A REMARK ON LONG-RANGE SCATTERING FOR THE HARTREE TYPE EQUATION

Takeshi WADA

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1. Introduction and main results

In this paper we study the scattering problem for the following nonlinear Schrödinger equation of Hartree type:

$$\begin{aligned} iu_t &= -(1/2)\Delta u + f(|u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) &= u_0(x). \end{aligned} \tag{1.1}$$

Here, Δ denotes the Laplacian in $\mathbb{R}^n$, $f(\psi) = \lambda \int_{\mathbb{R}^n} |x - y|^{-\gamma} \psi(y) dy$, $\lambda \in \mathbb{R}$, and $\gamma > 0$. From the viewpoint of scattering theory, the interaction $f(|u|^2)$ is said to be of short range if $\gamma > 1$ and long range if $0 < \gamma \leq 1$.

In the short-range case, we construct the scattering theory by comparing the solution of (1.1) with the free solution $U(t)\phi_+ = \exp(it\Delta/2)\phi_+$ as $t \rightarrow \infty$. Provided a function ϕ_+ satisfies the asymptotic relation

$$\lim_{t \rightarrow \infty} \|U(-t)u(t) - \phi_+\|_2 = 0, \tag{1.2}$$

we call ϕ_+ the scattering state of $u(t)$ (for positive time). Let X be a Banach space densely embedded in L^2 , or a neighborhood of the origin in such a space. If every $\phi_+ \in X$ is the scattering state of some solution $u(t)$ of (1.1), then we can define the wave operator $W_+ : X \rightarrow L^2$ by $W_+\phi_+ = u(0)$. We can also define the operator W_- by considering the case $t \rightarrow -\infty$. For short-range scattering, see, for example, [3, 10, 12, 14].

However, in the long-range case the scattering state of nontrivial solutions cannot exist in the usual sense (see [5, 12]), so here we consider another asymptotic profile. We call ϕ_+ the modified scattering state of $u(t)$ (for positive time) if there exists a real-valued function $S_+(t, \xi)$ satisfying

$$\lim_{t \rightarrow \infty} \|U(-t)u(t) - \exp[-iS_+(t, -i\nabla)]\phi_+\|_2 = 0. \tag{1.3}$$

The phase function $S_+(t, \xi)$ may depend on ϕ_+ . We define the modified wave operator W_+ by $W_+\phi_+ = u(0)$, if it is well defined. In recent years, the long-range scattering

for this equation has advanced rapidly. The existence of the modified wave operator for (1.1) was shown in [2, 4], and the asymptotic behavior of the solution of (1.1) was studied in [6–9].

Now we shall state the theorems of this paper. In what follows we always assume that

$$n \geq 3 \quad \text{and} \quad 1/2 < \gamma < 1.$$

THEOREM 1.1. *Let $u_0 \in L^{2,s}$, $s > 1 + \gamma/2$, and let $\|u_0\|_{L^{2,s}}$ be sufficiently small. Then the solution u of (1.1) satisfies the estimate*

$$\|u(t)\|_p \leq C \|u_0\|_{L^{2,s}} |t|^{-n/2+n/p} \quad (1.4)$$

for $2 \leq p \leq 2n/(n-2s)$ if $s < n/2$, and for $2 \leq p \leq \infty$ if $s > n/2$.

THEOREM 1.2. *Let $u_0 \in L^{2,2}$ and let $\|u_0\|_{L^{2,2}}$ be sufficiently small. Then there exists a function $\phi_{\pm} \in L^{2,2}$ satisfying*

$$\|U(-t)u(t) - \exp[\mp i S_{\pm}(t, -i\nabla)]\phi_{\pm}\|_2 = O(|t|^{1-2\gamma}) \quad (1.5)$$

as $t \rightarrow \pm\infty$, where $S_{\pm}(t, \xi) = (1 - \gamma)^{-1} |t|^{1-\gamma} f(|\hat{\phi}_{\pm}|^2)(\xi)$.

