove that $\eta[G/S, L] = 0$ it suffices to show that $[PG, L]_{(2)} = [G, L]_{(2)}$. Consider the circle subgroup S' consisting of all matrices $\text{diag}(z, \bar{z}, \dots, z, \bar{z})$, where $z \in S^1$, instead of S above. Then analogously we get

$$(2n + 1)(E^{\otimes 2} + \mathbb{C}) \cong (4n + 2)E.$$

Let C' be the central subgroup generated by $-I$, where I is the identity matrix. Then $C' \subset S'$ as a subgroup and, further, S' is contained in the diagonal subgroup of the product group of $2n + 1$ copies of $SU(2)$. Then from the above isomorphism we have

$$(2n + 1)[G/C', L] = (4n + 2)[G, L].$$

Since $(2n + 1)\tilde{J}(\beta E^{\otimes(4n+2)})_{(2)} = \tilde{J}(\beta E^{\otimes 2})_{(2)}$ we have $(2n + 1)[G/C', L]_{(2)} = [PG, L]_{(2)}$ by applying formula (2.2) and the solution of the Adams conjecture. By comparing the two equalities above we see that $\eta[G/S, L] = 0$ so that $[PG, L] = (4n + 2)[G, L]$.

Case $G = SO(4n + 2)$. Define S to be the circle subgroup generated by all matrices $\text{diag}(D(\theta), \dots, D(\theta))$, where $D(\theta)$ is the realification matrix of $(e^{\theta i})$. Then since C is of order 2 and generated by $-I$, $C \subset S$ obviously as a subgroup. Define α to be an isomorphism mapping $\text{diag}(D(\theta), \dots, D(\theta))$ to $e^{\theta i}$. The rest of the proof proceeds analogously to the above cases. Consider the complexification of the identity representation of G into itself. Then we have

$$(2n + 1)(E^{\otimes 2} \oplus \mathbb{C}) \cong (4n + 2)E.$$

Applying this to (2.2) we get $[PG, L]_{(2)} + \eta[G/S, L]_{(2)} = 2[G, L]_{(2)}$. Since we see that $[PG, L]_{(3)} = 2[G, L]_{(3)}$ using the solution of the Adams conjecture, it follows that $[PG, L] + \eta[G/S, L] = 2[G, L]$. Thus we have the proposition. $\square$

We complete the paper with a further remark. We have such an example as $2[SO(5), L] = 4[Sp(2), L]$ [4]. Since $SO(5) \cong PSp(2)$, this means that $2[PSp(2), L] = 4[Sp(2), L]$. In a similar way to as above, this can be generalized to be $2n[PSp(2n), L] = 4n[Sp(2n), L]$.

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