NORMAL HYPERSURFACES AS A
COMPACTIFICATION OF $\mathbb{C}^2$

Tomoaki OHTA

(Received 12 October 1999 and revised 23 February 2000)

1. Introduction

A pair $(X, Y)$ of a projective normal Gorenstein surface over $\mathbb{C}$ and its closed subvariety is called a compactification of $\mathbb{C}^2$ if $X \setminus Y$ is biholomorphic to $\mathbb{C}^2$. The Hartogs theorem shows that the boundary $Y$ is of pure codimension one; that is, $Y$ is a Weil divisor on $X$. A compactification $(X, Y)$ of $\mathbb{C}^2$ is said to be primitive if the second Betti number $b_2(X) = \dim H_2(X, \mathbb{R}) = 1$, which is equivalent to saying that $Y$ is irreducible.

A primitive compactification of $\mathbb{C}^2$ gives an example of a rational $\mathbb{Z}$-homology plane; that is, a normal rational surface $X$ with $H_i(X, \mathbb{Z}) = H_i(\mathbb{P}^2, \mathbb{Z})$ for $i \geq 0$. Conversely, we have the following.

Question 1.1. [Fu1] Does there exist a rational $\mathbb{Z}$-homology plane which is not a compactification of $\mathbb{C}^2$?

In this paper, we shall mainly study the following.

Problem 1.2. [Fu1] Determine the primitive compactifications of $\mathbb{C}^2$ which are hypersurfaces in the projective three-space $\mathbb{P}^3$.

In the rest of this paper, we assume that $(X, Y)$ is a primitive compactification of $\mathbb{C}^2$ which is a hypersurface of degree $d \geq 1$ in $\mathbb{P}^3$. We put $x := \text{Sing } X = \{x_1, \ldots, x_t\}$ for $t \geq 0$. Let $\pi : M \rightarrow X$ be the minimal resolution of $X$ with exceptional set $E = \bigcup_{j=1}^t E_j := \text{Exc } \pi = \pi^{-1}(x)$ for $s \geq 0$, where each $E_j$ is irreducible. Let $\hat{Y}$ be the proper transform of $Y$ by $\pi$. We set $A := \hat{Y} \cup E$. We put $p_g(x) := \sum_{i=1}^t p_g(x_i)$, where $p_g(x_i) := \dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_M)_{x_i}$ is the geometric genus of the singular point $x_i$. Then we easily see that

$$(X, Y) = \begin{cases} 
\text{(a hyperplane, a line)} & \text{(if } d = 1) \\
\text{(a quadric cone, a generating line)} & \text{(if } d = 2). 
\end{cases}$$
For $d \geq 3$, we see later that $X$ is a singular surface which is not a cone. In particular, we obtain $2 \leq \text{mult}_x X \leq d - 1$ for $1 \leq i \leq t$ (cf. Proposition 2.5(iv)). Then we have the following.

**Conjecture 1.3.** ([Fu1]) Assume that $d \geq 3$ and $x$ contains at least a $(d - 1)$-ple point. Then the following hold:

(C1) $Y$ is a line in $\mathbb{P}^3$;

(C2) $x$ is one point with $p_g(x) = (d - 1)(d - 2)(d - 3)/6$;

(C3) the weighted dual graph of $A$ is

\[
\begin{array}{c}
\text{d vertices} & \text{(d}^2 - 2d - 1\text{) vertices} \\
-2 & -2 & \cdots & -2 & -2 & \cdots & -2 & -1 \\
\circ & \cdots & \circ & \cdots & \circ & \circ \\
-(d - 1) & \\
\end{array}
\]

where $\bullet$, $\circ$ denote smooth rational curves;

(C4) $(X, Y) \cong (V_d, \Delta_d)$ (up to automorphisms of $\mathbb{P}^3$), where

\[
V_d := \left\{ (z_0 : z_1 : z_2 : z_3) \in \mathbb{P}^3 ; \ z_2^{d-1}z_3 = \sum_{0 \leq i+j \leq d} c_{ij} z_0^i z_1^j z_2^{d-i-j} \right\},
\]

\[
\Delta_d := \{ z_0 = z_2 = 0 \}
\]

and $c_{ij} \in \mathbb{C}$ satisfy

\[
c_{0,d} = c_{1,d-1} = c_{2,d-2} = \cdots = c_{d-1,1} = 0,
\]

\[
c_{d,0} = c_{0,d-1} = 1.
\]

**Remark.** For any $d \geq 3$, one has a birational map $\Phi_d : \mathbb{P}^2 \rightarrow V_d \hookrightarrow \mathbb{P}^3$ such that

\[
\Phi_d : \begin{cases}
  z_0 = w_0 w_2^{d-1} \\
  z_1 = w_1 w_2^{d-1} \\
  z_2 = w_2^d \\
  z_3 = \sum_{0 \leq i+j \leq d} c_{ij} w_0^i w_1^j w_2^{d-i-j},
\end{cases}
\]

which gives an isomorphism $\mathbb{P}^2 \setminus \{ w_2 = 0 \} \cong V_d \setminus \Delta_d$.