Hayashi and Naumkin [7, 8] found the asymptotic profile in the L^{∞} -sense for small solutions of (1.1) in the case $0 < \gamma < 1$, and showed the existence of the modified scattering state in the case $1/2 < \gamma < 1$. However, they assumed that $u_0 \in L^{2,s}$, where s is an integer larger than $(n/2) + 2$. In the case $n \geq 3$, our assumption for initial data is weaker than that of [8], and our proof is simpler than the previous one.

In this paper, we use the following notation. $(\cdot, \cdot)$ denotes the inner-product in L^2 , $\|\cdot\|_p$ means the usual L^p -norm. H_p^l and $\dot{H}_p^l$ denote the Sobolev and the homogeneous Sobolev space of l th order with index p , respectively. H_2^l (respectively $\dot{H}_2^l$) is abbreviated to H^l (respectively $\dot{H}^l$). $L^{2,s}$ denotes the weighted L^2 -space of s th order. $\partial_j = \partial/\partial x_j$, $\nabla = (\partial_1, \dots, \partial_n)$, $J = (1 - \Delta)^{1/2}$, $U(t) = \exp(it\Delta/2)$, $M(t) = \exp(i|x|^2/2t)$.

2. Proof of theorems

We shall introduce the known result on the Cauchy problem for (1.1) obtained in [1, 3, 11].

PROPOSITION 2.1. *Let $0 < \gamma < 2$. If $u_0 \in L^2$, then the Cauchy problem (1.1) has a unique solution in $C(\mathbb{R}; L^2) \cap L_{\text{loc}}^{8/\gamma}(\mathbb{R}; L^{4n/(2n-\gamma)})$. Moreover:*

- (i) $u \in L_{\text{loc}}^q(\mathbb{R}; L^r)$ for any (q, r) satisfying $0 < 2/q = n/2 - n/r < 1$;
- (ii) if $u_0 \in L^{2,s}$, $s > 0$, then $U^{-1}u \in C(\mathbb{R}; L^{2,s})$;
- (iii) if $\|u_0\|_{L^{2,s}}$ is sufficiently small, then $\sup_{|t| \leq 1} \|U(-t)u(t)\|_{L^{2,s}} \leq 2\|u_0\|_{L^{2,s}}$.

The following lemma obtained in [13] is a generalization of Leibniz's rule.

LEMMA 2.2. Let $s > 0$, $1 < p, q_1, q_2 < \infty$, $1 < r_1, r_2 \leq \infty$ and $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$. Then:

- (i) $\|J^s(uv)\|_p \leq C\|J^s u\|_{q_1}\|v\|_{r_1} + C\|J^s v\|_{q_2}\|u\|_{r_2}$;
- (ii) $\|J^s(uv) - uJ^s v\|_p \leq C\|J^s u\|_{q_1}\|v\|_{r_1} + C\|J^{s-1}v\|_{q_2}\|\nabla u\|_{r_2}$.

We shall proceed to the proof of our theorems. For simplicity, we only consider the case $t \rightarrow \infty$, so we may assume $t \geq 1$. We transform the solution of (1.1) as follows. First, we put

$$v(t) = \mathcal{F} M(t) U(-t) u(t) = (it)^{n/2} \exp(-it|x|^2/2) u(t, tx),$$

where $\mathcal{F}$ denotes the Fourier transform. Then $v(t) \in C([1, \infty); H^s)$ and satisfies the equation

$$iv_t = -(2t^2)^{-1} \Delta v + t^{-\gamma} f(|v|^2)v. \quad (2.1)$$

Next, we put $g(t) = \int_1^t \tau^{-\gamma} f(|v|^2) d\tau$ and $w(t) = e^{ig} v(t)$. Then $g(t) = \int_1^t \tau^{-\gamma} f(|w|^2) d\tau$ and w satisfies

$$iw_t = (2t^2)^{-1} (-\Delta w + 2i \nabla g \cdot \nabla w + i \Delta g w + |\nabla g|^2 w). \quad (2.2)$$

