INTERTWINING OPERATORS OF THE HEISENBERG GROUP

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(Received 22 June 1998 and revised 13 October 1998 and 4 January 1999)

0. Introduction

It is well known that the finite-dimensional Heisenberg group, denoted by G throughout this paper, has three typical and concrete realizations of irreducible unitary representations of infinite dimension, i.e. the Schrödinger representation, the Fock representation, and the Lattice representation. All the three representations can be constructed as induced representations (cf. [Car66]). The celebrated Stone–von Neumann theorem assures that all irreducible unitary representations are equivalent to each other if the actions of the center T are identical. We remark that the element of the Lattice representation corresponding to the constant function of the Fock representation under this equivalence is a θ -series (see [Mum91]).

The purpose of this paper is to construct intertwining operators among these representations via integral operators on G with respect to the Haar measure, the kernel function of which is a function characterized by certain properties related to a polarization in the cases of the Schrödinger and Fock representations, or a maximal isotropic subgroup in the case of the Lattice representation, employed in the construction of each representation. One can see that our strategy agrees with the principle described in [Oka80]: Let us be given irreducible representations (π_i, V_i) ($i = 1, 2$) of G which are obtained as induced representations $\text{ind}_{H_i}^G \rho_i$ with H_i subgroups of G and ρ_i representations of H_i . The elements of V_i are regarded as sections of vector bundles on G/H_i . Suppose that π_1 and π_2 are equivalent to each other. Then, in order to construct an intertwining operator between π_1 and π_2 , one lifts elements of V_1 to functions on G , and then projects them onto V_2 (see [Oka80, ch. 4] for more details).

More precisely, let P be one of the following: (1) a subgroup of G corresponding to the polarization for the Schrödinger representation, (2) a subgroup of $G_{\mathbb{C}}$ corresponding to the polarization for the Fock representation, or (3) a maximal isotropic subgroup of G modulo the center T for the Lattice representation. We denote by L_{χ} the line bundle associated to the principal bundle $G \rightarrow G/P$ in the

cases of (1) and (3), or $G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}/P$ in the case of (2), induced by a one-dimensional representation χ of P , and by π_χ the representation on the sections of L_χ defined by left translation. Furthermore, let $\mathfrak{n}$ be a closed subspace transverse to $\mathfrak{p} = \text{Lie } P$, N a submanifold of G containing the identity element $1 \in G$ whose tangent space at 1 is isomorphic to $\mathfrak{n}$, and B a symplectic form on $\mathfrak{g}/\mathfrak{t}$, where $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{t} = \text{Lie } T$. Then we take a section ψ_χ of L_χ which is characterized by the following conditions:

$$\pi_\chi(t)\psi_\chi = \chi(t)\psi_\chi \quad (t \in T), \quad (0.1a)$$

$$d\pi_\chi(X)\psi_\chi(\exp Y) = 2\pi i B(X, Y)\psi_\chi(\exp Y) \quad (X \in \mathfrak{n}), \quad (0.1b)$$

$$\psi_\chi(1) = 1, \quad (0.1c)$$

where $Y \in \mathfrak{n}$ such that $\exp Y \in N$ (see below for more details). We call the element ψ_χ the *primitive vector* of $\Gamma(L_\chi)$. Note that if $\mathfrak{n}$ is a subalgebra, then the condition (0.1b) reduces to

$$d\pi_\chi(X)\psi_\chi = 0 \quad (X \in \mathfrak{n}) \quad (0.1b')$$

since $\mathfrak{n} \cap \mathfrak{t} = \{0\}$. Then the integral operator with ψ_χ as kernel function gives the desired intertwining operator.

If we replace G by a semisimple Lie group having holomorphic discrete series, T by its compact Cartan subgroup, and so on, then the conditions (0.1a), (0.1b'), and (0.1c) characterizing the primitive vector correspond to those for the highest weight vector. We showed in [HOOS93] that, for such Lie groups, the integral operator with the highest weight vector as kernel function is the identity operator on the holomorphic discrete series representation that is constructed by Borel–Weil theory. This fact seems to indicate the possibility that a similar construction of intertwining operators may be applicable to semisimple Lie groups.

The content of this paper is as follows. In Section 1, we introduce our notation and realize the Schrödinger representation as an induced representation. In Section 2, we construct the Fock representation in the same way, and then show that the integral operator whose kernel is the primitive vector for the Fock representation coincides with the intertwining operator given in [Bar61]. In Section 3, we construct the Lattice representation similarly, and then show that the integral operators whose kernels are the primitive vectors for the Lattice representation and for the Schrödinger representation coincide with the intertwining operators given in [LV80, Mum91]. Finally, in Section 4, taking the opposite real polarization to the one in Section 1, we realize another Schrödinger representation, and construct intertwining operators from the Fock representation onto each of the two Schrödinger representations using the

primitive vectors for the Schrödinger representations. Then we show that the Jacobi identity for θ -series is equivalent to the commutativity of these intertwining operators.

1. Preliminaries

We will treat the case where $n = 1$ in this paper for simplicity; however, it is easy to extend our result to a general n .

