NOTE ON HOMOTOPY NORMALITY AND THE n -CONNECTED FIBER SPACE

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1. Introduction

Given an H -map $i : Y \rightarrow X$, we say that i is mod p homotopy normal if the commutator map from $X_{(p)} \times Y_{(p)}$ to $X_{(p)}$ can be deformed into $Y_{(p)}$ [J, Mi]. In the paper [KY], conditions of mod p homotopy normality are studied by using the Morava K -theory. For example, if $H^*(Y; \mathbb{Z}/3)$ does not have any 19-dimensional primitive elements and if an H -map $i : Y \rightarrow F_4$ is mod 3 homotopy normal, then $i^*H^*(F_4; \mathbb{Z}/3)$ is isomorphic to one of the mod 3 cohomologies of F_4 , $Spin(9)$, G_2 and a *point*.

On the other hand Hemmi [He] used the 3-connected fiber space to show the non-existence of some types of loop spaces ($Q_{\mathbb{Z}/p}^*$ in Section 3 for $p \geq 7$). We will see that the above 19-dimensional condition is necessary by using the 3-connective fiber of F_4 . Indeed the natural map $i_{(n)}$ from the n -connected space $X\langle n \rangle$ to X is homotopy normal. Using this property, we study curious relations between Morava K -theory and the ordinary mod p -cohomology of $X\langle n \rangle$. For example, $i_{(n)}^*H^*(F_4; \mathbb{Z}/3) = 0$ for $n \geq 11$, but $i_{(n)}^*K(2)^*(F_4) \cong K(2)^* \otimes H^*(Spin(9); \mathbb{Z}/3)$ for $n \geq 3$. In this paper, the prime p is always assumed *odd*.

2. mod p homotopy normality

Throughout this paper, let Y and X be simply connected homotopy associative H -spaces such that $H^*(X; \mathbb{Z}/p)$ and $H^*(Y; \mathbb{Z}/p)$ are finitely generated as rings, and $i : Y \rightarrow X$ be an H -map. The commutator map $c_X : X \times X \rightarrow X$ is defined by

$$c_X = \mu(\mu \times \mu)(1 \times 1 \times \sigma \times \sigma)(1 \times tw \times 1)(d_X \times d_X)$$

where d_X is the diagonal, tw is the twisting map, σ is the inverse and μ is the multiplication map of X . Of course, when X is a topological group, $c_X(g, h) = ghg^{-1}h^{-1}$ for $g, h \in X$.

Define an H -map $i : Y \rightarrow X$ is mod p homotopy normal if there exists a map $\tilde{c} : X_{(p)} \times Y_{(p)} \rightarrow Y_{(p)}$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X_{(p)} \times Y_{(p)} & \xrightarrow{\tilde{c}} & Y_{(p)} \\ \downarrow id \times i & & \downarrow i \\ X_{(p)} \times X_{(p)} & \xrightarrow{c_X} & X_{(p)}. \end{array}$$

Let $h_*(-)$ be the mod p ordinary homology $H_*(-; Z/p)$ or the Morava K -theory $K(n)_*(-)$ with the coefficient $K(n)_* = Z/p[v_n, v_n^{-1}]$, $|v_n| = 2(p^n - 1)$. For these theories, $h_*(X)$ are Hopf algebras with the multiplication μ_* and the comultiplication d_X^* . Hence they are cocommutative but, in general, not commutative.

LEMMA 2.1. [R] *If $x, y \in h_*(X)$ are primitive, then*

$$c_{X*}(x \otimes y) = [x, y] = xy - (-1)^{|x||y|}yx.$$

COROLLARY 2.2. *If $i : Y \rightarrow X$ is mod p homotopy normal and if $x \in h_*(X)$ and $y \in h_*(Y)$ are primitive, then $[x, i_*(y)] \in i_*h_*(Y)$.*

3. H -spaces with one even degree generator

If X is a finite H -space, then by the Borel theorem, the mod p cohomology $H^*(X; Z/p)$ is a tensor algebra of truncated polynomial and exterior algebras generated by even and odd-dimensional elements respectively. In this section, we consider the case of having one even degree generator and the form

$$H^*(X; Z/p) \cong Z/p[y]/(y^p) \otimes \Lambda, \quad |y| = 2p + 2 \quad (3.1)$$

where Λ is an exterior algebra generated by odd degree elements. Then it is also known by Kane [Ka] that there exist generators $x_0, x'_0 \in \Lambda$ such that

$$\wp^1 x_0 = x'_0 \quad \text{and} \quad \beta x'_0 = y \quad \text{with} \quad |x_0| = 3, \quad |x'_0| = 2p + 1.$$

For such H -space X , the Morava K cohomology ring $K(2)^*(X)$ is just a tensor product $K(2)^*(X) \cong K(2)^* \otimes H^*(X; Z/p)$, and the Hopf algebra structure is given in [Y].