Proof of Theorem 1.1. We fix $\rho \in (0, 1]$ and put $T^* = \sup\{T; \sup_{1 \leq t \leq T} \|J^s w(t)\|_2 \leq \rho\}$. Then for $1 \leq t < T^*$, equation (2.2) implies

$$\begin{aligned} & \frac{d}{dt} \|J^s w(t)\|_2^2 \\ &= t^{-2} \left\{ 2\text{Re} \sum_{k=1}^n (J^s (\partial_k g \partial_k w) - \partial_k g J^s \partial_k w, J^s w) + 2\text{Re} \sum_{k=1}^n (\partial_k g J^s \partial_k w, J^s w) \right. \\ & \quad \left. + \text{Re}(J^s (\Delta g w), J^s w) + \text{Im}(J^s (|\nabla g|^2 w), J^s w) \right\} \\ &= t^{-2} \{\text{I} + \text{II} + \text{III} + \text{IV}\}. \end{aligned} \quad (2.3)$$

We shall estimate the braces in the right-hand side term by term. By Lemma 2.2,

$$\begin{aligned} \text{I} &\leq 2 \sum_{k=1}^n \|J^s(\partial_k g \partial_k w) - \partial_k g J^s \partial_k w\|_2 \|J^s w\|_2 \\ &\leq C \sum_{k=1}^n (\|J^s \partial_k g\|_{2n/\gamma} \|\partial_k w\|_{2n/(n-\gamma)} + \|\partial^2 g\|_\infty \|J^{s-1} \partial_k w\|_2) \|J^s w\|_2. \end{aligned}$$

Here $\|\partial^2 g\|_\infty$ denotes $\max_{|\beta|=2} \|\partial^\beta g\|_\infty$. Using the Hardy–Littlewood–Sobolev inequality and Lemma 2.2, we have

$$\begin{aligned} \|J^s \partial_k f(|w|^2)\|_{2n/\gamma} &\leq C \|J^s |w|^2\|_{2n/(2n-\gamma-2)} \\ &\leq C \|J^s w\|_2 \|w\|_{2n/(n-\gamma-2)} \leq C \|J^s w\|_2^2 \leq C \rho^2, \end{aligned}$$

thus we get $\|J^s \partial_k g(|w|^2)\|_{2n/\gamma} \leq C \rho^2 t^{1-\gamma}$. We choose the numbers α_j , $j = 1, 2$, satisfying $2 < \alpha_1 < \alpha_2 < 2n/(n-2s)_+$, $1/\alpha_1 + 1/\alpha_2 = 1 - (\gamma + 2)/n$. Then by Hölder's inequality and Sobolev's inequality,

$$\|\partial^2 f(|w|^2)\|_\infty \leq C \| |\cdot|^{-(\gamma+2)} * |w|^2 \|_\infty \leq C \|w\|_{\alpha_1} \|w\|_{\alpha_2} \leq C \|J^s w\|_2^2 \leq C \rho^2,$$

and hence $\|\partial^2 g(|w|^2)\|_\infty \leq C \rho^2 t^{1-\gamma}$. Therefore we obtain

$$\text{I} \leq C \rho^2 t^{1-\gamma} \|J^s w\|_2^2.$$

By integration by parts,

$$\begin{aligned} \text{II} &= - \int_{\mathbb{R}^n} \Delta g |J^s w|^2 dx \\ &\leq \|\Delta g\|_\infty \|J^s w\|_2^2 \leq C \rho^2 t^{1-\gamma} \|J^s w\|_2^2. \end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned} \text{III} &\leq C \{ \|J^s \Delta g\|_{2n/(\gamma+2)} \|w\|_{2n/(n-\gamma-2)} + \|\Delta g\|_\infty \|J^s w\|_2 \} \|J^s w\|_2 \\ &\leq C \rho^2 t^{1-\gamma} \|J^s w\|_2^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{IV} &\leq C \{ \|J^s(|\nabla g|^2)\|_{2n/(\gamma+2)} \|w\|_{2n/(n-\gamma-2)} + \|\nabla g\|_\infty^2 \|J^s w\|_2 \} \|J^s w\|_2 \\ &\leq C \rho^4 t^{2(1-\gamma)} \|J^s w\|_2^2. \end{aligned}$$