Let $V = \mathbb{R}^2$, and B be the canonical symplectic form on V given by $B(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Our Heisenberg group G is then $V \times U(1)$ as set, with multiplication law defined by

$$(\mathbf{x}, \lambda)(\mathbf{y}, \mu) = (\mathbf{x} + \mathbf{y}, \lambda\mu e^{\pi i B(\mathbf{x}, \mathbf{y})}).$$

Namely, we have a central extension of V by $U(1)$:

$$1 \longrightarrow U(1) \longrightarrow G \longrightarrow V \longrightarrow 0. \quad (\text{exact})$$

Throughout this paper, let us fix the following subspaces of V once and for all:

$$l_1 = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}; x_1 \in \mathbb{R} \right\}, \quad l_2 = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}; x_2 \in \mathbb{R} \right\}.$$

Let us identify $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; x_1, x_2 \in \mathbb{R} \right\}$ with the vector space consisting of upper triangular matrices given by

$$\mathfrak{n}_V = \left\{ \begin{pmatrix} 0 & x_1 & 0 \\ & 0 & x_2 \\ & & 0 \end{pmatrix}; x_1, x_2 \in \mathbb{R} \right\},$$

and denote also by B the induced symplectic form on $\mathfrak{n}_V$ under this identification. Then the universal covering group $\tilde{G}$ of G is given by

$$\tilde{G} = \left\{ \exp \begin{pmatrix} 0 & p & r \\ & 0 & q \\ & & 0 \end{pmatrix}; p, q, r \in \mathbb{R} \right\}.$$

If we embed the additive group $\mathbb{Z}$ into the center $\tilde{T}$ of $\tilde{G}$ by

$$\mathbb{Z} \longrightarrow \tilde{T}, \quad m \mapsto \begin{pmatrix} 1 & 0 & m \\ & 1 & 0 \\ & & 1 \end{pmatrix},$$

then G is isomorphic to $\tilde{G}/\mathbb{Z}$. Under this identification, we write an element of G as $g = n_{p,q}t_r$, where

$$n_{p,q} = \exp \begin{pmatrix} 0 & p & 0 \\ & 0 & q \\ & & 0 \end{pmatrix}, \quad t_r = \exp \begin{pmatrix} 0 & 0 & r \\ & 0 & 0 \\ & & 0 \end{pmatrix} \quad (1.1)$$

with $p, q \in \mathbb{R}$ and $r \in \mathbb{R}/\mathbb{Z}$. The element $n_{p,q}t_r$ of G corresponds to $((\frac{p}{q}), e^{2\pi i r})$ in the above notation. We will mainly use the matrix notation for elements of G in what follows.

With the above notation, the Schrödinger representation, the Fock representation, and the Lattice representation are then presented as follows.

(1) *Schrödinger representation* $(U_S, \mathcal{H}_S)$. The Hilbert space $\mathcal{H}_S$ is $L^2(\mathbb{R})$, where $\mathbb{R}$ is identified with l_2 , and the representation operator is given by

$$U_S(g)F(q) = e^{-2\pi i c} e^{2\pi i(-aq+ab/2)} F(q-b)$$

for $g = n_{a,b}t_c \in G$ and $F \in \mathcal{H}_S$.

(2) *Fock representation* $(U_F, \mathcal{H}_F)$. Fix a positive real number $t > 0$. The Hilbert space $\mathcal{H}_F$ is the space of holomorphic and square-integrable functions on $\mathbb{C}$ with respect to $e^{-\pi H(z,z)} dp dq$ ($z = q + itp$), where H is a Hermitian form on $\mathbb{C}$ given by $H(z, w) = z\bar{w}/t$, and $dx dy$ denotes the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. The representation operator is given by

$$U_F(g)F(z) = e^{-2\pi i c} e^{\pi H(z,\gamma) - \pi H(\gamma,\gamma)/2} F(z - \gamma)$$

for $g = n_{a,b}t_c \in G$ and $F \in \mathcal{H}_F$, where $\gamma = b + ita$.

(3) *Lattice representation* $(U_L, \mathcal{H}_L)$. The Hilbert space $\mathcal{H}_L$ is given by

$$\mathcal{H}_L = \left\{ F : \mathbb{R}^2 \rightarrow \mathbb{C}; \begin{array}{l} F(x+m, y+n) = e^{-\pi i(mn+nx-my)} F(x, y) \ (m, n \in \mathbb{Z}), \\ \int_{\mathbb{R}^2/\mathbb{Z}^2} dx dy |F(x, y)|^2 < \infty \end{array} \right\},$$

and the representation operator is given by

$$U_L(g)F(x, y) = e^{-2\pi i c} e^{\pi i(bx-ay)} F(x-a, y-b)$$

for $g = n_{a,b}t_c \in G$ and $F \in \mathcal{H}_L$.

Remark 1.1. In Section 4 below, we will construct another realization of the Schrödinger representation whose Hilbert space is also $L^2(\mathbb{R})$, but this $\mathbb{R}$ will be identified with l_1 .

Here in this section, we only show how the Schrödinger representation can be obtained as an induced representation. As for the others, we will give their similar construction in Sections 2–4.

To begin with, we define a central character by

$$T \longrightarrow U(1), \quad t_r \mapsto e^{-2\pi i r}. \quad (1.2)$$

Let $\mathfrak{p}_S$ be a subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{p}_S = \left\{ \begin{pmatrix} 0 & p & r \\ & 0 & 0 \\ & & 0 \end{pmatrix}; p, r \in \mathbb{R} \right\}$$

and P_S its analytic subgroup. Note that $\mathfrak{p}_S/\mathfrak{t}$ is a real polarization of $\mathfrak{g}/\mathfrak{t}$ corresponding to l_1 . Then one can uniquely extend the central character (1.2) to a one-dimensional representation λ of P_S by

$$\lambda : P_S \longrightarrow U(1), \quad n_{p,0}t_r \mapsto e^{-2\pi i r}.$$

Associated to the one-dimensional representation λ , there exists a line bundle L_λ on $G/P_S \simeq \mathbb{R}$. The space of smooth sections of L_λ is identified with

$$\Gamma(L_\lambda) = \{f \in C^\infty(G); f(xp) = \lambda(p)^{-1}f(x) (x \in G, p \in P_S)\}.$$