THEOREM 3.1. [Y, Ka] *Let X be an H -space satisfying (3.1). Then $H^*(X; Z/p)$ (respectively $K(2)^*(X)$) has a quotient Hopf algebra*

$$Q_K^* = K[y]/(y^p) \otimes \Lambda(x_i, x'_i \mid 0 \leq i \leq p-2) \quad \text{with } K = Z/p \text{ (respectively } K(2)^*),$$

$|x_i| = 2(p+1)i+3$, $|x'_i| = 2(p+1)(i+1)-1$. Here the dual Hopf algebra $Q_{K(2)*}$ is multiplicatively generated by z, z', y with the relations $ad^{p-1}(y)(z) = -v_2z$, $ad^{p-1}(y)(z') = -v_2z'$, $y^p = -v_2y$ and $ad(z)(z') = 0$ where $ad(y)(z) = [y, z]$. The elements x_i, x'_i are dual of $ad(y)^i(z), ad(y)^i(z')$ respectively. The relation in $H_*(X; Z/p)$ is given from that of $K(2)_*(X)$ by letting $v_2 = 0$.

THEOREM 3.2. [KY] Let $i : Y \rightarrow X$ be mod p homotopy normal for an H -space X satisfying (3.1). Let $Q^* = Q_{K(2)}^*$. Suppose that the quotient map to Q^* splits, i.e. $Q^* \subset K(2)_*(X)$ as Hopf algebras. Then i^*Q^* is isomorphic to one of the following Hopf algebras:

$$Q^*, \quad Q^*/(y), \quad K(2)^* \otimes \Lambda(x_i \mid 0 \leq i \leq p-2), \\ K(2)^* \otimes \Lambda(x'_i \mid 0 \leq i \leq p-2), \quad K(2)_*.$$

The idea of the above theorem is the following. Suppose $i^*(x_j) \neq 0 \in K(2)^*(Y)$ for some j . Let us write by $\tilde{z}_j$ the dual of $i^*(x_j)$. From Corollary 2.2, we see $[y, i_*(\tilde{z}_j)] \in \text{Im}(i_*)$. Take $\tilde{z}_{j+1} \in K(2)_*(Y)$ such that $i_*\tilde{z}_{j+1} = [y, i_*(\tilde{z}_j)]$. Similarly we can take $\tilde{z}_k$ for all $k \geq j$. Letting $\tilde{z}_0 = v_2^{-1}\tilde{z}_{p-1}$, we can prove

$$K(2)_* \otimes \Lambda(\tilde{z}_i \mid 0 \leq i \leq p-2) \subset K(2)_*(Y).$$

By taking this dual, we can prove the above theorem.

THEOREM 3.3. [KY] Let $i : Y \rightarrow X$ be mod p homotopy normal for an H -space X satisfying (3.1). Let $Q^* = Q_{Z/p}^*$. Suppose that there does not exist any primitive element of degree $2(p^2-1)+3$ nor $2(p^2-1)+2p+1$ in $H^*(Y; Z/p)$ and that $Q^* \subset H^*(X; Z/p)$ as Hopf algebras. Then $i^*(Q^*)$ is isomorphic to one of the following Hopf algebras:

$$Q^*, \quad Q^*/(y), \quad \Lambda(x_i \mid 0 \leq i \leq p-2), \quad Z/p.$$

The idea of arguments for the dimensional conditions $2(p^2+1)+3$ or $2(p^2-1)+2p+1$ in $H^*(Y; Z/p)$ is the following. Suppose that $i : Y \rightarrow X$ is mod p homotopy normal. From the Pontrjagin structure of $H_*(X; Z/p)$, there is the possibility of $i^*(Q^*)$ isomorphic to

$$\Lambda(x_i \mid k \leq i \leq p-2), \quad \text{for } 1 \leq k \leq p-2.$$

Suppose this. However note that $i^*(Q_K^*) = K(2)^* \otimes \Lambda(x_i \mid 0 \leq i \leq p-2)$ in $K(2)^*(Y)$. Hence there exists $\tilde{x}_{p-1}$ in $H^*(Y; Z/p)$ such that

$$\tilde{x}_{p-1} = v_2^{-1}i^*(x_0) \quad \text{in } K(2)^*(Y), \quad \text{and} \quad |\tilde{x}_{p-1}| = 2(p^2-1)+3. \quad (3.2)$$