Consequently we obtain under $\gamma < 1$,

$$\frac{d}{dt} \|J^s w(t)\|_2^2 \leq C \rho^2 t^{-2\gamma} \|J^s w\|_2^2, \quad 1 \leq t < T^*,$$

since $\rho \leq 1$ and $t \geq 1$. Thus, if $\gamma > 1/2$, Gronwall's inequality yields

$$\begin{aligned} \|J^s w(t)\|_2 &\leq \|J^s w(1)\|_2 \exp(C\rho^2 t^{1-2\gamma}) \\ &\leq C \exp(C\rho^2) \|u_0\|_{L^{2,s}}, \quad 1 \leq t < T^*. \end{aligned} \quad (2.4)$$

Here we have used the local well-posedness for the Cauchy problem (Proposition 2.1(iii)).

Now if we choose u_0 sufficiently small such that $C \exp(C\rho^2) \|u_0\|_{L^{2,s}} < \rho$, then $T^* = \infty$. Indeed, if we assume that $T^* < \infty$, then (2.4) contradicts the definition of T^* . The theorem immediately follows from (2.4) by the definition of $w(t)$ and Sobolev's inequality. $\square$

Proof of Theorem 1.2. By virtue of Theorem 1.1 for $s = 2$ and the estimate

$$\|\nabla g\|_\infty, \|\partial^2 g\|_\infty \leq C t^{1-\gamma}, \quad (2.5)$$

$\|w_t(t)\|_2 \leq C t^{2(1-\gamma)}$ immediately follows from (2.2). Thus

$$\begin{aligned} \|w(t') - w(t)\|_2 &\leq \int_t^{t'} \tau^{-2} \|w_\tau(\tau)\|_2 d\tau \\ &\leq C(t^{1-2\gamma} - t'^{1-2\gamma}) \end{aligned}$$

for $1 < t < t'$. Hence, by L^2 continuity of $w(t)$ in t , there exists a function $\Phi \in L^2$ satisfying

$$\|w(t) - \Phi\|_2 \leq C t^{1-2\gamma}. \quad (2.6)$$

Moreover, (2.6) and the boundedness of $\|w(t)\|_{H^2}$ show that $w\text{-}\lim_{t \rightarrow \infty} w(t) = \Phi$ in H^2 , and hence $\Phi \in H^2$. Here we consider the function

$$\tilde{w}(t) = w(t) \exp \left[i \int_1^t (2\tau^2)^{-1} |\nabla g|^2 d\tau \right].$$

Then $\tilde{w}$ satisfies

$$i \tilde{w}_t = (2t^2)^{-1} (-\Delta w + 2i \nabla g \cdot \nabla w + i \Delta g w) \exp \left[i \int_1^t (2\tau^2)^{-1} |\nabla g|^2 d\tau \right], \quad (2.7)$$

and hence the estimate

$$\|\tilde{w}_t(t)\|_2 \leq C t^{-1-\gamma}$$

follows from Theorem 1.1 and (2.5). So, there exists a function $\tilde{\Phi} \in L^2$ satisfying

$$\|\tilde{w}(t) - \tilde{\Phi}\|_2 \leq C t^{-\gamma}. \quad (2.8)$$

We note that $|\Phi| = |\tilde{\Phi}|$, since $|w(t)| = |\tilde{w}(t)|$.