Then one obtains a representation π_λ of G on $\Gamma(L_\lambda)$ by left translation:

$$\pi_\lambda(g)f(x) = f(g^{-1}x) \quad (f \in \Gamma(L_\lambda), x \in G).$$

Let $\mathfrak{n}_S$ be a subalgebra transverse to $\mathfrak{p}_S$ given by

$$\mathfrak{n}_S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ & 0 & q \\ & & 0 \end{pmatrix}; q \in \mathbb{R} \right\} \simeq \mathbb{R}.$$

Since each $f \in \Gamma(L_\lambda)$ is determined by its values on $\exp \mathfrak{n}_S$, the restriction map to $\exp \mathfrak{n}_S$ induces an isomorphism:

$$\Gamma(L_\lambda) \xrightarrow{\simeq} C^\infty(\mathbb{R}), \quad f \mapsto F, \quad (1.3)$$

where $F(q) = f(n_{0,q})$. We denote its inverse by R_λ .

Let $U_S(g)$ denote the action of G on $C^\infty(\mathbb{R})$ such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} \Gamma(L_\lambda) & \xleftarrow{R_\lambda} & C^\infty(\mathbb{R}) \\ \pi_\lambda(g) \downarrow & & \downarrow U_S(g) \\ \Gamma(L_\lambda) & \xleftarrow{R_\lambda} & C^\infty(\mathbb{R}), \end{array}$$

where the horizontal maps are the isomorphism in (1.3). The Hilbert-space completion of $C_c^\infty(\mathbb{R})$ with respect to the norm defined by

$$\|F\| = \int_{\mathbb{R}} |F(q)|^2 dq$$

gives the Schrödinger representation $(U_S, \mathcal{H}_S)$.

Finally, let ψ_λ be an element of $\Gamma(L_\lambda)$ satisfying the following conditions:

$$\pi_\lambda(t)\psi_\lambda = \lambda(t)\psi_\lambda \quad (t \in T), \quad (1.4a)$$

$$d\pi_\lambda(X)\psi_\lambda = 0 \quad (X \in \mathfrak{n}_S), \quad (1.4b)$$

$$\psi_\lambda(1) = 1. \quad (1.4c)$$

Note that it is uniquely determined by these conditions. We call the element ψ_λ satisfying (1.4) the *primitive vector* of $\Gamma(L_\lambda)$.

2. Schrödinger and Fock representations

In this section, we show how the Fock representation is obtained as an induced representation and construct the intertwining operator from the Schrödinger representation onto the Fock representation. We will treat the reverse direction in Section 4.

First, let us introduce the complexification $G_{\mathbb{C}}$ of G and its Lie algebra $\mathfrak{g}_{\mathbb{C}}$. By abuse of notation, we also write $n_{p,q}t_r$ as in (1.1) for $p, q \in \mathbb{C}, r \in \mathbb{C}/\mathbb{Z}$. Then, $G_{\mathbb{C}}$ is just

$$G_{\mathbb{C}} = \{n_{p,q}t_r; p, q \in \mathbb{C}, r \in \mathbb{C}/\mathbb{Z}\},$$

and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. This time we take a totally complex polarization of $\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}}$, where $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$. Namely, let $\mathfrak{p}_F$ be a subalgebra $\mathfrak{g}_{\mathbb{C}}$ given by

$$\mathfrak{p}_F = \left\{ \begin{pmatrix} 0 & ib/t & c \\ & 0 & b \\ & & 0 \end{pmatrix}; b, c \in \mathbb{C} \right\}$$

and P_F its analytic subgroup. Furthermore, let $\mathfrak{n}_F$ be a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ transverse to $\mathfrak{p}_F$:

$$\mathfrak{n}_F = \left\{ \begin{pmatrix} 0 & -iz/2t & 0 \\ & 0 & z/2 \\ & & 0 \end{pmatrix}; z \in \mathbb{C} \right\}.$$

For $z, w \in \mathbb{C}$, let us denote $n_{-iz/2t, z/2}$ and $n_{iw/2t, w/2}$ by n_z and n_w^- , respectively, for brevity.

The central character (1.2) uniquely extends to a one-dimensional representation ξ of P_F :

$$\xi : P_F \longrightarrow \mathbb{C}^\times, \quad n_b^- t_c \mapsto e^{-2\pi i c}.$$

Construct a holomorphic line bundle L_ξ on $G_\mathbb{C}/P_F \simeq \mathbb{C}$ and a representation π_ξ of $G_\mathbb{C}$ on the space $\Gamma(L_\xi)$ by left translation, where

$$\Gamma(L_\xi) = \{f : G_\mathbb{C} \rightarrow \mathbb{C} : \text{holomorphic}; f(xp) = \xi(p)^{-1} f(x) (x \in G_\mathbb{C}, p \in P_F)\},$$

in the same way as the Schrödinger representation given in Section 1. If $\Gamma(\mathbb{C})$ denotes the space of holomorphic functions on $\mathbb{C}$, the restriction map induces an isomorphism between $\Gamma(L_\xi)$ and $\Gamma(\mathbb{C})$ as before:

$$\Gamma(L_\xi) \xrightarrow{\sim} \Gamma(\mathbb{C}), \quad f \mapsto F,$$

where $F(z) = f(n_z)$. We denote its inverse by R_ξ . Now one can define a representation of $G_\mathbb{C}$ on $\Gamma(\mathbb{C})$ such that the following diagram commutes for all $g \in G_\mathbb{C}$:

$$\begin{array}{ccc} \Gamma(L_\xi) & \xleftarrow{R_\xi} & \Gamma(\mathbb{C}) \\ \pi_\xi(g) \downarrow & & \downarrow U_F(g) \\ \Gamma(L_\xi) & \xleftarrow{R_\xi} & \Gamma(\mathbb{C}). \end{array}$$