COROLLARY 3.4. *Let $i : Y \rightarrow X$ be mod p homotopy normal for an H -space satisfying (3.1). If $i^*(x_i) = 0$ for all $0 \leq i < j$ but $i^*(x_j) \neq 0$ for some j in $H^*(Y; Z/p)$, then there exist primitive elements $\tilde{x}_{i+p-1} \in H^*(Y; Z/p)$ satisfying $i^*(x_i) = v_2 \tilde{x}_{i+p-1} \in K(2)^*(Y)$.*

We remark that the above corollary also holds if we exchange x_i to x'_i .

Hemmi [He] showed that there does not exist a loop space X such that $H^*(X; Z/p)$ is isomorphic to just Q^* for $p \geq 7$. On the other hand, for the cases $p = 3, 5$, the quotient Hopf algebra Q^* are isomorphic to $H^*(F_4; Z/3)$, $H^*(E_8; Z/5)$ respectively [Mi, KM]:

$$\begin{aligned} H^*(F_4; Z/3) &\cong Z/3[y_8]/(y_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}) \\ H^*(E_8; Z/5) &\cong Z/5[y_{12}]/(y_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \end{aligned}$$

where the subscript means the degree, i.e. $|x_i| = i$. The reduced powers are also known $\wp^1 x_i = x_{i+2(p-1)}$ (namely, this means $\wp^1 x_i = x'_i$ for the notations in Theorem 3.1).

COROLLARY 3.5. [KY] *Let $i : Y \rightarrow F_4$ (respectively E_8) be a mod 3 (respectively 5) homotopy normal map. Suppose that there does not exist any primitive element in $H^*(Y; Z/p)$ of degree 19 (respectively 51). Then $i^*H^*(F_4; Z/3)$ (respectively $i^*H^*(E_8; Z/5)$) is isomorphic to one of the following Hopf algebras*

$$\begin{aligned} &H^*(F_4; Z/3), \quad H^*(F_4; Z/3)/(y_8), \quad \Lambda(x_3, x_{11}), \quad Z/3 \\ &\text{(respectively } H^*(E_8; Z/5), \quad H^*(E_8; Z/5)/(y_{12}), \quad \Lambda(x_3, x_{15}, x_{27}, x_{39}), \quad Z/5). \end{aligned}$$

4. H -spaces with two even degree generators

In this section, we consider an H -space X having two different even degree generators and the form of

$$H^*(X; Z/p) \cong Z/p[y, u]/(y^p, u^p) \otimes \Lambda, \quad |y| = 2p + 2, \quad |u| = 2p^2 + 2. \quad (4.1)$$

Here also by Kane [Ka], we can write $\Lambda = \Lambda(x, x', w, w') \otimes \Lambda'$ with $\wp^1 x = x'$, $\beta x' = y$, $\wp^p x = w$, $\wp^1 w = w'$, $\beta w' = u$. Hence we know that $Q_1 x = y$, $Q_1 w = u$ and $Q_2 x = u$ for the Milnor operation. Then we can compute the Morava K cohomology ring.

LEMMA 4.1. [Y] For an H -space X in (4.1), the Morava K cohomology ring is

$$\begin{aligned} K(3)^*(X) &\cong K(3)^* \otimes H^*(X; \mathbb{Z}/p) \\ K(2)^*(X) &\cong K(2)^* \otimes \mathbb{Z}/p[y]/(y^p) \otimes \Lambda(xu^{p-1}, x', w, w') \otimes \Lambda'. \end{aligned}$$

Let us write the dual elements of x, x', w, w', y, u by z, z', v, v', y, u respectively in the Morava K -homology. From [Y], we also know that

$$y^p = -v_2 y \quad \text{in } K(2)_*(X) \quad (4.2)$$

$$u^p = -v_3 y \quad \text{in } K(3)_*(X). \quad (4.3)$$

The following theorem is an extension of a theorem of Kane [Ka, Theorem C, p. 392] for type (4.1).