**Theorem 1.4.** ([Fu1, Fu2]) For $d = 3, 4$, Conjecture 1.3 is true.
Theorem 1.5. [Fu1, Fu2] Assume that $d = 4$ and $x$ contains no triple points; that is, $x$ consists of at most double points. Then $(X, Y)$ satisfies the conditions (C1), (C2), (C3*) and (C4*).

(C3*) The weighted dual graph of $A$ is

$$
\begin{array}{cccccccc}
-2 & -2 & -2 & -2 & -2 & -3 & -2 & -1 \\
\end{array}
$$

$$
\begin{array}{cccccccc}
-2 & -2 \\
\end{array}
$$

(C4*) $(X, Y) \cong (V_4^*, \Delta_4^*)$ (up to automorphisms of $\mathbb{P}^3$), where

$$V_4^* := \{z_2(z_2^2z_3 + z_0^3 + c_3 z_0^2z_2 + c_4z_0z_2^2)\}
= (z_1z_2 + z_3^2)(z_1z_2 + z_3^2 + c_1z_0z_2 + c_2z_0^2),$$

$$\Delta_4^* := \{z_2 = z_3 = 0\}$$

with $c_i \in \mathbb{C}$.

Remark. One has a birational map $\Phi_4^* : \mathbb{P}^2 \rightarrow V_4^* \hookrightarrow \mathbb{P}^3$ such that

$$\Phi_4^* : \begin{cases}
  z_0 = w_0 w_2^5 \\
  z_1 = w_1 w_2^3 - \{w_1 w_2(w_1 + c_1 w_0 + c_2 w_2) - w_0(w_0^2 + c_3 w_0 w_2 + c_4w_2^2)\}^2 \\
  z_2 = w_2^6 \\
  z_3 = w_1 w_2^5(w_1 + c_1 w_0 + c_2 w_2) - w_0 w_2^3(w_0^3 + c_3 w_0 w_2 + c_4w_2^2),
\end{cases}$$

which gives an isomorphism $\mathbb{P}^2 \setminus \{w_2 = 0\} \cong V_4^* \setminus \Delta_4^*$.

For $d \geq 4$, we obtain the structure theorems of $(X, Y)$ as follows.

Theorem 1.6. Assume that $d \geq 4$. Then the following hold.

(i) $Y$ is a line in $\mathbb{P}^3$.

(ii) $x = \{x_1\}$ or $x = \{x_1, x_2\}$, where $p_5(x_1) = (d - 1)(d - 2)(d - 3)/6$ and $x_2$ is a rational double point of $A_n$ type.

(iii) The weighted dual graph of $A$ is a tree of smooth rational curves with $\tilde{Y}^2 = -1$ and $E_i^2 \leq -2$ for each $i$ as in Figure 1, where $*$ denotes a smooth rational curve with a self-intersection number less than or equal to $-2$ and $\alpha$ or $\beta$ may be zero.

(iv) Let $\tau : M \rightarrow M^*$ be a blowing-down on $A$ to a minimal normal compactification $(M^*, A^*)$ of $\mathbb{C}^2$. Then the weighted dual graph of $A^*$ is either $\frac{1}{6}$ or $\frac{6}{6-a}$ (a $\geq 2$). In particular, there exists a fibration $f : M \rightarrow \mathbb{P}^1$. 
whose general fiber is $\mathbb{P}^1$ with the following properties: (1) $f$ has a unique singular fiber $F$; (2) $E$ has a unique irreducible component $E_{ls}$ is a section of $f$; (3) $A = E_{ls} \cup F$.

![Diagram](image)

**Figure 1.**

**Theorem 1.7.** Conjecture 1.3 is true.

**Notation.**
- $(-1)$-curve.
- $(-2)$-curve.
- $(-m)$-curve; that is, a smooth rational curve with self-intersection number $-m$.
- smooth rational curve with self-intersection number less than or equal to $-2$.

2. Preliminaries

In this section we shall study properties of primitive compactifications of $\mathbb{C}^2$ that are hypersurfaces in $\mathbb{P}^3$. We start with some definitions and well-known results related to compactifications of $\mathbb{C}^2$ and $\mathbb{C} \times \mathbb{C}^*$.

A pair $(V, \Delta)$ of a projective normal Gorenstein surface over $\mathbb{C}$ and its closed subvariety is called a *compactification of $\mathbb{C}^2$* (respectively $\mathbb{C} \times \mathbb{C}^*$) if $V \setminus \Delta$ is biholomorphic to $\mathbb{C}^2$ (respectively $\mathbb{C} \times \mathbb{C}^*$). The Hartogs theorem shows that the boundary $\Delta$ is of pure codimension one; that is, $\Delta$ is a Weil divisor on $V$. We denote by $\Delta = \bigcup_i \Delta_i$ the irreducible decomposition of $\Delta$ and we put $\nu := \text{Sing } V$. We denote by $\omega_V$ and $K_V$ the dualizing sheaf and the canonical divisor of $V$ respectively.

