A SIMPLE PFAFFIAN FORM REPRESENTING THE HYPERGEOMETRIC DIFFERENTIAL EQUATION OF TYPE (3, 6)

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1. Introduction

The system $E(k, n) = E(k, n; a_1, a_2, \dots, a_m)$ of differential equations of type (k, n) is defined on the Grassmanian $G_{k,n}$ and has many symmetries. Thanks to these symmetries, we can reduce the system $E(k, n)$ to the system $E(k, n)' = E(k, n; a_1, a_2, \dots, a_m)'$ of differential equations on $\mathbb{C}^{(k-1)(n-k-1)}$ of rank $\binom{n-2}{k-1}$.

In [MSY], Matsumoto *et al* studied the system $E(3, 6) = E(3, 6; a_1, a_2, \dots, a_6)$ and explicitly obtained the reduced differential equations $E(3, 6)' = E(3, 6; a_1, a_2, \dots, a_6)'$ on $\mathbb{C}^4$ with rank 6. They represented $E(3, 6)'$ in the form

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = G_{ij} \frac{\partial^2 u}{\partial x^1 \partial x^4} + \sum_{k=1}^4 A_{ij}^k \frac{\partial u}{\partial x^k} + A_{ij}^0 u \quad (1 \leq i, j \leq 4) \quad (1.1)$$

(Theorem 1.7.1 in [MSY]). They also obtained an equivalent Pfaffian form

$$d\vec{u} = \omega \vec{u}, \quad (1.2)$$

where

$$\vec{u} = {}^t(u, u_1, u_2, u_3, u_4, u_{14}), \quad u_j = \frac{\partial u}{\partial x^j}, \quad u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$$

(see [MSY, pp. 52–55]). Unfortunately, this Pfaffian form is somewhat complicated.

In this paper, we replace $\vec{u}$ with $\tilde{u} := \Lambda \vec{u}$, where Λ is the diagonal matrix with elements $1, x^1, x^2, x^3, x^4, D_2 := x^1 x^4 - x^2 x^3$. Then the Pfaffian form (1.2) changes to the simple form

$$d\tilde{u} = \tilde{\omega} \tilde{u}, \quad \tilde{\omega} = \sum_j P_j d \log f_j, \quad (1.3)$$

where P_j are constant 6×6 matrices and f_j are defining functions of irreducible components of singular locus of $E(3, 6)'$ in $\mathbb{C}^4$ (see Theorem below).

2. Main theorem

The system $E(3, 6)$ has six parameters a_j , $1 \leq j \leq 6$, satisfying the relation $\sum a_j = 3$. We denote

$$a_{ij} = a_i + a_j \quad \text{and} \quad a_{ijk} = a_i + a_j + a_k.$$

The reduced system $E(3, 6)'$ on $\mathbb{C}^4$ is written in the form of (1.1), where x^j ($1 \leq j \leq 4$) are the coordinates on $\mathbb{C}^4$ and $u = u(x)$ is the unknown function. Let

$$u_j = \partial u / \partial x^j, \quad u_{ij} = \partial^2 u / \partial x^i \partial x^j, \\ D_1 = (x^1 - 1)(x^4 - 1) - (x^2 - 1)(x^3 - 1), \quad D_2 = x^1 x^4 - x^2 x^3.$$

The singular locus of $E(3, 6)'$ are defined by

$$\prod_{j=1}^4 x^j (x^j - 1) \cdot (x^1 - x^2)(x^1 - x^3)(x^2 - x^4)(x^3 - x^4) D_1 D_2 = 0$$

(see [MSY, p. 51]).