THEOREM 4.2. Let X be an H -space of type (4.1). Then $ad(u)^{p^2-p-1}(v') \neq 0$ and $ad^{p-1}(u)(v) = v_3 z$ in $K(3)_*(X)$. The cohomology $H^*(X; \mathbb{Z}/p)$ contains as an algebra

$$\mathcal{Q}_{\mathbb{Z}/p}^* = \mathbb{Z}/p[u, y]/(u^p, y^p) \otimes \Lambda(w'_j, w_i \mid 0 \leq j \leq p^2 - p - 1, 0 \leq i \leq p - 1)$$

or

$$P_{\mathbb{Z}/3}^* = \mathcal{Q}_{\mathbb{Z}/3}^*/(w'_{p^2-p-1} = w_0) \quad \text{and} \quad p = 3.$$

Here in $K(3)^*(X)$, w'_j (respectively w_j) is the dual of $v_3^{-s} ad(u)^j(v')$ = $(ad(u)^{k+1} ad(y)^{s-1}(z'))$ when $s \geq 1$ (respectively $v_3^{-s} ad(u)^j(v)$) for $j = sp + k$ with $-1 \leq k \leq p - 2$, and $|w'_j| = 2((k+1)p^2 + ps + k + s) + 1$, $|w_j| = |w'_j| - 2p$.

Proof. (See [Y] for $p = 3$.) In $K(2)_*(X)$, apply Q_0 to the equation $y^p = -v_2 y$. Since $Q_0 y = z'$, we have $ad^{p-1}(y)(z') = -v_2 z'$. Hence $ad^{p-2}(y)(z') \neq 0$ in $H_*(X; \mathbb{Z}/p) \cong k(2)_*(X)$ for $0 < * < 2(p^2 - 1)$, where $k(2)_*(-)$ is the connective Morava K homology theory. Therefore, they are also non-zero in $K(3)_*(X)$. Since $v_3 y = -u^p$, we see $ad(u)^{p(p-2)}(z') = -v_3 ad(y)^{p-2}(z') \neq 0$ in $K(3)_*(X)$. Apply Q_0 to $v_3 y = -u^p$, we get $v_3 z' = -ad^{p-1}(u)(v')$. Therefore in $K(3)_*(X)$, we have $ad^{p(p-2)+p-1}(u)(v') \neq 0$. Applying Q_1 to $v_3 y = -u^p$, we get $v_3 z = -ad^{p-1}(u)(v)$ and so $ad(u)^{p-2}(z) \neq 0$. $\square$

COROLLARY 4.3. Let $i : Y \rightarrow X$ be mod p homotopy normal map for an H -space of type (4.1). If $i^*(w'_j) \neq 0 \in K(3)^*(X)$, then so is $i^*(w'_k)$ for all $p^2 - p - 1 \geq k \geq j$.

COROLLARY 4.4. Suppose the same assumption as the above corollary. Given $j = sp + k$, $-1 \leq k \leq p - 2$, suppose also that there is not any primitive element

in $H^*(Y; Z/p)$ of dimension $|v_3 z|$ nor $|v_3 \text{ad}^i(y)(v')|$ for all $s \leq i \leq p-2$. Then if $i^*(w'_j) \neq 0$ in $H^*(Y; Z/p)$, then so is $i^*(w'_h)$ for all $p^2 - p - 1 \geq h \geq j$.

However, the known example for (4.1) is only the case $p = 3$ and $X = E_8 \times X'$ for some X' such that $H^*(X'; Z/3)$ is isomorphic to an exterior algebra. The ordinary mod 3 cohomology of E_8 is isomorphic to $P_{Z/3}^*$ defined in Theorem 4.2 [Mi, KM].

$$H^*(E_8; Z/3) \cong Z/3[y_8, u_{20}]/(y_8^3, u_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}).$$

THEOREM 4.5. [Y] *The Pontrjagin ring $K(3)_*(E_8)$ is generated by two elements $u, z' = z_{19}$, with the relations $u^9 = 0$, $\text{ad}(u)^8(z') = 0$, $(z')^2 = 0$ (and $u^3 = -v_3 y$). The adjoint map is given by the following arrows, i.e. $z \rightarrow [u, z]$,*

$$\begin{aligned} z_{19} &\rightarrow z_{39} \rightarrow -v_3 z_7, & z_7 &\rightarrow z_{27} \rightarrow z_{47} \rightarrow -v_3 z_{15} \\ z_{15} &\rightarrow z_{35} \rightarrow -v_3 z_3, & z_3 &\rightarrow 0. \end{aligned}$$

