SOLVABILITY OF THE TANGENTIAL $\bar{\partial}$ COMPLEX ON HYPERSURFACES WITH GENERAL LEVI SIGNATURE

Luca BARACCO and Giuseppe ZAMPIERI

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Let M be a real, smooth (C^∞) hypersurface of $\mathbb{C}^N$ in a neighborhood of a point z_o . By a choice of complex coordinates $z = (z_1, z', z'', z''')$ with $z = x + \sqrt{-1}y$, we can assume that $z_o = 0$ and M is defined by $\rho_1 = 0$ with

$$\rho_1(z) = y_1 + x_1^2 + |z'|^2 - |z''|^2 + o(\cdot)^2, \quad z' \in \mathbb{C}^{s_M^+}, \quad z'' \in \mathbb{C}^{s_M^-},$$

where $\cdot$ denotes all arguments but y_1 . Here $s_M^\pm = s_M^\pm(z_o)$ are the numbers of positive/negative eigenvalues of the Levi form $L_M(z_o) := \partial\bar{\partial}\rho_1(z_o)|_{T_{z_o}^{\mathbb{C}}M}$. We shall assume throughout the paper that $s_M^- \leq s_M^+$. By replacing ρ_1 by $\rho_2 := \rho_1 - 2\rho_1^2$ (which is still a defining function of M), we get

$$\rho_2(z) = y_1 - x_1^2 + |z'|^2 - |z''|^2 + 2\Re z_1^2 + o(\cdot)^2.$$

We define $\Omega^\mp = \{\pm\rho < 0\}$ ($\rho = \rho_1$ or ρ_2). For small real positive λ , let $\sigma_i = \sigma_i^\lambda$ ($i = 1, 2$) be the functions defined by

$$\begin{aligned}\sigma_1 &= -y_1 - \lambda + 2|z''|^2 + |z'''|^2 \\ \sigma_2 &= +y_1 - \lambda + 2|z'|^2 + |z'''|^2 + 2\Re(z_1 - \sqrt{-1}\lambda)^2.\end{aligned}$$

Define

$$\begin{aligned}u_1 &= -\partial\rho_1(z) + \overline{(2(\xi'' - z''), \xi''' - z''')}, \\ u_2 &= +\partial\rho_2(z) + (\xi_1 - z_1, 0, \dots) + \overline{(2(\xi' - z'), \xi''' - z''')}, \\ u_3 &= (\sqrt{-1}/2, 0, \dots) + \overline{(2\xi'', \xi''')} (= \partial\sigma_1(\xi)), \\ u_4 &= ((-\sqrt{-1}/2 + 2(\xi_1 - \sqrt{-1}\lambda)) + \xi_1 - z_1, \dots) \\ &\quad + \overline{(2\xi', \xi''')} (= \partial\sigma_2(\xi) + (\tfrac{1}{2}\partial_{z_1}^2\sigma_2(\xi)\xi_1 - z_1), 0, \dots),\end{aligned}$$

where we used in the definition of u_2 that $\frac{1}{2}\partial_{z_1}^2\rho_2|_0 = 1$. Let us define the kernels $E_i = E(u_i)$ by

$$E_i = (2\pi\sqrt{-1})^{-N} \left(\frac{u_i \cdot d(\zeta - z)}{u_i \cdot (\zeta - z)} \right) \wedge \left(\frac{\bar{\partial}u_i \cdot d(\zeta - z)}{u_i \cdot (\zeta - z)} \right)^{N-1},$$

and $E_{ij} = E(u_i, u_j)$ by

$$E_{ij} = (2\pi\sqrt{-1})^{-N} \left(\frac{u_i \cdot d(\zeta - z)}{u_i \cdot (\zeta - z)} \right) \wedge \left(\frac{u_j \cdot d(\zeta - z)}{u_j \cdot (\zeta - z)} \right) \\ \wedge \sum_{h+k=N-2} \left(\frac{\bar{\partial}u_i \cdot d(\zeta - z)}{u_i \cdot (\zeta - z)} \right)^h \wedge \left(\frac{\bar{\partial}u_j \cdot d(\zeta - z)}{u_j \cdot (\zeta - z)} \right)^k.$$