Let $(V, \Delta)$ be a compactification of $\mathbb{C}^2$ (respectively $\mathbb{C} \times \mathbb{C}^*$). $(V, \Delta)$ is said to be nonsingular if $V$ is nonsingular. $(V, \Delta)$ is said to be minimally normal if the following conditions are satisfied: (1) $V$ is nonsingular; (2) $\Delta$ is of normal crossing
type; that is, \( \Delta \) has at most ordinary double points; and (3) if \( \Delta_j \) is a \((-1)\)-curve, then \( \Delta_j \cap \Delta \setminus \Delta_j \) contains at least three points.

**Proposition 2.1.** [Ko, Ra] Let \((V, \Delta)\) be a compactification of \(\mathbb{C}^2\). Then

(i) \( V \setminus \Delta \) is biregular to \(\mathbb{C}^2\),
(ii) \( V \) is a rational surface.

**Proposition 2.2.** [Fu1] Let \((V, \Delta)\) be a compactification of \(\mathbb{C}^2\). Then

(i) \( H_0(V, \mathbb{Z}) \cong H_0(\Delta, \mathbb{Z}) = \mathbb{Z} \),
(ii) \( H_1(V, \mathbb{Z}) \cong H_1(\Delta, \mathbb{Z}) = 0 \),
(iii) \( H_2(V, \mathbb{Z}) \cong H_2(\Delta, \mathbb{Z}) = \bigoplus_i \mathbb{Z} \cdot \Delta_i \),
(iv) \( H_3(V, \mathbb{Z}) \cong H_3(\Delta, \mathbb{Z}) = 0 \),
(v) \( H^1(V, \mathcal{O}_V) = 0 \),
(vi) \( p_g(v) = h^2(V, \mathcal{O}_V) = h^0(V, \omega_V) \).

**Remark.** By (i), \( \Delta \) is connected. By (ii), \( \Delta \) cannot have any cycles and hence every \( \Delta_i \) is a rational curve without nodes.

**Proposition 2.3.** [Mo] Let \((V, \Delta)\) be a minimal normal compactification of \(\mathbb{C}^2\). Then \( V \) is a rational surface and the weighted dual graph of \( \Delta \) is a linear tree of smooth rational curves which is one of the following three types:

(i) \( \overset{1}{\bullet} \),
(ii) \( \overset{a}{\circ} \overset{-a}{\circ} \ (a \neq -1) \),
(iii) the weighted dual graph of \( \Delta \) contains a subgraph \( \overset{-a}{\circ} \overset{1}{\bullet} \overset{a}{\circ} \ (a > 0) \). If the weighted dual graph of \( \Delta \) has the other vertices, then their weights are less than or equal to \(-2\).

**Remark.** We note that \( V \cong \mathbb{P}^2 \) for type (i) and \( V \cong \mathbb{F}_a \) for type (ii), where \( \mathbb{F}_a \) is the Hirzebruch surface of degree \( a \).

**Proposition 2.4.** [Su2] Let \((V, \Delta)\) be a minimal normal compactification of \(\mathbb{C} \times \mathbb{C}^*\). Then \( V \) is a rational surface and the weighted dual graph of \( \Delta \) is a linear tree of smooth rational curves which is one of the following four types:

(i) \( \overset{1}{\bullet} \),
(ii) \( \overset{a}{\circ} \overset{-a}{\circ} \ (a \neq -1) \),
(iii) \( \overset{-a}{\circ} \overset{a}{\circ} \ (a \geq 2) \),
(iv) the weighted dual graph of \( \Delta \) contains a subgraph \( \overset{-a}{\circ} \overset{1}{\bullet} \overset{a}{\circ} \ (a > 0) \). If the weighted dual graph of \( \Delta \) has the other vertices, then their weights are less than or equal to \(-2\).