THEOREM 4.6. [KY] *Let $i : Y \rightarrow E_8$ be mod 3 homotopy normal map. Let $(a_0, \dots, a_7)$ be the ordered set $(19, 39, 7, 27, 47, 15, 35, 3)$. Suppose that there does not exist any primitive element of degree 55, 59 nor 67. Then $i^*H^*(E_8; Z/3)$ is isomorphic to one of the following Hopf algebras:*

$$\begin{aligned} &\Lambda(x_{a_j}, x_{a_{j+1}}, \dots, x_{a_7}), \quad \text{for } 0 \leq j \leq 7, \\ &Z/p[y]/(y^3) \otimes \Lambda(x_{a_j}, \dots, x_{a_7}), \quad \text{for } 0 \leq j \leq 2 \\ &Z/3 \quad \text{and} \quad H^*(E_8; Z/3). \end{aligned}$$

5. The 3-connective fiber space

We give examples of Corollary 3.6 and Theorem 4.6, in this section. Since X is simply connected, it is well known $\pi_3(X) \cong Z$. So we have two fiberings:

$$K(Z, 2) \rightarrow X\langle 3 \rangle \xrightarrow{i_X} X \quad (5.1)$$

$$X\langle 3 \rangle \xrightarrow{i_X} X \xrightarrow{x} K(Z, 3) \quad (5.2)$$

where $X\langle 3 \rangle$ is the 3-connective fiber space over X .

LEMMA 5.1. *The map $i_X : X\langle 3 \rangle \rightarrow X$ is homotopy normal.*

Proof. Consider the composition map c_X and x

$$xc_X : X \times X \rightarrow X \rightarrow K(Z, 3).$$

Since x is also a map of H -spaces, we see $xc_X \sim c_{K(Z,3)}(x \times x)$. This map is homotopic to zero, since $K(Z, 3)$ is a double loop space and a homotopy commutative H -spaces, i.e. $c_{K(Z,3)} \sim 0$. Since (5.2) is a fibering, there is a map $\tilde{c} : X \times X \rightarrow X\langle 3 \rangle$ such that $i_X \tilde{c} \sim c_X$. This means that $i_X : X\langle 3 \rangle \rightarrow X$ is homotopy normal. (More strongly commutator subgroup of X is contained homotopically in $X\langle 3 \rangle$.) $\square$

The cohomology of $X\langle 3 \rangle$ is computed in [KM, He]. Suppose X is of type (3.1), i.e.

$$H^*(X; Z/p) \cong Z/p[y]/(y^p) \otimes \Lambda(x_0, x'_0) \otimes \Lambda'. \quad (5.3)$$

We consider the Serre spectral sequence induce from (5.1)

$$E_2^{*,*} \cong H^*(X; Z/p) \otimes Z/p[t] \Rightarrow H^*(X\langle 3 \rangle; Z/p) \quad (5.4)$$

since $H^*(K(Z, 2); Z/p) \cong Z/p[t]$ with $|t| = 2$. The differential $d_3(t) = x_0$ and hence $E_4^{*,*} \cong Z/p[y, t^p]/(y^p) \otimes \Lambda(x_0 t^{p-1}, x'_0) \otimes \Lambda'$. The next non-zero differential is the Kudo transgression, $d_{2p-1}(x_0 t^{p-1}) = \beta \wp^1 x_0 = y$. Hence we get $E_{2p}^{*,*} \cong Z/p[t^p] \otimes \Lambda(x_0 t^{p-1} y^{p-1}, x'_0) \otimes \Lambda'$. The last differential is $d_{2p+1}(t^p) = \wp^1 x_0 = x'_0$. Thus we have

$$E_{2p+2}^{*,*} \cong Z/p[t^{p^2}] \otimes \Lambda(x_0 y^{p-1} t^{p-1}, x'_0 t^{p(p-1)}) \otimes \Lambda'.$$

Therefore we have the following.

THEOREM 5.2. *Let X be an H -space of type (3.1) such that $\wp^p \wp^1(x_0) = 0$ and Λ' does not contain generators of dimension $2s(p^2 - 1) + 3$ for $s \geq 2$. Then*

$$\begin{aligned} H^*(X\langle 3 \rangle; Z/p) &\cong Z/p[t_{2p^2}] \otimes \Lambda(x_{p-1}, x'_{p-1}) \otimes \Lambda', \\ K(2)^*(X\langle 3 \rangle) &\cong K(2)^* \otimes H^*(X\langle 3 \rangle; Z/p) \end{aligned}$$

such that in $K(2)^*(X\langle 3 \rangle)$, $x_{p-1} = v_2^{-1} i^*(x_0)$ and $x'_{p-1} = v_2^{-1} i^*(x'_0)$.