The meaning of any other kernel E_{ijh} , E_{ijhk} ... is clear. For a form f on M (with compact support), we define a form (of the same bi-degree) $E_i(f) = E_i([M]^{(0,1)} \wedge f)$, where $[M]^{(0,1)}$ is the $(0, 1)$ component of the current of integration along M . Recall (cf., for example, [2]) that when E_i is not locally integrable in $M \times M$, then the above form has to be understood in some generalized sense: for instance as boundary value from (one side of) $\mathbb{C}^N \setminus M$. We define $A_i = A_i^\lambda$ by

$$A_i^\lambda = M \cap \{\sigma_i < 0\}.$$

We remark that

$$\sigma_1 + \rho_1 = |x_1|^2 + |(z_2, \dots)|^2 - \lambda + o(\cdot)^2, \\ \sigma_2 - \rho_2 = |x_1|^2 + |(z_2, \dots)|^2 - \lambda + o(\cdot)^2.$$

Let $\omega' = A_1 \cap A_2$ and let ω be any domain with C^1 boundary such that $\omega \supset \overline{A_1 \cup A_2}$. Denote by $C = C_r$ the cylinder $C = \{x_1^2 + |z_2|^2 + \dots < r^2\}$. Clearly, for a convenient positive c ,

$$M \cap C_{\lambda^{1/2}-c\lambda} \subset \omega' \subset \omega \subset M \cap C_{\lambda^{1/2}+c\lambda}.$$

We shall consider forms $f = \sum'_{|I| \leq j} f_I d\bar{z}_I$ of type $(0, j)$ with C^∞ coefficients. (Here $\sum'$ denotes the sum restricted to ordered multi-indices $I = (i_1, \dots, i_j)$ with $i_1 \leq i_2 \leq \dots$ and $d\bar{z}_I$ stands for $d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_j}$.) Since all forms shall be supposed to be anti-holomorphic, we shall only refer to their degree j instead of their type $(0, j)$. We shall denote by $\bar{\partial}_M$ the tangential Cauchy–Riemann operator.

THEOREM 1. *Let $j < s_M^-$. Then the equation $\bar{\partial}_M u = f$ has a solution u in ω' for f in ω of degree j verifying $\bar{\partial}_M f = 0$. Moreover, if $\chi = \chi_\omega$ is the characteristic function*

of ω , we have the integral representation for the solution:

$$u = E_{12}(\chi f) - (E_{123}([\partial\omega]^{(0,1)} \wedge f) - E_{234}([\partial\omega]^{(0,1)} \wedge f)) = a_1 + a_2 \quad (1)$$

with the estimates

$$|D^\alpha a_1| \leq c |D^\alpha f|, \quad |a_2| \leq c \operatorname{dist}_{\partial\omega}^{-2N}. \quad (2)$$

Proof. For $i = 1, 2$ we have, for any $c < 1$ and in a suitable neighborhood of $z_o = 0$,

$$\begin{aligned} 2 \Re \langle u_i, \zeta - z \rangle &\geq |\rho_i(z) - \rho_i(\zeta)| + c[\Re(\zeta_1 - z_1)^2 + |\zeta_2 - z_2|^2 + \dots] \\ &\text{if } \rho_1(\zeta) \leq 0, \rho_1(z) \geq 0 \text{ (respectively } \rho_2(\zeta) \geq 0, \rho_2(z) \leq 0). \end{aligned} \quad (3)$$

For $i = 3, 4$, we have

$$\begin{aligned} 2 \Re \langle u_i, \zeta - z \rangle &= 2 \Re \langle \partial\sigma_{i-2}(\zeta), \zeta - z \rangle \geq |\sigma_{i-2}(z) - \sigma_{i-2}(\zeta)| \\ &\text{if } \sigma_{i-2}(\zeta) \geq 0, \sigma_{i-2}(z) \leq 0. \end{aligned} \quad (4)$$

Both (3) and (4) are elementary. As a consequence of (3), $E_1(g)|_{\Omega^-}$ and $E_2(g)|_{\Omega^+}$ extend continuously to M (cf. [2, Theorem 2, p. 304]); we shall denote such extensions by $E_i(g)^\mp$. As a consequence of (4), the kernels $E_{13}, E_{14}, E_{23}, E_{24}$ are bi-regular in $\{(\zeta, z) \in M \times M : \sigma_i(\zeta) \geq 0, \sigma_i(z) < 0\}$; in particular in $\partial\omega \times \omega'$. We also have the following.

LEMMA 2.