**Remark.** We note that \( V \cong \mathbb{P}^2 \) for type (i) and \( V \cong \mathbb{F}_a \) for types (ii) and (iii).
Now we assume that $(X, Y)$ is a primitive compactification of $\mathbb{C}^2$ which is a hypersurface of degree $d \geq 1$ in $\mathbb{P}^3$. We use the same notation as in Section 1. We put $x := \text{Sing} \, X = \{x_1, \ldots, x_t\}$ and $p_g(x) := \sum_{i=1}^{t} p_g(x_i)$ for $t \geq 0$. Since $X \setminus Y \cong \mathbb{C}^2$ is nonsingular, we obtain $x \subset Y$. Let $\Gamma$ be a smooth hyperplane section of $X$ with $\Gamma \cap x = \emptyset$. By the adjunction formula, we obtain the dualizing sheaf $\omega_X = \mathcal{O}_X(K_X) = \mathcal{O}_X((d-4)\Gamma)$. Let $\pi : M \to X$ be the minimal resolution of $X$ with the exceptional set $E = \bigcup_{j=1}^{s} E_j = \pi^{-1}(x)$ for $s \geq 0$. Let $\tilde{C}$ be the proper transform of a curve $C$ on $X$ by $\pi$. We set $A := \tilde{Y} \cup E$. Since $\pi : M \setminus E \cong X \setminus x$, the pair $(M, A)$ is a nonsingular compactification of $\mathbb{C}^2$.

**Proposition 2.5.**

(i) $Y$ is a line in $\mathbb{P}^3$.

(ii) There exists a hyperplane $H$ in $\mathbb{P}^3$ such that $H|_X = dY$.

(iii) $p_g(x) = (d-1)(d-2)(d-3)/6$.

(iv) If $d \geq 3$, then $X$ is a singular surface which is not a cone. In particular, one has $2 \leq \text{mult}_x X \leq d - 1$ ($1 \leq i \leq t$).

**Proof.** (i) Let us consider the following exact sequence:

$$\longrightarrow H_2(X; \mathbb{Z}) \overset{\iota_*}{\longrightarrow} H_2(\mathbb{P}^3; \mathbb{Z}) \longrightarrow H_2(\mathbb{P}^3, X; \mathbb{Z}) \longrightarrow .$$

Since $H_2(X; \mathbb{Z}) = \mathbb{Z} \cdot Y$ by Proposition 2.2(iii) and $H_2(\mathbb{P}^3; \mathbb{Z}) = \mathbb{Z} \cdot l$ where $l$ is a line in $\mathbb{P}^3$, it is sufficient to show that $\iota_*$ is surjective. Now we show that $H_2(\mathbb{P}^3, X; \mathbb{Z}) = 0$ by using the following facts.

1. The universal coefficient theorem says

$$H^i(\mathbb{P}^3, X; \mathbb{Z}) \cong H_i(\mathbb{P}^3, X; \mathbb{Z})_{\text{free}} \oplus H_{i-1}(\mathbb{P}^3, X; \mathbb{Z})_{\text{tor}} \quad (\forall i).$$

2. The fact that $\mathbb{P}^3 \setminus X$ is Stein implies

$$H^i(\mathbb{P}^3, X; \mathbb{Z}) \cong H_{6-i}(\mathbb{P}^3 \setminus X; \mathbb{Z}) = 0 \quad (i = 1, 2).$$

$$H^3(\mathbb{P}^3, X; \mathbb{Z}) \cong H_3(\mathbb{P}^3 \setminus X; \mathbb{Z}) : \text{free}.$$ 

By (2) and the case $i = 3$ in (1), we obtain $H_2(\mathbb{P}^3, X; \mathbb{Z})_{\text{tor}} = 0$. Similarly, by (2) and the case $i = 2$ in (1), we obtain $H_2(\mathbb{P}^3, X; \mathbb{Z})_{\text{free}} = 0$.

(ii) Since $M$ is a nonsingular rational surface and $(M, A)$ is a compactification of $\mathbb{C}^2$, we obtain the isomorphism

$$c_1 : \text{Pic}(M) \cong H^2(M; \mathbb{Z}) = H^2(\tilde{Y}; \mathbb{Z}) \oplus (\oplus_i H^2(E_i; \mathbb{Z})).$$
where \( c_1 \) is the Chern map. Hence, by this isomorphism, there exist integers \( a \) and \( b_i \) such that \( \hat{\Gamma} \sim a \hat{Y} + \sum b_i E_i \). By using the equalities \((\hat{\Gamma}^2) = d\), \((\hat{\Gamma} \cdot \hat{Y}) = 1\) and \((\hat{\Gamma} \cdot E_i) = 0\), we obtain \( a = d \).

Now we note that the homomorphism of rational function fields \( \pi^* : \mathbb{C}(X) \to \mathbb{C}(M), \varphi \mapsto \varphi \circ \pi \), is an isomorphism. Then we obtain a rational function \( \varphi \) on \( X \) such that \( \hat{\Gamma} - (d \hat{Y} + \sum b_i E_i) = \text{div}(\pi^* \varphi) \). Since \( \pi : M - E \cong X - x \), we obtain \( \Gamma - dY = \text{div}(\varphi) \). Hence we have that \( dY \) is a Cartier divisor and \( \mathcal{O}_X(dY) = \mathcal{O}_X(\Gamma) = \mathcal{O}_X(1) \).

Since the restriction map \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(X, \mathcal{O}_X(1)) \) is an isomorphism, we obtain (ii).