Proof. Since $\wp^p \wp^1(x_0) = 0$, we know $d_{2p^3+1}(t^{p^2}) = 0$ in the spectral sequence (5.4). Hence $E_{2p^3+1}^{*,*} \cong E_{\infty}^{*,*}$. The map $i_X : X\langle 3 \rangle \rightarrow X$ is homotopy normal and $i_X^* H^*(X; Z/p) = \Lambda'$. Hence from Corollary 3.4, there are the elements x_{p-1} and x'_{p-1} in $H^*(X\langle 3 \rangle; Z/p)$ such that in $K(2)^*(X\langle 3 \rangle)$ they are represented by $v_2^{-1} i^*(x)$ and $v_2^{-1} i^*(x')$. Since $|x_0 y^{p-1} t^{p-1}| = 2(p^2 - 1) + 2p + 1$ and $|x'_0 t^{p(p-1)}| = 2(p^2 - 1) + 3$, we have the isomorphism of mod p -cohomology. The isomorphism for $K(2)^*$ -theory is shown by the dimensional reason such that $d_r(t^{p^2}) = 0$ for all $r = |v_2^s| + 1$ in the Atiyah–Hirzebruch spectral sequence converging $K(2)^*(X\langle 3 \rangle)$. $\square$

For example $H^*(F_4\langle 3 \rangle; \mathbb{Z}/3) \cong \mathbb{Z}/3[t_{18}] \otimes \Lambda(x_{19}, x_{23}) \otimes \Lambda'$. In $K(2)^*(F_4\langle 3 \rangle)$ we see that $x_{19} = v_2^{-1}i^*(x_3)$ and $x_{23} = v_2^{-1}i^*(x_7)$.

COROLLARY 5.3. *Suppose that $i : Y \rightarrow X$ is mod p homotopy normal map of loop spaces such that X is of type (3.1). Suppose also $i^*(x_0) = 0$, but $i^*(x_i) \neq 0$ for some $i > 0$. Then there is a map $\tilde{i} : Y \rightarrow X\langle 3 \rangle$ such that $i_X \tilde{i} = i$ and $\tilde{i}^*(x_{p-1}) \neq 0$.*

Next we also consider 3-connective cover of X of type (4.1).

The cohomology $H^*(E_8\langle 3 \rangle; \mathbb{Z}/3)$ is computed in [KM], by using the spectral sequence induced from (4.1). Similarly we have

$$E_2^{*,*} = H^*(X; \mathbb{Z}/p) \otimes \mathbb{Z}/3[t] \Rightarrow H^*(X\langle 3 \rangle; \mathbb{Z}/p).$$

The differentials are given

$$\begin{aligned} d_3(t) &= x, & d_{2l+2}(xt^{p-1}) &= \beta \wp^1 x = y, & d_{2p+1}(t^p) &= \wp^1 x = x', \\ d_{2p^2+2}(x't^{p(p-1)}) &= \beta \wp^p \wp^1 x = u, & d_{2p^2+1}(t^{p^3}) &= \wp^p \wp^1 x = w'. \end{aligned}$$

By the arguments similar to that in Section 4, we can prove

$$E_{2p^2+2}^{*,*} \cong \mathbb{Z}/p[w^{p^3}] \otimes \Lambda(y^{p-1}t^{p-1}x, u^{p-1}t^{p(p-1)}x', t^{p^2(p-1)}w', w) \otimes \Lambda'.$$

THEOREM 5.4. *Let X be an H -space of type (4.1) such that $\wp^{p^2}\wp^p\wp^1(x) = 0$ and Λ' does not have generators of dimension $2s(p^3 - 1) + 3$ for $s \geq 2$. Then*

$$\begin{aligned} H^*(X\langle 3 \rangle; \mathbb{Z}/p) &\cong \mathbb{Z}/p[t_{2p^3}] \otimes \Lambda(x'_{(2)}, x'_{(3)}, x_{(3)}, w) \otimes \Lambda', \\ K(3)^*(X\langle 3 \rangle) &\cong K(3)^* \otimes H^*(X\langle 3 \rangle; \mathbb{Z}/p) \end{aligned}$$

such that $x'_{(3)} = v_3^{-1}i^*(x')$ and $x_{(3)} = v_3^{-1}i^*(x)$ in $K(3)^*(X\langle 3 \rangle)$ and $x'_{(2)} = v_2^{-1}i^*(x')$ in $K(2)^*(X\langle 3 \rangle)$.