If $\mathcal{H}_F$ denotes the Hilbert space consisting of holomorphic and square-integrable functions on $\mathbb{C}$ with respect to the Gaussian measure $e^{-\pi H(z,z)} dp dq$ ($z = q + itp$), then it is easy to see that $U_F(g)$ is unitary if g is in G , i.e. $(U_F, \mathcal{H}_F)$ is a unitary representation of G , called the Fock representation.

We call an element ψ_ξ which satisfies the following conditions the *primitive vector* of $\Gamma(L_\xi)$:

$$\pi_\xi(t)\psi_\xi = \xi(t)\psi_\xi \quad (t \in T), \quad (2.1a)$$

$$d\pi_\xi(X)\psi_\xi = 0 \quad (X \in \mathfrak{n}_F), \quad (2.1b)$$

$$\psi_\xi(1) = 1. \quad (2.1c)$$

PROPOSITION 2.1. *Let ψ_ξ be the primitive vector of $\Gamma(L_\xi)$. Then the integral operator*

$$\mathcal{A}_{\xi\lambda} F(w) = \int_G \psi_\xi(g^{-1}g') R_\lambda F(g) dg \quad (2.2)$$

gives an intertwining operator from $(U_S, \mathcal{H}_S)$ onto $(U_F, \mathcal{H}_F)$, where $g' = n_w$ with $w \in \mathbb{C}$ and dg is the Haar measure on G .

Proof. Set $g = n_{p,q} t_r$ for $p, q \in \mathbb{R}$ and $r \in \mathbb{R}/\mathbb{Z}$. If we put $z = q + itp$, then we see that $g = n_z n_{\bar{z}}^{-1} t_{r+iH(z,z)/4}$, and

$$g^{-1} g' = n_{-z+w} n_{-\bar{z}}^{-1} t_{-r+i(H(z,z)/4-H(w,z)/2)}.$$

Therefore, we obtain that

$$\psi_\xi(g^{-1} g') = e^{-2\pi i r} e^{-\frac{\pi}{2} H(z,z) + \pi H(w,z)}.$$

Since $R_\lambda F$ is given by

$$R_\lambda F(g) = e^{2\pi i(r+pq/2)} F(q),$$

for $F \in C_c^\infty(\mathbb{R})$, the right-hand side of (2.2) equals

$$\begin{aligned} & \int_{\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}} dp dq dr e^{-\frac{\pi}{2} H(z,z) + \pi H(w,z)} e^{\pi i p q} F(q) \\ &= \int_{\mathbb{R}} dq e^{-\frac{\pi}{2i}(q^2 - 2wq)} F(q) \int_{\mathbb{R}} dp e^{-\frac{\pi}{2}(tp^2 + 2i(w-q)p)}. \end{aligned} \quad (2.3)$$

Now, using the formula

$$\int_{\mathbb{R}} dx e^{-ax^2} e^{ixy} = \left(\frac{\pi}{a}\right)^{1/2} e^{-y^2/4a}$$

with $a > 0$, it follows that (2.3) equals

$$\left(\frac{2}{i}\right)^{1/2} \int_{\mathbb{R}} dq e^{-\frac{\pi}{i}(q^2 - 2wq + w^2/2)} F(q), \quad (2.4)$$

which is the intertwining operator, up to constant multiple, given in [Bar61]. $\square$

Remark 2.2. In view of (2.2), it is immediate that our integral operator $\mathcal{A}_{\xi\lambda}$ intertwines between the two representations, since both are induced from left translation of G and dg is left invariant.

3. Schrödinger and Lattice representations

In this section, we show that it is possible to construct intertwining operators between the Schrödinger representation and the Lattice representation in the same way as in Section 2. In order to realize the Lattice representation as an induced representation, however, one has to take a maximal isotropic subgroup of G/T instead of a polarization of $\mathfrak{g}/\mathfrak{t}$ (see [Car66, Mum91]).

Set $P_L = HT$, where

$$H = \{\gamma_{n_1, n_2} := n_{n_1, n_2} t_{n_1 n_2 / 2}; n_1, n_2 \in \mathbb{Z}\}.$$

Then the central character (1.2) uniquely extends to a one-dimensional representation η :

$$\eta : P_L \longrightarrow U(1), \quad \gamma_{n_1, n_2} t_r \mapsto e^{-2\pi i r}.$$

Construct a line bundle L_η on $G/P_L \simeq \mathbb{R}^2/\mathbb{Z}^2$ and a representation π_η of G on the space of $\Gamma(L_\eta)$ by left translation, where

$$\Gamma(L_\eta) = \{f : G \rightarrow \mathbb{C} : \text{measurable}; f(xp) = \eta(p)^{-1} f(x) (x \in G, p \in P_L)\},$$

in the same way as above. Let $\Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta)$ be the space consisting of all measurable functions F on $\mathbb{R}^2$ satisfying

$$F(x + n_1, y + n_2) = e^{-\pi i(n_1 n_2 + n_2 x - n_1 y)} F(x, y) \quad (n_1, n_2 \in \mathbb{Z}).$$