(iii) By Proposition 2.2(vi) and an easy computation, we obtain \( p_g(x) = h^0(X, \omega_X) = h^0(X, \mathcal{O}_X(d - 4)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d - 4)) = (d - 1)(d - 2)(d - 3)/6 \).

(iv) It is obvious that \( X \) is singular. If \( X \) is a cone, then \( X \) is a cone over a smooth plane curve of degree \( d \geq 3 \). Hence \( X \) is not rational. This is a contradiction. \( \Box \)

To the end of this section we assume that \( d \geq 4 \). Then, by Proposition 2.5(iii), we obtain \( p_g(x) > 0 \). By Proposition 2.2(ii), \( A \) cannot have any cycles and hence every irreducible component of \( A \) is a rational curve without nodes. In particular, since \( E \) is the exceptional set of the minimal resolution \( \pi \), we have one of the following for each irreducible component \( E_i \) of \( E \): (1) \( E_i^2 \leq -2 \); (2) \( E_i^2 = -1 \) and \( E_i \) is a singular rational curve without nodes.

**Proposition 2.6.** \((M, A)\) is not a minimal normal compactification of \( \mathbb{C}^2 \).

**Proof.** If \((M, A)\) is a minimal normal compactification of \( \mathbb{C}^2 \), then the weighted dual graph of \( A \) is a linear tree by Proposition 2.3 and thus the weighted dual graph of \( E \) is a disjoint union of linear trees. Then, since \( E_i^2 \leq -2 \) for each \( i \), the weighted dual graph of \( A \) is \( a \geq 2 \) \( \mathbb{C}^2 \) as in Proposition 2.3(ii). Since \( \hat{Y} \) is the 0-curve and \( E \) is the \( (-a) \)-curve, we obtain \( p_g(x) = 0 \). This is a contradiction. \( \Box \)

**Proposition 2.7.**
(i) \( A \) is of normal crossing type.
(ii) \( \hat{Y} \) is a \((-1)\)-curve and each \( E_i \) is isomorphic to \( \mathbb{P}^1 \) with \( E_i^2 \leq -2 \).

**Proof.** (i) We assume that \( A \) is not of normal crossing type. We note that any irreducible component \( E_i \) of \( E \) with \( E_i^2 = -1 \) is a singular rational curve without nodes. Now there exists a birational morphism \( \tau' : M' \to M \) with the following properties:
(1) \( A' := \tau'^{-1}(A) \) is of normal crossing type;
\[(2) \quad E_i^2 \leq -2 \text{ and } B_j^2 \leq -1 \quad (\forall i, j),\] where \(E_i\) is the proper transform of \(E_i\) by \(\tau'\) and \(B_j = \bigcup_j B_j^j\) is the exceptional set of \(\tau'\);

\[(3) \quad \text{if } B_j^j \text{ is a } (-1)\text{-curve, then } B_j \text{ intersects at least three irreducible components of } A'.\]

Then we have that \((M', A')\) is a nonsingular compactification of \(\mathbb{C}^2\) with \(A' = \hat{Y}' \cup (\bigcup_i E_i^j) \cup (\bigcup_j B_j^j)\), where \(\hat{Y}'\) is the proper transform of \(\hat{Y}\) by \(\tau'\).

By Proposition 2.3, any minimal normal compactifications of \(\mathbb{C}^2\) have no \((-1)\text{-curves in their boundaries. On the other hand, since } \tau'\text{ is not isomorphism, there exists at least one } (-1)\text{-curve in } B', \text{ and hence in } A'. \text{ Thus we obtain the following.}\)

**Claim 1.** \((M', A')\) is not a minimal normal compactification of \(\mathbb{C}^2\).

If \(\hat{Y}'\) is not a \((-1)\text{-curve}, then } (M', A')\) is a minimal normal compactification of \(\mathbb{C}^2\) by (1), (2) and (3). This is a contradiction. Thus we obtain the following.

**Claim 2.** \(\hat{Y}'\) is a \((-1)\text{-curve.}\)

Now, since \(\tau'\) is not an isomorphism, we take a \((-1)\text{-curve } B_j^j \text{ in } B'. \text{ Then } B_j^j \text{ intersects at least three irreducible components of } A'. \text{ Hence we obtain the connected components } C_1, C_2, \ldots, C_r \text{ of the closure of } A' \setminus B_j^j \text{ in } M' \text{ for some } r \geq 3. \text{ We may assume that } \hat{Y}' \subset C_1.\)

We contract the \((-1)\text{-curve } \hat{Y}'\) and blow-down on \(A'\) repeatedly. Finally, we obtain a minimal normal compactification \((M'', A'')\) of \(\mathbb{C}^2\). Let us denote by \(\tau'': M' \to M''\) the blowing-down to \(M''\) with \(A'' = \tau''(A').\) By (3), we note that any \((-1)\text{-curve } B_j \text{ in } C_2 \cup \cdots \cup C_r \text{ cannot be contracted by } \tau''.\)