Proof. The map $i : X\langle 3 \rangle \rightarrow X$ is also homotopy normal and $i^*H^*(X; \mathbb{Z}/p) = \Lambda' \otimes \Lambda(w)$. Hence $i^*(x) = i^*(x') = i^*(w') = 0$. The theorem is proved similar to that of Theorem 5.2. $\square$

For example we consider the case $X = E_8$. The cohomology is $H^*(E_8\langle 3 \rangle; \mathbb{Z}/3) \cong \mathbb{Z}/3[t_{54}] \otimes \Lambda(x_{23}, x_{55}, x_{59}, x_{15}) \otimes \Lambda'$. We see $x_{55} = v_3^{-1}i^*(x_3)$ and $x_{59} = v_3^{-1}i^*(x_7)$; but there need not exist the generator of degree 67 in $H^*(E_8\langle 3 \rangle; \mathbb{Z}/3)$, since $i^*(x_{15}) \neq 0$. The element x_{23} corresponds to $v_2^{-1}i^*(x_7)$ in $K(2)^*(E_8\langle 3 \rangle)$.

COROLLARY 5.5. *Suppose that $i : Y \rightarrow X$ is mod p homotopy normal and a map of loop spaces. Suppose also $i^*(x) = 0$, but $i^*(x_i) \neq 0$ for some i . Then there is the map $\tilde{i} : Y \rightarrow X\langle 3 \rangle$ such that $i_X \tilde{i} = i$ and $\tilde{i}^*(x_{(3)}) \neq 0$.*

6. Higher connected fiber space

In this section we consider n -connected fiber space $X\langle n \rangle$ of an H -space for $n > 3$. First, we assume the space X of type in Theorem 5.2. It is well known that [RW] if $m < n$,

$$K(m)^*(K(Z, n)) \cong K(m)^*(K(Z/(p^r), n+1)) \cong K(m)^*. \quad (6.1)$$

The fibering

$$K(\pi_n(X), n-1) \rightarrow X\langle n \rangle \rightarrow X\langle n-1 \rangle \quad (6.2)$$

induces the spectral sequence

$$E_2^{*,*} \cong H^*(X\langle n-1 \rangle; K(2)^*(K(\pi_n(X), n-1))) \Rightarrow K(2)^*(X\langle n \rangle). \quad (6.3)$$

From (6.1), this spectral sequence is isomorphic to the Atiyah–Hirzebruch spectral sequence converging to $K(2)^*(X\langle n-1 \rangle)$. Hence the following is immediate from (6.1).

LEMMA 6.1. *Let X be a space such that $H^4(X)$ has no p -torsion. Then for all $n \geq 3$, we have the isomorphism $K(2)^*(X\langle n \rangle) \cong K(2)^*(X\langle 3 \rangle)$.*

In particular, when X is an H -space in Theorem 5.2 and $n \geq 3$, we have the isomorphism $i_{\langle n \rangle}^* K(2)^*(X) \cong K(2)^*(X)/(y)$ for $i_{\langle n \rangle} : X\langle n \rangle \rightarrow X$. Hence there are $x(n)_0, \dots, x(n)_{p-2}$ in $H^*(X\langle n \rangle; Z/p)$ such that

$$i^* K(2)^*(X) \cong K(2)^* \otimes \Lambda(x(n)_0, \dots, x(n)_{p-2}) \quad (6.4)$$

with $v_2^{s_k}(x(n)_k) = i_{\langle n \rangle}^*(x_k)$ for $0 \leq s_k$.

CONJECTURE 6.2. *Let $s(p-1) + j \leq n/(2(p-1)) < s(p-1) + j+1$ for $0 \leq j \leq p-2$. Then we can take s_k satisfying (6.4) such that $s_k = s$ if $k \geq j$ and $s_k = s-1$ if $k < j$.*

We want to seek for the element $x(n)_k \in H^*(X\langle n \rangle; Z/p)$ in the above conjecture.