- (i) $E_1(g) = 0, E_{13}(g) = 0$ if $\deg(g) \leq s_M^+ - 1$. Moreover, $E_{13}(g) = 0$ for $g = [\partial\omega]^{(0,1)} \wedge f$ with $\deg(f) = s_M^+ - 1$ and $\bar{\partial}_M(f) = 0$.
- (ii) $E_2(g) = 0, E_{24}(g) = 0$ if $\deg(g) \leq s_M^- - 1$. Moreover, $E_{24}(g) = 0$ if $g = [\partial\omega]^{(0,1)} \wedge f$ with $\deg(f) = s_M^- - 1$ and $\bar{\partial}_M(f) = 0$.

Proof. We shall only prove (i). ((ii) is similar.) With $E = E_1$ or $E = E_{13}$, we have $\deg(E) \leq s_M^- + s_M^0$. Hence the first half of the statement is obvious.

As for the second-half part, it is a ‘version with parameters’ of the approximation method by Henkin for local solvability in degree $N - 2$ when M is strongly pseudoconvex (cf. [2, Lemma 2, p. 331]). Let $\Omega^- = \{\rho_1 < 0\}$, $D^- = \{\sigma_1 < 0\}$ and write, for $z \in \omega', \zeta \in \partial\Omega$, $[E]^{(0, s_M^- + s_M^0)} = h d\bar{\zeta}'' \wedge d\bar{\zeta}'''$, where $\bar{\partial}_{\zeta_1, \zeta'} h = 0$ (i.e. h is partially holomorphic in ζ_1, ζ'). We begin with by observing that

$$\bar{\partial}_M([E]^{(0, s_M^- + s_M^0)} \wedge f)|_{\Omega \setminus \bar{A}_1} = 0.$$

This follows from the fact that h is holomorphic in a neighborhood of $\omega \setminus \bar{A}_1$ (while $\bar{\partial}_M(f) = 0$ there). Next we remark that if we fix $z \in \omega'$, then the coefficient $h = h_z$ can be extended to a neighborhood of $B_1 := \Omega^- \cap \{\sigma_1 = 0\}$ (depending on z) as a holomorphic function of ζ_1, ζ' . This follows from (3) for $i = 1$ with the role of ζ and z interchanged. On the other hand, the sections $\hat{B}_1$ of B_1 with the planes $\zeta'', \zeta''' \equiv \text{const.}$ are convex. It follows that for fixed ζ'', ζ''' there exists a sequence of entire functions $\{h_\nu\}$ such that $h_\nu \rightarrow h$ on $\hat{B}_1$. By a partition of the unity, for any ϵ there exists h_ϵ , entire in ζ_1, ζ' and C^∞ in ζ'', ζ''' such that $|h_\epsilon - h| < \epsilon$ in $\hat{B}_1$. It follows, if $\text{degree}(f) = s_M^+ - 1$ that

$$\begin{aligned} \int_M [E_{13}]^{(0, s_M^- + s_M^0)} \wedge [\partial\omega]^{(0,1)} \wedge f &= \int_{\partial\omega} [E]^{(0, s_M^- + s_M^0)} \wedge f \\ &= \int_{\partial\omega} h_\epsilon d\bar{\zeta}'' \wedge d\bar{\zeta}''' \wedge f \pm c\epsilon \\ &= \int_\omega \bar{\partial}_M(h_\epsilon d\bar{\zeta}'' \wedge d\bar{\zeta}''' \wedge f) \pm c\epsilon \\ &= \pm c\epsilon \quad \forall \epsilon. \end{aligned} \tag{5}$$

This concludes the proof of (i). The proof of (ii) is similar. $\square$

End of proof of Theorem 1. We observe that the coefficients of $u_1 d(\zeta - z) \wedge u_2 d(\zeta - z)$ are estimated by $c|\zeta - z|$ (due to $|u_1 + u_2| \leq c|\zeta - z|$). Therefore [2, Lemma 1, pp. 319–322] applies and leads to [2, Theorem 1, p. 318]:

$$I = E_2^+ - E_1^- + \bar{\partial}_M \circ E_{12} - E_{12} \circ \bar{\partial}_M. \tag{6}$$

(Here the superscripts $\pm$ mean that we are taking boundary values from $\Omega^\pm$.) Also by *loc. cit.* (second statement in Theorem 1, p. 318) we have for any m

$$|E_{12}(f)|_{C^m} \leq c|\chi f|_{C^m}. \tag{6'}$$

We now want to solve the term $E_{12} \circ \bar{\partial}_M$. By [2, Theorem 1, p. 297] we have

$$\begin{aligned} E_{12} &= -\bar{\partial}_M E_{123} + E_{13} - E_{23} \\ E_{23} &= E_{24} - E_{34} - \bar{\partial}_M E_{234}, \end{aligned}$$

whence

$$E_{12} = \bar{\partial}(-E_{123} + E_{234}) + E_{13} - E_{24} + E_{34}. \tag{7}$$