Since the weighted dual graph of \(A''\) is a linear tree, we obtain the following: (a) \(\text{Exc } \tau'' = C_1;\) (b) \(B_j^j'' = \tau''(B_j^j)\) is a smooth rational curve with \(B_j^j'' \geq 0;\) and (c) \(r = 3.\) By (a), we note that \(\tau''\) is an isomorphism in an open neighborhood of \(C_2 \cup C_3.\)

By Proposition 2.3, there exist no \((-1)\text{-curves in } A''\) and hence in \(\tau''(C_2 \cup C_3).\) Thus the weighted dual graph of \(A''\) is a linear tree of a unique smooth rational curve with a self-intersection number greater than or equal to zero and smooth rational curves with self-intersection numbers less than or equal to -2. However, the weighted dual graph of this type is not found in Proposition 2.3. This is a contradiction.

(ii) Since \(A\) is of normal crossing type by (i), we have that \(\hat{Y} \cong E_i \cong \mathbb{P}^1\) and \(E_i^2 \leq -2\) for each \(i.\) If \(\hat{Y}\) is not a \((-1)\text{-curve, then } (M, A)\) is a minimal normal compactification of \(\mathbb{C}^2.\) This contradicts Proposition 2.6. Hence \(\hat{Y}\) is a \((-1)\text{-curve.}\)

Now we contract the \((-1)\text{-curve } \hat{Y}\) and blow-down on \(A\) repeatedly. Finally, we obtain a minimal normal compactification \((M^*, A^*)\) of \(\mathbb{C}^2.\) Let us denote by
\( \tau = \tau_1 \circ \cdots \circ \tau_m : M^{(m)} := M \to \cdots \to M^{(0)} := M^* \) the blowing-down to \( M^* \) with \( A^* = \tau(A) \). Conversely, we obtain \( M \) by blowing-up \( M^* \) on \( A^* \) repeatedly. We denote by \( P_{i-1} \) the center of the blowing-up \( \tau_i \) and by \( B_i := \tau_i^{-1}(P_{i-1}) \) the exceptional curve of \( \tau_i \), which is a \((-1)\)-curve on \( M^{(i)} \), for \( 1 \leq i \leq m \). We set \( A^{(m)} := A \) and \( A^{(i)} := (\tau_{i+1} \circ \cdots \circ \tau_m)(A) \) for \( 0 \leq i \leq m - 1 \). We note that \( B_m = \hat{Y} \) and \( A^{(0)} = A^* \).

**Lemma 2.8.**

(i) \( \tau(\text{Exc } \tau) = \{P_0\}, P_i \in B_i \ (1 \leq i \leq m - 1) \). In particular, \( \text{Exc}(\tau_1 \circ \cdots \circ \tau_m) \) is connected for \( 1 \leq i \leq m \).

(ii) \( A^* \) contains a unique irreducible component with self-intersection number greater than or equal to zero.

**Proof.** (i) Since \( \hat{Y} \) is a unique \((-1)\)-curve in \( A \) and since \( \hat{Y} \subset \text{Exc } \tau \subset A \), we have that \( \hat{Y} \) is a unique \((-1)\)-curve in \( \text{Exc } \tau \). From this, we easily obtain (i).

(ii) By Proposition 2.3, \( A^* \) contains at least one such irreducible component. Now we assume that there exist two irreducible components \( A_1^* \) and \( A_2^* \) of \( A^* \) such that \( A_i^{*2} \geq 0 \) for \( i = 1, 2 \). By Proposition 2.3 again, \( A_1^* \) and \( A_2^* \) intersect transversally at one point. Since \( \hat{Y} \) is contracted by \( \tau \), the proper transforms of \( A_1^* \) and \( A_2^* \) by \( \tau \) are contained in \( E \). Thus we may assume that \( E_i \) is the proper transform of \( A_i^* \) by \( \tau \) for \( i = 1, 2 \). We note that \( E_i^2 \leq -2 \) for \( i = 1, 2 \). Since the centers of \( \tau_1, \ldots, \tau_m \) are infinitely near \( P_0 \) by (i) and since \( A_i^{*2} \geq 0 \) and \( E_i^2 < 0 \) for \( i = 1, 2 \), we obtain that \( \tau_1^* \cap \tau_2^* = \{P_0\} \). Since the centers of \( \tau_2, \ldots, \tau_m \) are infinitely near \( P_1 \) by (i), we have that \( E_1^2 \geq -1 \) or \( E_2^2 \geq -1 \). This is a contradiction. \( \Box \)

**Proposition 2.9.** One of the following holds:

(i) \( M^* \cong \mathbb{P}^2 \) and the weighted dual graph of \( A^* \) is \( \overset{1}{\circ} \);

(ii) \( M^* \cong \mathbb{P}_a \) and the weighted dual graph of \( A^* \) is \( \overset{a}{\circ} \overset{0}{\circ} \overset{0}{\circ} \) for some \( a \geq 2 \).