Now we put $S(t) = S_+(t) = (1 - \gamma)^{-1} t^{1-\gamma} f(|\Phi|^2)$. By the Hardy–Littlewood–Sobolev inequality, Sobolev’s inequality and (2.8),

$$\begin{aligned} \|g(t') - S(t') - g(t) + S(t)\|_\infty &\leq \int_t^{t'} \tau^{-\gamma} \|f(|w(\tau)|^2) - f(|\Phi|^2)\|_\infty d\tau \\ &\leq C \int_t^{t'} \tau^{-\gamma} \|\tilde{w}(\tau) - \tilde{\Phi}\|_2 (\|w(\tau)\|_{H^2} + \|\Phi\|_{H^2}) d\tau \\ &\leq C \int_t^{t'} \tau^{-2\gamma} d\tau = C(t^{1-2\gamma} - t'^{1-2\gamma}) \end{aligned}$$

for $1 < t < t'$. Thus, there exists a function $\psi \in L^\infty$ satisfying

$$\|g(t) - S(t) - \psi\|_\infty \leq Ct^{1-2\gamma}. \quad (2.9)$$

Moreover, let σ satisfy $(n/2) - \gamma < \sigma < \min\{n; (n/2) + 2\} - \gamma$. For this σ , we have

$$\|g(t) - S(t) - \psi\|_{\dot{H}^\sigma} \leq Ct^{1-2\gamma} \quad (2.10)$$

by (2.8) and (2.9). Indeed, Sobolev’s inequality implies that $\|w\|_p \leq C\|w\|_{H^2}$ for $p = n/(n - \gamma - \sigma)$, because σ lies in the interval mentioned above. Thus the Hardy–Littlewood–Sobolev inequality and Theorem 1.1 show that

$$\begin{aligned} \|g(t) - S(t) - \psi\|_{\dot{H}^\sigma} &\leq \int_t^\infty \tau^{-\gamma} \|f(|w(\tau)|^2) - f(|\Phi|^2)\|_{\dot{H}^\sigma} d\tau \\ &\leq C \int_t^\infty \tau^{-\gamma} \| |\cdot|^{-(\gamma+\sigma)} * \{|\tilde{w}(\tau) - \tilde{\Phi}|(|w(\tau)| + |\Phi|)\}\|_2 d\tau \\ &\leq C \int_t^\infty \tau^{-\gamma} \|\tilde{w}(\tau) - \tilde{\Phi}\|_2 (\|w(\tau)\|_p + \|\Phi\|_p) d\tau \\ &\leq C \int_t^\infty \tau^{-2\gamma} d\tau = Ct^{1-2\gamma}. \end{aligned}$$

To show that $e^{-i\psi} \Phi \in H^2$, we choose the number q satisfying

$$\max\{2/n - 1/2; 0\} + \gamma/n < 1/q < \min\{1/2; 2/n\}.$$

Then $L^\infty \cap \dot{H}^\sigma \subset \dot{H}_q^2$ and $H^2 \subset L^{2q/(q-2)}$. Thus Leibniz’s rule and Sobolev’s inequality show that

$$\|e^{-i\psi} \Phi\|_{H^2} \leq C\|\Phi\|_{H^2} + C(\|\psi\|_\infty + 1)\|\Delta\psi\|_q \|\Phi\|_{2q/(q-2)} < \infty.$$

Now we define $\phi_+ = \mathcal{F}^{-1} e^{-i\psi} \Phi \in L^{2,2}$. Then

$$\begin{aligned}
 & \|U(-t)M(t)u(t) - \exp[-iS(t, -i\nabla)]\phi_+\|_2 \\
 &= \|e^{-ig(t)}w(t) - e^{-iS(t)-i\psi}\Phi\|_2 \\
 &\leq \|e^{-ig(t)}w(t) - e^{-iS(t)-i\psi}w(t)\|_2 + \|w(t) - \Phi\|_2 \\
 &\leq C\|g(t) - S(t) - \psi\|_\infty\|w(t)\|_2 + \|w(t) - \Phi\|_2 \\
 &\leq Ct^{1-2\gamma}.
 \end{aligned} \tag{2.11}$$