If we apply (6) and (7) to χf , observe that $\bar{\partial}_M(\chi f) = [\omega]^{(0,1)} \wedge f$, recall Lemma 2 (and remark that clearly E_{34} vanishes over any form of degree ≥ 1), we conclude that

$$\chi f = \bar{\partial}_M(E_{12}(\chi f) - (E_{123} - E_{234})([\partial\omega]^{(0,1)} \wedge f)).$$

If we denote by $u = a_1 + a_2$ the term between brackets in the above equality, we now want to prove estimate (2) of Theorem 1. The first is just another way of writing (6'). As for the second, recall (3), (4) and observe that when $z \in \omega'$ and $\zeta \in \partial\omega$, then

$$|\sigma_{i-2}(z) - \sigma_{i-2}(\zeta)| \geq \epsilon|(z - \zeta)| \\ \epsilon \operatorname{dist}_{\partial\omega}(z),$$

where the first inequality follows from the fact that M intersects transversally the hypersurface $\{\sigma_{i-2} = 0\}$. $\square$

In a similar way we can prove the following theorem for a fundamental system of neighborhoods $\{\omega\}$ of z_0 .

THEOREM 3. *Let f have degree j and satisfy $\bar{\partial}_M f = 0$ in ω . Then if $j > s_M^+ + s_M^0$ or $s_M^- + s_M^0 < j < s_M^+$, the equation $\bar{\partial}_M u = f$ has a solution u in ω . Moreover, u has a description by kernels of type (1) which satisfies (2). (Note that in both cases there is no shrinking in the domain of existence of the solution.)*

Proof. (a) Assume first that $j > s_M^+ + s_M^0$. Define

$$\begin{aligned} u_1 &= \partial\rho_1(\zeta) + \overline{(2(\zeta'' - z''), \zeta''' - z''')} \\ u_2 &= -\partial\rho_2(\zeta) + (\tfrac{1}{2}\partial_{z_1}^2 \rho_2(\zeta)(\zeta_1 - z_1), 0, \dots) + \overline{(2(\zeta' - z'), \zeta''' - z''')} \\ u_3 &= \overline{(\zeta_1, \zeta_2, \dots)}. \end{aligned} \quad (8)$$

Note that

$$2\Re\langle u_i, \zeta - z \rangle \geq |\rho(\zeta) - \rho(z)| + c[\Re(\zeta_1 - z_1)^2 + |\zeta_2 - z_2|^2 + \dots], \quad (9)$$

for $\rho_1(\zeta) \geq 0$ and $\rho_1(z) \leq 0$ (respectively $\rho_2(\zeta) \leq 0$ and $\rho_2(z) \geq 0$). We then reason as in Theorem 1. Note, however, that we obtain the vanishing of $E_{i3}([\partial\omega]^{(0,1)} \wedge f)$ only for reason of degree (without using the approximation argument).

(b) For $s_M^- + s_M^0 < j < s_M^+$, we define, with $\rho = \rho_1$,

$$\begin{aligned} u_1 &= \partial\rho(\zeta) + \overline{(2(\zeta'' - z''), \zeta''' - z''')} \\ u_2 &= -\partial\rho(z) + \overline{(2(\zeta'' - z''), \zeta''' - z''')} \\ u_3 &= \tfrac{1}{2}(\sqrt{-1}, 0, \dots) + \overline{(2\zeta'', \zeta''')}. \end{aligned}$$

We still have (9). We then carry out the proof as in Theorem 1, and use approximation for showing that $E_{i3}([\partial\omega]^{(0,1)} \wedge f) = 0$. $\square$

Remark 4. In Theorem 1 (respectively Theorem 3) we do not need to assume $s_M^- \leq s_M^+$. In the general setting we can modify the above proofs and prove solvability for $j \leq \inf(s_M^-, s_M^+) - 1$ (respectively $\inf(s_M^-, s_M^+) + s_M^0 < j < \sup(s_M^-, s_M^+)$ and $j \geq \sup(s_M^-, s_M^+) + s_M^0 + 1$).

We now discuss solvability at the top degree. The main statement has been given in [6] but we want to recover it in our setting as an application of the above preparation.

We assume, for simplicity, $s_M^- < s_M^+$. For $\omega' \subset \omega$ as in Theorem 1, and with $\chi = \chi_\omega$, recall the representation by kernels of χf .