For each case, there exists a fibration \( f : M \to \mathbb{P}^1 \) whose general fiber is \( \mathbb{P}^1 \) with the following properties: (1) \( f \) has a unique singular fiber \( F \); (2) \( E \) has a unique irreducible component \( E_0 \), which is a section of \( f \); and (3) \( A = E_0 \cup F \).

**Proof.** This is an easy consequence of Proposition 2.3 and Lemma 2.8(ii). \( \Box \)

**Corollary 2.10.** There exists an integer \( 2 \leq j_1 \leq 3 \) such that:

(i) the weighted dual graph of \( A^{(j_1)} \) is \( \overset{-1}{\circ} \overset{-2}{\circ} \overset{b}{\circ} \overset{0}{\circ} \) for some \( b \geq 2 \). Moreover, the image \( (\tau_{j_1+1} \circ \cdots \circ \tau_m)(E_{j_1}) \) is the \((-b)\)-curve of \( A^{(j_1)} \).
(ii) for any $j \geq j_1$, the weighted dual graph of $A^{(j)}$ is a tree of a unique $(-1)$-curve $B_j$ and smooth rational curves with self-intersection numbers less than or equal to $-2$.

Proof. This is an easy consequence of Lemma 2.8(i) and Proposition 2.9. □

**Proposition 2.11.**

(i) $x = \{x_1\}$ or $x = \{x_1, x_2\}$, where $p_{\mathbf{g}}(x_1) = (d - 1)(d - 2)(d - 3)/6$ and $x_2$ is a rational double point of $A_n$ type.

(ii) The weighted dual graph of $A$ is

![Diagram showing the weighted dual graph of $A$]

where $\alpha$ or $\beta$ may be zero.

Proof. (i) If $\hat{Y}$ intersects at least three irreducible components of $E$, then $(M, A)$ is a minimal normal compactification of $\mathbb{C}^2$ by Proposition 2.7. This contradicts Proposition 2.6. Thus $\hat{Y}$ intersects at most two irreducible components of $E$ and hence $x$ consists of either one point or two points.

Now it suffices to consider the case where $x = \{x_1, x_2\}$. Since $p_{\mathbf{g}}(x) > 0$, we may assume that $p_{\mathbf{g}}(x_1) > 0$. Then, by Brieskorn’s theorem [Br], the weighted dual graph of $\pi^{-1}(x_1)$ is not a linear tree. We assume that the weighted dual graph of $\pi^{-1}(x_2)$ is neither a linear tree. Then there exists the largest connected curve $D = \bigcup_i D_i$ in $A$ with the following properties:

1. $\hat{Y}$ is contained in $D$;
2. $D_i$ intersects exactly two irreducible components of $A$ for each $i$.

We note that the weighted dual graph of $D$ is a connected linear tree and the case $D = \hat{Y}$ may occur. Now we take a unique irreducible component $E_{k_l}$ of the closure of $\pi^{-1}(x_i) \setminus D$ in $M$ which intersects $D$ for $i = 1, 2$. Then $E_{k_l}$ intersects at least three irreducible components of $A$ for $i = 1, 2$.

We contract the $(-1)$-curve $\hat{Y}$ and blow-down on $D$ as repeatedly as possible. Then, by noticing the shape of the weighted dual graph of $A$, we obtain a minimal
normal compactification \((M^*, A^*)\) of \(\mathbb{C}^2\). However, the weighted dual graph of \(A^*\) is not a linear tree. This is a contradiction. Thus the weighted dual graph of \(\pi^{-1}(x_2)\) is a linear tree and hence \(p^*_\pi(x_2) = 0\). Since \(X\) is a hypersurface in \(\mathbb{P}^3\), we have that \(x_2\) is a rational double point of \(A_n\) type and \(p^*_\pi(x_1) = (d - 1)(d - 2)(d - 3)/6\).

(ii) We note that \(\tilde{Y}\) is a unique \((-1)\)-curve in \(A\) and \(E_i^2 \leq -2\) for each \(i\), and that the weighted dual graph of \(A^*\) is a linear tree. Then, by the same argument above, we easily obtain the weighted dual graph of \(A\) as required. \(\square\)

3. Proof of theorems

In this section we will prove Theorems 1.6 and 1.7. Since we obtain Theorem 1.6 by combining Propositions 2.5, 2.7, 2.9 and 2.11, we prove Theorem 1.7.