We also have

$$\|(1 - M(t))U(-t)u(t)\|_2 \leq Ct^{-1}\|x\|^2 U(-t)u(t)\|_2 \leq Ct^{-1}\|\Delta v(t)\|_2 \leq Ct^{1-2\gamma}.$$

Therefore

$$\begin{aligned}
 & \|U(-t)u(t) - \exp[-iS(t, -i\nabla)]\phi_+\|_2 \\
 &\leq \|(1 - M(t))U(-t)u(t)\|_2 + \|M(t)U(-t)u(t) - \exp[-iS(t, -i\nabla)]\phi_+\|_2 \\
 &\leq Ct^{1-2\gamma}.
 \end{aligned}$$

Thus we have just proved (1.5). $\square$

COROLLARY 2.3. *Under the assumption of Theorem 1.2, we have*

$$\|u(t) - \exp[\mp iS_\pm(t, x/t)]U(t)\phi_\pm\|_2 = O(|t|^{1-2\gamma}) \tag{2.12}$$

as $t \rightarrow \pm\infty$.

Proof. We shall prove only the case $t \rightarrow \infty$. Since

$$U(-t)\exp[-iS(t, \cdot/t)]U(t) = M(-t)\exp[-iS(t, -i\nabla)]M(t),$$

the left-hand side of (2.12) is estimated by

$$\begin{aligned}
 & \|U(-t)u(t) - M(-t)\exp[-iS(t, -i\nabla)]M(t)\phi_+\|_2 \\
 &\leq \|U(-t)u(t) - M(-t)\exp[-iS(t, -i\nabla)]\phi_+\|_2 + \|(1 - M(t))\phi_+\|_2.
 \end{aligned}$$

The first term is estimated by (2.11), and the second term by $(2t)^{-1}\|x\|^2\phi_+\|_2$. Thus we obtain (2.12). $\square$

COROLLARY 2.4. *Let $u_0 \in L^{2,s}$, $s > n/2$, and let $\|u_0\|_{L^{2,s}}$ be sufficiently small. Then*

$$u(t, x) = (it)^{-n/2}\hat{\phi}_\pm(x/t)\exp[(i|x|^2/2t) \mp iS_\pm(t, x/t)] + o(|t|^{-n/2}) \tag{2.13}$$

uniformly in $\mathbb{R}^n$ as $t \rightarrow \pm\infty$.

Proof. We shall prove only the case $t \rightarrow \infty$. Using Sobolev's inequality, we have

$$\|w(t) - \Phi\|_\infty \leq C \|w(t) - \Phi\|_{H^s}^{n/2s} \|w(t) - \Phi\|_2^{1-(n/2s)} \rightarrow 0$$

as $t \rightarrow \infty$, so that

$$\begin{aligned} & \|u(t) - (it)^{-n/2} \hat{\phi}_+(\cdot/t) M(t) \exp[-iS(t, \cdot/t)]\|_\infty \\ &= t^{-n/2} \|v(t) - e^{-iS(t)} \hat{\phi}_+\|_\infty \\ &= t^{-n/2} \|e^{-ig(t)} w(t) - e^{-iS(t)-i\psi} \Phi\|_\infty \\ &\leq t^{-n/2} \{\|g(t) - S(t) - \psi\|_\infty \|w(t)\|_\infty + \|w(t) - \Phi\|_\infty\} \\ &= o(t^{-n/2}). \end{aligned} \quad \square$$

Remark. If $s > (n/2) + 2$, then we can easily show that $\|w(t) - \Phi\|_\infty = O(t^{1-2\gamma})$. Thus the remainder term in the right-hand side of (2.13) is actually $O(t^{-(n/2)+1-2\gamma})$.

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Takeshi Wada
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka 560–0043
Japan
(E-mail: wada@math.sci.osaka-u.ac.jp)