LEMMA 6.3. *Let $Q_n x \neq 0 \in H^*(X; Z/p)$. Then $Q_n x$ is in the image of the natural map $\rho : k(n)^*(-) \rightarrow H^*(-; Z/p)$, and we can take q (with modulo v_n) in $k(n)^*(X)$ such that $v_n q = 0$ and $\rho(q) = Q_n x$.*

Proof. The Sullivan–Bockstein exact sequence

$$k(n)^*(-) \xrightarrow{\rho} H^*(-; Z/p) \xrightarrow{\delta} k(n)^*(-) \xrightarrow{v_n} k(n)^*(-)$$

and the fact $Q_n = \rho\delta$ implies the lemma. $\square$

Let us write by $[a] \in k(m)^*(-)$ an element (with modulo (v_n)) such that $\rho([a]) = a \in H^*(-; Z/p)$. For example $v_m[Q_mx] = 0$.

LEMMA 6.4. *Let $K(Z, n-1) \rightarrow Y \xrightarrow{i} X$ be a fibration and X be an $(n-1)$ -connected space. In the induced Serre spectral sequence $E_n^{*,*}$ converging to $H^*(Y; Z/p)$, let $d_n(t) = x \neq 0$ for $x \in H^n(X; Z/p)$ and $t \in H^{n-1}(K(Z, n-1))$. Suppose that there is $q \in H^*(Y; Z/p)$ such that $v_m[q] = i^*([x]) \neq 0$ in $K(m)^*(Y) \cong K(m)^* \otimes H^*(Y; Z/p)$. Then q is represented by λQ_mt , $0 \neq \lambda \in Z/p$ in the spectral sequence $E_\infty^{*,*}$.*

Proof. Consider the spectral sequence

$$\bar{E}_2^{*,*} = H^*(X; k(m)^*(K(Z, n-1))) \implies k(m)^*(Y).$$

By dimensional reason, q is represented as a non-zero element in $E_2^{0,*} = H^*(K(Z, n-1); Z/p)$. Since $0 \neq i^*(x) \in K(m)^*(Y)$, x is a permanent cycle in this spectral sequence. Hence $v_m[q] = 0 \in k(m)^*(K(Z, n-1))$. This means $q \in \text{Im}(Q_m)$ in $H^*(K(Z, n-1); Z/p)$. $\square$

Using this lemma and the fact $X\langle n \rangle \rightarrow X$ is homotopy normal, we get the following theorem.

THEOREM 6.5. *Let X be an H -space in Theorem 5.2, $p > 3$, and $|(\Delta')^+| \geq 2p+5$. Then there are elements $x_p, x'_p \in H^*(X\langle 2p+5 \rangle; Z/p)$ such that $v_2[x_p] = i_{(2p+5)}^*(x_1)$, $v_2[x'_p] = i_{(2p+5)}^*(x'_1)$ and $j^*x_p = Q_2t$, $j^*x'_p = Q_2\wp^1t$ for $t \in H^{2p+4}(K(Z, 2p+4); Z/p)$ and $j : K(Z, 2p+4) \rightarrow X\langle 2p+5 \rangle$ corresponding the element $x_1 : X \rightarrow K(Z, 2p+5)$.*

For example, the case $p = 5$ and $X = E_8$ there exist elements $q_{63}, q_{79} \in H^*(E_8\langle 15 \rangle; Z/5)$ with $v_2q_{63} = i^*(x_{15})$, $v_2q_{79} = i^*(x_{31})$ and $j^*(q_{63}) = Q_2t_{14} \in H^*(K(Z, 14); Z/5)$. The case $p = 3$, $X = F_4$ is not a direct corollary of the above theorem, however we show easily the following result by the similar arguments. There are $q_{27}, q_{31} \in H^*(F_4\langle 11 \rangle; Z/3)$ such that $v_2q_{27} = i^*(x_{11})$, $v_2q_{31} = i^*(x_{15})$ and $j^*q_{27} = Q_2t \in H^*(K(Z, 10); Z/3)$.

Lastly, we consider the case X in Theorem 5.4. The following lemma is immediate from (6.1).

LEMMA 6.6. *Let X be an H -space in Theorem 5.4. Then $K(3)^*(X\langle n \rangle) \cong K(3)^*(X\langle 3 \rangle)$ for $n \geq 3$. In particular $i^*K(3)^*(X\langle n \rangle) \cong K(3)^*(X)/(y, u, w')$.*

We give here only an example. From Lemma 6.5, we can easily prove the following facts. There is the element $q_{67} \in H^*(E_8(15); \mathbb{Z}/3)$ such that $v_3 q_{67} = i^* x_{15} \in K(3)^*(E_8(15))$ and $j^*(q_{67}) = Q_3 t \in H^*(K(\mathbb{Z}, 14); \mathbb{Z}/3)$. This is the reason of dimension 67 which appears in Theorem 4.6.

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