Then the restriction map

$$\Gamma(L_\eta) \xrightarrow{\simeq} \Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta), \quad f \mapsto F$$

gives an isomorphism between $\Gamma(L_\eta)$ and $\Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta)$, where $F(x, y) = f(n_{x, y})$. We denote its inverse by R_η . Now one can define a representation of G on $\Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta)$ such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} \Gamma(L_\eta) & \xleftarrow{R_\eta} & \Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta) \\ \pi_\eta(g) \downarrow & & \downarrow U_L(g) \\ \Gamma(L_\eta) & \xleftarrow{R_\eta} & \Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta). \end{array}$$

Let $\mathcal{H}_L$ be a Hilbert space defined by

$$\mathcal{H}_L = \left\{ f \in \Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta); \|F\|^2 = \int_{\mathbb{R}^2/\mathbb{Z}^2} |F(x, y)|^2 dx dy < \infty \right\}.$$

Then $(U_L, \mathcal{H}_L)$ is a unitary representation of G , called the Lattice representation.

If we take representatives of elements of $\mathbb{R}/\mathbb{Z}$ from $(-1/2, 1/2]$, and put

$$N_L = \{n_{x, y} \in G; x, y \in (-1/2, 1/2]\},$$

then any element $g \in G$ uniquely decomposes as

$$g = n_{x, y} \gamma_{n_1, n_2} t_r \tag{3.1a}$$

with $n_{x,y} \in N_L$, $\gamma_{n_1,n_2} \in H$, and $t_r \in T$. Namely, we have a decomposition of G as a manifold:

$$G = N_L P_L \simeq N_L \times P_L. \quad (3.1b)$$

With the decomposition (3.1), we define a function ψ_η on G by

$$\psi_\eta(n_{x,y}\gamma_{n_1,n_2}t_r) = e^{2\pi i r}. \quad (3.2)$$

Then it is easy to see that ψ_η belongs to $\Gamma(L_\eta)$ and is characterized by the following conditions: the restriction $\psi_\eta|_{N_L^\circ \times \{1\}}$ is smooth,

$$\pi_\eta(t)\psi_\eta = \eta(t)\psi_\eta \quad (t \in T), \quad (3.3a)$$

$$d\pi_\eta(X)\psi_\eta(\exp Y) = 2\pi i B(X, Y)\psi_\eta(\exp Y) \quad (X \in \mathfrak{n}_V), \quad (3.3b)$$

$$\psi_\eta(1) = 1 \quad (3.3c)$$

for $Y \in \mathfrak{n}_V$ with $\exp Y \in N_L^\circ$, where N_L° is the interior of N_L . We call the element ψ_η satisfying the conditions (3.3) the *primitive vector* of $\Gamma(L_\eta)$.

Now we are ready to state one of our main results.

THEOREM 3.1. *Let ψ_λ and ψ_η be the primitive vectors of $\Gamma(L_\lambda)$ and $\Gamma(L_\eta)$, respectively. Then the integral operator defined by*

$$\mathcal{A}_{\eta\lambda}\Phi(u, v) = \int_G \psi_\eta(g^{-1}g') R_\lambda\Phi(g) dg \quad (3.4)$$

gives an intertwining operator from $(U_S, \mathcal{H}_S)$ onto $(U_L, \mathcal{H}_L)$, where $g' = n_{u,v}$ with $u, v \in \mathbb{R}$ and dg is the Haar measure on G .

Conversely, the integral operator defined by

$$\mathcal{A}_{\lambda\eta}\Psi(q) = \int_G \psi_\lambda(g^{-1}g') R_\eta\Psi(g) dg \quad (3.5)$$

gives an intertwining operator from $(U_L, \mathcal{H}_L)$ onto $(U_S, \mathcal{H}_S)$, where $g' = n_{0,q}$ with $q \in \mathbb{R}$ and dg is as above.

Remark 3.2. It is known that the following operators

$$\mathcal{A}_{LS}\Phi(u, v) = \sum_{n \in \mathbb{Z}} e^{2\pi i(nu + \frac{1}{2}uv)} \Phi(v + n), \quad (3.6)$$

$$\mathcal{A}_{SL}\Psi(q) = \int_{\mathbb{R}/\mathbb{Z}} dx e^{-\pi i q x} \Psi(x, q) \quad (3.7)$$

give intertwining operators from $(U_S, \mathcal{H}_S)$ onto $(U_L, \mathcal{H}_L)$, and from $(U_L, \mathcal{H}_L)$ onto $(U_S, \mathcal{H}_S)$, respectively. Furthermore, it is immediate to see that these two operators are isometric and inverse to each other (see [LV80, Mum91]).

Therefore, the theorem follows from the propositions below.

PROPOSITION 3.3. *With the notation above, we see that the integral operator (3.4) coincides with (3.6). Namely, we have*

$$\int_G \psi_\eta(g^{-1}g') R_\lambda \Phi(g) dg = \sum_{n \in \mathbb{Z}} e^{2\pi i(nu + \frac{1}{2}uv)} \Phi(v+n) \quad (3.8)$$

for $\Phi \in C_c^\infty(\mathbb{R})$, where $g' = n_{u,v}$.