PROPOSITION 5. *In the situation of Theorem 1, suppose $\text{degree}(f) = s_M^-$ (and $\bar{\partial}_M f = 0$). Then the equation $\bar{\partial}_M u = f$ has solution u in a neighborhood of z_o if and only if the form*

$$g := E_2(\chi f) + E_{12}([\partial\omega]^{(0,1)} \wedge f), \quad (10)$$

which is $\bar{\partial}_M$ closed in the same way as f , is induced by G defined and $\bar{\partial}$ closed in a full neighborhood of z_o .

This means that there is G with

$$\bar{\partial}G = 0 \quad \text{and} \quad j^*G = g, \quad (11)$$

where $j : M \hookrightarrow \mathbb{C}^N$ is the natural embedding. Note that in the above statement there is no control of the domain of existence of the solution.

Proof. If (10) is induced by G , we integrate (in a ball) $\bar{\partial}_{\mathbb{C}^N} U = G$, and define

$$u = E_{12}(\chi f) + j^*U.$$

This is clearly a solution of $\bar{\partial}_M u = f$.

Conversely, if u solves $\bar{\partial}_M u = f$, we write $u' = u - E_{12}(\chi f)$, extend u' to U' arbitrarily, and define $G := \bar{\partial}U'$. $\square$

We assume henceforth that M is real-analytic.

Remark 6. $E_{12}([\partial\omega]^{(0,1)} \wedge f)$ is real-analytic in ω' for any degree j (including $j = s_M^-$). In fact

$$\begin{cases} u_i \text{ is real-analytic} \\ \langle u_i, \zeta - z \rangle \neq 0 \text{ if } \zeta \in \partial\omega \text{ and } z \in \omega'. \end{cases}$$

Let f have degree $j = s_M^-$ and solve $\bar{\partial}_M f = 0$. Note that by the same argument as in Remark 6, the condition

$$E_2(\chi f) \text{ is real-analytic at } z_o \quad (12)$$

is independent of χ .

THEOREM 7. *The equation $\bar{\partial}_M u = f$ has a solution u at z_o if and only if (12) is satisfied.*

Proof. When M is strongly pseudoconvex, Henkin obtained the above criterion for $N - 1$ forms. Let us sketch his proof (adapted to general Levi signatures). Note that by Remark 6, (12) is equivalent to the fact that

$$g \text{ is real-analytic.} \quad (13)$$

Assume (13). By the Cauchy–Kowalewsky theorem, we can find u real-analytic such that $\bar{\partial}_M u = g$.

Conversely, assume the solvability of $\bar{\partial}_M u = f$. Then

$$\begin{aligned} E_2(\chi f) &= E_2(\chi \bar{\partial}_M u) = \int_{\omega} \bar{\partial}_M (E_2 \wedge u) - \int_{\omega} (\bar{\partial}_M E_2) \wedge u \\ &= E_2([\partial \omega]^{(0,1)} \wedge u), \end{aligned} \quad (14)$$

(due to $\bar{\partial}_{\zeta}[E_2]^{(0, s_M^+ + s_M^0)} = 0$).

Finally the last term in (14) is real-analytic by the same argument as in Remark 6. $\square$

Remark 7. When $s_M^- > s_M^+$ (respectively $s_M^- = s_M^+$), then in (12) we have to replace $E_2(\chi f)$ by $E_1(\chi f)$ (respectively $E_1(\chi f) + E_2(\chi f)$). Here, $E_2 = E(u_2)$ is the kernel defined in part (b) of the proof of Theorem 3.

Remark 8. When the degree j of f equals s_M^+ , one proves that there is solution u of $\bar{\partial}_M u = f$ if and only if $E_2(\chi f)$ is real-analytic. In particular, when $s_M^+ = N - 1$ (i.e. Ω^- is strictly pseudoconvex), one recovers the criterion by Henkin (cf., for example, [2, Theorem 1, p. 338]). Note that there are no reasonable conditions of solvability for $s_M^- < j \leq s_M^- + s_M^0$ and $s_M^+ < j \leq s_M^+ + s_M^0$ (i.e. the condition of real-analyticity is still sufficient but no longer necessary for existence of solutions).

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Luca Baracco and Giuseppe Zampieri

Dipartimento di Matematica

Universita di Padova

via Belzoni 7

35131 Padova

Italy

(E-mail: {baracco,zampieri}@math.unipd.it)