THEOREM. *The system $E(3, 6; a_1, a_2, \dots, a_6)'$ is equivalent to the following Pfaffian form:*

$$d\tilde{u} = \tilde{\omega}\tilde{u},$$

where

$$\tilde{u} = {}^t(u, x^1 u_1, x^2 u_2, x^3 u_3, x^4 u_4, D_2 u_{14})$$

and

$$\begin{aligned} \tilde{\omega} = & \sum_{j=1}^4 P_j d \log x^j + \sum_{j=1}^4 Q_j d \log (x^j - 1) \\ & + P_{12} d \log (x^1 - x^2) + P_{13} d \log (x^1 - x^3) + P_{24} d \log (x^2 - x^4) \\ & + P_{34} d \log (x^3 - x^4) + R_1 d \log D_1 + R_2 d \log D_2, \end{aligned}$$

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 - a_{246} & 0 & 0 & 0 & 0 \\ 0 & a_6 - 1 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(a_6 - 1) & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_5 - 1 & 0 & 0 & 0 \\ 0 & 0 & 2 - a_{245} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & a_3(1 - a_5) & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 - a_{346} & 0 & 0 \\ 0 & 0 & 0 & a_6 - 1 & 0 & 0 \\ 0 & 0 & 0 & a_2(1 - a_6) & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & a_5 - 1 & 0 \\ 0 & 0 & 0 & 0 & 2 - a_{345} & 0 \\ 0 & 0 & 0 & 0 & a_2(a_5 - 1) & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_2(a_5 - 1) & -(2 - a_{345}) & a_5 - 1 & -a_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_2(a_6 - 1) & a_6 - 1 & -(2 - a_{346}) & 0 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_3(a_5 - 1) & -a_3 & 0 & -(2 - a_{245}) & a_5 - 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_3(a_6 - 1) & 0 & -a_3 & a_6 - 1 & -(2 - a_{246}) & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 - 1 & 1 - a_5 & 0 & 0 & -1 \\ 0 & 1 - a_6 & a_5 - 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1 - a_6) & a_3(a_5 - 1) & 0 & 0 & a_3 \end{pmatrix},$$

$$P_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & a_2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & -a_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(1 - a_6) & 0 & -a_2(1 - a_6) & 0 & 1 - a_6 \end{pmatrix},$$

$$P_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 & 0 & a_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & -a_2 & -1 \\ 0 & 0 & -a_3(1 - a_5) & 0 & a_2(1 - a_5) & 1 - a_5 \end{pmatrix},$$

$$\begin{aligned}
 P_{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 - 1 & 1 - a_5 & 1 \\ 0 & 0 & 0 & 1 - a_6 & a_5 - 1 & -1 \\ 0 & 0 & 0 & a_2(a_6 - 1) & a_2(1 - a_5) & a_2 \end{pmatrix}, \\
 R_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3(a_6 - 1) & a_3(1 - a_5) & a_2(1 - a_6) & a_2(a_5 - 1) & -a_{123} \end{pmatrix}, \\
 R_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 - a_{234} \end{pmatrix}.
 \end{aligned}$$

Proof. Since $\tilde{u} = \Lambda \bar{u}$ (see Section 1 for the terminologies), the Pfaffian form (1.2) given in [MSY, pp. 52–55] changes to

$$\tilde{u} = (d\Lambda + \Lambda\omega)\Lambda^{-1}\tilde{u}.$$

A direct computation proves the theorem. $\square$

Remark. The $(5, 5)$ -element ω_5^5 of ω given in [MSY, p. 55] is wrong; it should be corrected to

$$\begin{aligned}
 \omega_5^5 &= -\alpha_{123}d \log D_1 - \alpha_{234}d \log D_2 + \alpha_2 d \log(x^4 - x^3) + \alpha_3 d \log(x^1 - x^2) \\
 &\quad + (1 - \alpha_5)d \log(x^4 - x^2) + (1 - \alpha_6)d \log(x^1 - x^3).
 \end{aligned}$$

The characteristic exponents along the singular locus of $E(3, 6)'$ in $\mathbb{C}^4$ are given in [MSY]. Those along the plane at infinity L_∞ can be easily obtained by use of the Pfaffian form in the above theorem.

COROLLARY. Assume $a_{23} + a_{56}$ is not an integer. Then the characteristic exponents of $E(3, 6)'$ along L_∞ are

$$a_{23}, \quad a_{23}, \quad a_{23}, \quad 2 - a_{56}, \quad 2 - a_{56}, \quad 2 - a_{56}.$$

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