Proof. Put $g = n_{p,q}t_r$ and $g' = n_{u,v}$. Choosing integers n_1 and n_2 such that both $x := u - p - n_1$ and $y := v - q - n_2$ are in $(-1/2, 1/2]$, we see that

$$g^{-1}g' = n_{x,y} \gamma_{n_1,n_2} t_{\frac{1}{2}(n_1y - n_2x - n_1n_2)} t_{-r + \frac{1}{2}((x+n_1)v - (y+n_2)u)}.$$

Hence, it follows from (3.2) that

$$\psi_\eta(g^{-1}g') = e^{2\pi i(\frac{1}{2}(n_1y - n_2x - n_1n_2) - r + \frac{1}{2}((x+n_1)v - (y+n_2)u))}.$$

On the other hand, the element of $\Gamma(L_\lambda)$ corresponding to $\Phi \in C_c^\infty(\mathbb{R})$ is given by

$$R_\lambda \Phi(g) = e^{2\pi i(r + \frac{1}{2}pq)} \Phi(q).$$

Therefore, since $p = u - x - n_1$ and $q = v - y - n_2$, we obtain that the left-hand side of (3.8) equals

$$\begin{aligned} & \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}/\mathbb{Z}} dx dy dr e^{2\pi i(\frac{1}{2}xy + n_1y - (y+n_2)u + \frac{1}{2}uv)} \Phi(v - y - n_2) \\ &= \sum_{n_1, n_2} \int_{-1/2}^{1/2} dy \frac{e^{\pi iy/2} - e^{-\pi iy/2}}{\pi iy} e^{2\pi i(n_1y - (y+n_2)u + \frac{1}{2}uv)} \Phi(v - y - n_2) \end{aligned} \quad (3.9)$$

for $\Phi \in C_c^\infty(\mathbb{R})$. Here, as usual, we understand that the function

$$\frac{e^{\pi iy/2} - e^{-\pi iy/2}}{\pi iy} = \frac{\sin(\pi y/2)}{\pi y/2}$$

takes the value 1 at $y = 0$ and, hence, is continuous on $\mathbb{R}/\mathbb{Z}$ and smooth on $(-1/2, 1/2]$. Now, summing up with respect to n_1 , we obtain that (3.9) equals

$$\begin{aligned} & \sum_{n_2} \int_{-1/2}^{1/2} dy \delta(y) \frac{e^{\pi iy/2} - e^{-\pi iy/2}}{\pi iy} e^{2\pi i(-(y+n_2)u + \frac{1}{2}uv)} \Phi(v - y - n_2) \\ &= \sum_{n_2} e^{2\pi i(-n_2u + \frac{1}{2}uv)} \Phi(v - n_2) \\ &= \sum_{n \in \mathbb{Z}} e^{2\pi i(nu + \frac{1}{2}uv)} \Phi(v + n), \end{aligned}$$

where $\delta(y)$ denotes the Dirac delta function on $\mathbb{R}/\mathbb{Z}$ supported at $y = 0$. This completes the proof of the proposition. $\square$

Now the other direction.

PROPOSITION 3.4. *With the notations above, we see that the integral operator (3.5) coincides with (3.7). Namely, we have*

$$\int_G \psi_\lambda(g^{-1}g') R_\eta \Psi(g) dg = \int_{\mathbb{R}/\mathbb{Z}} dx e^{-\pi i q x} \Psi(x, q) \quad (3.10)$$

for $\Psi \in \Gamma(\mathbb{R}^2/\mathbb{Z}^2, L_\eta)$, where $g' = n_{0,q}$.

Proof. Write $g = n_{x,y} \gamma_{n_1, n_2} t_r$ as in (3.1a). Then we have

$$g^{-1}g' = n_{0,q-y-n_2} n_{-x-n_1, 0} t_{-r-q(x+n_1)+n_1(y+n_2)+\frac{1}{2}xy}$$

and

$$\psi_\lambda(g^{-1}g') = e^{2\pi i(-r-q(x+n_1)+n_1(y+n_2)+\frac{1}{2}xy)}. \quad (3.11)$$

Setting $f := R_\eta \Psi \in \Gamma(L_\eta)$, we see that it satisfies

$$f(g) = e^{2\pi i r} f(n_{x,y}),$$

from which, considering (3.11), it follows that the left-hand side of (3.10) equals

$$\begin{aligned} & \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}/\mathbb{Z}} dx dy dr e^{2\pi i(-q(x+n_1)+n_1(y+n_2)+\frac{1}{2}xy)} f(n_{x,y}) \\ &= \sum_{n_1, n_2 \in \mathbb{Z}} \int_{\mathbb{R}^2/\mathbb{Z}^2} dx dy e^{2\pi i(n_1(y+n_2-q)+\frac{1}{2}xy-qx)} f(n_{x,y}). \end{aligned} \quad (3.12)$$

From the fact that f belongs to $\Gamma(L_\eta)$, it follows that the function

$$\sum_{n_2 \in \mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} dx e^{2\pi i(n_1(y+n_2-q)+\frac{1}{2}xy-qx)} f(n_{x,y})$$

is smooth on $\mathbb{R}/\mathbb{Z}$ as a function of y . Therefore, summing up with respect to n_1 , we obtain that (3.12) equals

$$\sum_{n_2} \int_{\mathbb{R}^2/\mathbb{Z}^2} dx dy e^{2\pi i(-qx+\frac{1}{2}xy)} f(n_{x,y}) \delta(y+n_2-q) \quad (3.13)$$

$$= \int_{\mathbb{R}/\mathbb{Z}} dx e^{2\pi i(-qx+\frac{1}{2}x(q-n_2))} f(n_{x,q-n_2}), \quad (3.14)$$

where n_2 is chosen so that $q - n_2 \in (-1/2, 1/2]$ when we integrate (3.13) with respect to y . Again, since $f \in \Gamma(L_\eta)$, we see that

$$f(n_{x,q-n_2}) = e^{\pi i n_2 x} f(n_{x,q}),$$

which implies that (3.14) equals

$$\int_{\mathbb{R}/\mathbb{Z}} dx e^{-\pi i q x} f(n_{x,q}) = \int_{\mathbb{R}/\mathbb{Z}} dx e^{-\pi i q x} \Psi(x, q).$$

This completes the proof of the proposition and, thus, of the theorem. $\square$

4. Schrödinger and Fock representations revisited

In this section, we construct intertwining operators from the Fock representation onto two Schrödinger representations, one of which we constructed in Section 1 with polarization corresponding to l_1 , and the other we will construct below with polarization corresponding to l_2 , using the respective primitive vectors as above, and then show that the unitarity of the intertwining operators leads to the Jacobi identity for θ -series:

$$\vartheta(t) = t^{-1/2} \vartheta(1/t), \quad (4.1)$$

where $\vartheta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$ with $t > 0$.

Before introducing another real polarization, we rewrite the character λ and Schrödinger representation defined in Section 1 as λ_1 and $(U_{S,\lambda_1}, \mathcal{H}_{S,\lambda_1})$, respectively, to distinguish them. The real polarization corresponding to l_2 is given by

$$\mathfrak{p}'_S = \left\{ \begin{pmatrix} 0 & 0 & r \\ & 0 & q \\ & & 0 \end{pmatrix}; q, r \in \mathbb{R} \right\}.$$

Let P'_S be its analytic subgroup. Define a one-dimensional representation λ_2 of P'_S by

$$\lambda_2 : P'_S \longrightarrow U(1), \quad n_{0,q} t_r \mapsto e^{-2\pi i r},$$

and form a line bundle L_{λ_2} and $\Gamma(L_{\lambda_2})$, the space of its sections, in the same way as above. Let $\mathfrak{n}'_S$ be a subalgebra transverse to $\mathfrak{p}'_S$ given by

$$\mathfrak{n}'_S = \left\{ \begin{pmatrix} 0 & p & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}; p \in \mathbb{R} \right\}.$$

Then we obtain an irreducible unitary representation of G , which we denote by $(U_{S,\lambda_2}, \mathcal{H}_{S,\lambda_2})$. Let us write the primitive vector of $\Gamma(L_{\lambda_2})$ as ψ_{λ_2} .

Remark 4.1. One can construct the intertwining operator $\mathcal{A}_{\eta\lambda_2}$ from $(U_{S,\lambda_2}, \mathcal{H}_{S,\lambda_2})$ onto the Lattice representation $(U_L, \mathcal{H}_L)$ using the primitive vector ψ_η of $\Gamma(L_\eta)$ as in Section 3. Explicitly, it is given by

$$(\mathcal{A}_{\eta\lambda_2}\Psi)(u, v) = \sum_{m \in \mathbb{Z}} e^{-2\pi i(mv + \frac{1}{2}uv)} \Psi(u + m) \quad (4.2)$$

for $\Psi \in \mathcal{H}_{S,\lambda_2}$. Note that the Fourier transform $\mathcal{F}_{\lambda_2\lambda_1}$ on $L^2(\mathbb{R})$ is the intertwining operator from $(U_{S,\lambda_1}, \mathcal{H}_{S,\lambda_1})$ onto $(U_{S,\lambda_2}, \mathcal{H}_{S,\lambda_2})$, and satisfies $\mathcal{A}_{\eta\lambda_1} = \mathcal{A}_{\eta\lambda_2} \mathcal{F}_{\lambda_2\lambda_1}$ (see [LV80]).

THEOREM 4.2. *Let ψ_{λ_j} be the primitive vector of $\Gamma(L_{\lambda_j})$ for $j = 1$ or 2 . Then the integral operators defined by*

$$\tilde{\mathcal{A}}_{\lambda_j\xi} F(q) = \int_G \psi_{\lambda_j}(g^{-1}g') R_\xi F(g) dg \quad (4.3)$$

give intertwining operators from $(U_F, \mathcal{H}_F)$ onto $(U_{S,\lambda_j}, \mathcal{H}_{S,\lambda_j})$ ($j = 1, 2$), where $g' = n_{0,x}$ with $x \in \mathbb{R}$ for $j = 1$, and $g' = n_{y,0}$ with $y \in \mathbb{R}$ for $j = 2$.

Proof. Since the commutativity $\tilde{\mathcal{A}}_{\lambda_j\xi} U_F(g) = U_{S,\lambda_j}(g) \tilde{\mathcal{A}}_{\lambda_j\xi}$ follows easily from a formal calculation (see Remark 2.2), it is sufficient to show that the integrals (4.3) absolutely converge.

Let us write down the integral (4.3) introducing coordinates in the case where $j = 1$. The proof for $j = 2$ is similar, which we will omit. We see that, for $g = n_{p,q}t_r$, $g' = n_{0,x} \in G$,

$$g^{-1}g' = n_{0,x-q}n_{-p,0}t_{-r-p(x-q/2)},$$

and

$$\psi_{\lambda_1}(g^{-1}g') = e^{2\pi i(-r-p(x-q/2))}.$$

On the other hand, since $g = n_{p,q}t_r$ decomposes as $g = n_z n_{\bar{z}}^{-1} t_{r+iH(z,z)/4}$ with $z = q + itp$,

$$R_\xi F(g) = e^{2\pi i(r+iH(z,z)/4)} F(z).$$

Hence, the right-hand side of (4.3) equals

$$\int_{\mathbb{C}} dp dq e^{-\frac{\pi}{2i}|z|^2} e^{-\frac{\pi}{i}((z-\bar{z})x - \frac{1}{4}(z^2 - \bar{z}^2))} F(z). \quad (4.4)$$

Therefore, by the Schwarz inequality, it is sufficient to show that the integral absolutely converges for $F(z) \equiv 1$.

For $\sigma, \tau, a, b \in \mathbb{C}$ given, the integral

$$I = \int_{\mathbb{C}} dx dy e^{-|z|^2 + \sigma z^2 + \bar{\tau} \bar{z}^2 + 2az + 2b\bar{z}} \quad (\text{with } z = x + iy) \quad (4.5)$$

converges absolutely if and only if $|\sigma + \tau| < 1$ (see [Bar61]) and, thus, so does the integral (4.4). This completes the proof. $\square$

Remark 4.3. The integral (4.5) is evaluated as follows:

$$I = \pi(1 - 4\sigma\bar{\tau})^{-1/2} \exp\left(\frac{4(\bar{\tau}a^2 + \sigma b^2 + ab)}{1 - 4\sigma\bar{\tau}}\right), \quad (4.6)$$

provided that the condition $|\sigma + \tau| < 1$ holds.

Note that the intertwining operators $\tilde{\mathcal{A}}_{\lambda_1\xi}$ and $\tilde{\mathcal{A}}_{\lambda_2\xi}$ do not satisfy $\mathcal{F}_{\lambda_2\lambda_1} \tilde{\mathcal{A}}_{\lambda_1\xi} = \tilde{\mathcal{A}}_{\lambda_2\xi}$. However, if we normalize them by

$$\mathcal{A}_{\lambda_1\xi} = \tilde{\mathcal{A}}_{\lambda_1\xi} / \|\tilde{\mathcal{A}}_{\lambda_1\xi} 1\|, \quad \mathcal{A}_{\lambda_2\xi} = \tilde{\mathcal{A}}_{\lambda_2\xi} / \|\tilde{\mathcal{A}}_{\lambda_2\xi} 1\|,$$

then it is easy to see that they do satisfy

$$\mathcal{F}_{\lambda_2\lambda_1} \mathcal{A}_{\lambda_1\xi} = \mathcal{A}_{\lambda_2\xi}.$$

Therefore, by Remark 4.1, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_F & \xrightarrow{\mathcal{A}_{\lambda_1\xi}} & \mathcal{H}_{S,\lambda_1} \\ \mathcal{A}_{\lambda_2\xi} \downarrow & & \downarrow \mathcal{A}_{\eta\lambda_1} \\ \mathcal{H}_{S,\lambda_2} & \xrightarrow{\mathcal{A}_{\eta\lambda_2}} & \mathcal{H}_L. \end{array} \quad (4.7)$$

In particular, we have an identity

$$\mathcal{A}_{\eta\lambda_1} \mathcal{A}_{\lambda_1\xi} 1 = \mathcal{A}_{\eta,\lambda_2} \mathcal{A}_{\lambda_2\xi} 1, \quad (4.8)$$

which, as we will see soon, is equivalent to the Jacobi identity for θ -series.

Let us compute both sides of (4.8). Using (4.6), we obtain

$$(\mathcal{A}_{\lambda_1\xi} 1)(x) = 2^{1/4} t^{-1/4} e^{-(\pi/t)x^2}, \quad (4.9)$$

$$(\mathcal{A}_{\lambda_2\xi} 1)(y) = 2^{1/4} t^{1/4} e^{-\pi t y^2}. \quad (4.10)$$

Applying $\mathcal{A}_{\eta\lambda_1}$ and $\mathcal{A}_{\eta\lambda_2}$ to (4.9) and (4.10), respectively, followed by evaluation at $(u, v) = (0, 0)$ yields

$$(\mathcal{A}_{\eta\lambda_1} \mathcal{A}_{\lambda_1\xi} 1)(0, 0) = 2^{1/4} t^{-1/4} \sum_{n \in \mathbb{Z}} e^{-(\pi/t)n^2} = 2^{1/4} t^{-1/4} \vartheta(1/t), \quad (4.11)$$

$$(\mathcal{A}_{\eta\lambda_2} \mathcal{A}_{\lambda_2\xi} 1)(0, 0) = 2^{1/4} t^{1/4} \sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = 2^{1/4} t^{1/4} \vartheta(t). \quad (4.12)$$

Thus, equating (4.11) and (4.12), we obtain the Jacobi identity (4.1).

Remark 4.4. Since both $\mathcal{F}_{\lambda_2\lambda_1} \tilde{\mathcal{A}}_{\lambda_1\xi}$ and $\tilde{\mathcal{A}}_{\lambda_2\xi}$ are the intertwining operators from $(U_F, \mathcal{H}_F)$ onto $(U_{S,\lambda_2}, \mathcal{H}_{S,\lambda_2})$, they are proportional by Stone-von Neumann theorem. Namely, there exists a nonzero complex number c such that

$$\mathcal{F}_{\lambda_2\lambda_1} \tilde{\mathcal{A}}_{\lambda_1\xi} = c \tilde{\mathcal{A}}_{\lambda_2\xi}.$$

Thus one can say that the Jacobi identity for θ -series is equivalent to the commutativity of the diagram (4.7).

Acknowledgement. I would like to thank Professor M. Wakayama for stimulating discussions and helpful suggestions. He also gave me warm hospitality with his family. Without any of these, this work would have been impossible.

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