From now on, we assume that \((X, Y)\) is a primitive compactification of \(\mathbb{C}^2\) which is a hypersurface of degree \(d \geq 4\) in \(\mathbb{P}^3\) with a \((d - 1)\)-ple point. We use the same notation as in Sections 1 and 2. Since we obtain the condition (C1) for \((X, Y)\) by Proposition 2.5(i), we only have to show the conditions (C2)–(C4) for \((X, Y)\). By Proposition 2.11(i), we have that \(x = \{x_1\}\) or \(x = \{x_1, x_2\}\), where \(p^*_\pi(x_1) > 0\) and \(x_2\) is a rational double point. Hence we note that \(x_1\) is the \((d - 1)\)-ple point.

Now we consider a projection and a blowing-up of \(\mathbb{P}^3\) to investigate the primitive compactification \((X, Y)\). Let \(\ell_1, \ldots, \ell_N\) be the lines in \(X\) through \(x_1\) for \(N \geq 0\). Since \(Y\) is such a line, we obtain \(N \geq 1\) and we may assume that \(\ell_1 = Y\). Let \(\sigma : \mathbb{P}^3 \to \mathbb{P}^3\) be the blowing-up at \(x_1\) with exceptional divisor \(\Sigma \cong \mathbb{P}^2\). We denote by \(\sigma^*\) the proper transform of a subvariety \(W\) of \(\mathbb{P}^3\) by \(\sigma\). We set \(\overline{X} : = \text{Sing } X\) and \(\overline{E} = \bigcup_i \overline{E}_i := \overline{X} \cap \Sigma\), where each \(\overline{E}_i\) is irreducible. Let \(\psi : \mathbb{P}^3 \to \mathbb{P}^2\) be the projection from \(x_1\) and \(\psi^* : \mathbb{P}^3 \to \mathbb{P}^2\) the resolution of indeterminacy of \(\psi\). We put \(\overline{W}^* : = \psi^*(W), \overline{E}_i^* : = \psi^*(E_i)\) and \(\overline{E}^* : = \bigcup_i \overline{E}_i^*\). Here we note that \(\psi^*|_{\Sigma} : \Sigma \cong \mathbb{P}^2\) and that \(\overline{E}^*\) is a smooth plane curve of degree \(d\). Then we obtain the following.

**Proposition 3.1.** [Oh]

1. \(\psi^*|_{\overline{X}} : \overline{X} \to \mathbb{P}^2\) is a birational morphism with the following properties:
   1. \(\psi^*|_{\overline{X}} : \overline{X} \setminus \{\bigcup_{i=1}^N \overline{E}_i\} \cong \mathbb{P}^2 \setminus \{\bigcup_{i=1}^N \overline{E}_i^*\}; \quad (2) \overline{x} \subset \bigcup_{i=1}^N \overline{E}_i\) and \(x \subset \bigcup_{i=1}^N \overline{E}_i^\circ\).
2. \(0 < N < +\infty, N = b_2(\overline{X}) - 1\).
3. \(\overline{X}\) is a normal hypersurface with, at most, rational double points of \(A_n\) type.
4. \((\overline{X}, \overline{Y} \cup \overline{E})\) is a compactification of \(\mathbb{C}^2\) with:
   1. \(\overline{x} \subset \overline{Y} \cup \overline{E}\); \(\overline{Y} \cup \overline{E}\) does not have any cycles; and (3) \(\omega_{\overline{X}} = \mathcal{O}_{\overline{X}}(K_{\overline{X}}) = \mathcal{O}_{\overline{X}}((d - 4)\overline{Y} - (d - 3)\Sigma|_{\overline{X}}).
5. There exists a birational morphism \(\overline{\pi} : M \to \overline{X}\) which is the minimal resolution of \(\overline{X}\) and satisfies the commutative diagram in Figure 2.
(vi) If \( l \) is a line in \( X \) through \( x_1 \), then \( \hat{l} \) is a \((-1)\)-curve in \( M \) and \( \overline{x} \cap \overline{l} \) consists of, at most, one rational double point of \( A_n \) type. Moreover, if \( \overline{x} \cap \overline{l} \neq \emptyset \), then the weighted dual graph of \( \hat{l} \cup \overline{\pi}^{-1}(\overline{x} \cap \overline{l}) \) is a linear tree \[ \bullet \longrightarrow \cdots \longrightarrow \bullet. \]

**Remark.** By (v), we may assume that, for each \( \overline{E}_i \), \( E_i \) is the proper transform of \( \overline{E}_i \) by \( \overline{\pi} \). We also note that, for a curve \( C \) on \( X \), \( \hat{C} \) coincides with the proper transform of \( \overline{C} \) by \( \overline{\pi} \).

**Lemma 3.2.** If there exists a line \( l \) (\( \neq Y \)) in \( X \) through \( x_1 \), then the following hold:

(i) \( \hat{l} \) is a \((-1)\)-curve in \( M \) with \( (\hat{Y} \cdot \hat{l}) = 0 \), \( (E \cdot \hat{l}) = 1 \);

(ii) \( (M, \hat{Y} \cup E \cup \hat{l}) \) is a nonsingular compactification of \( \mathbb{C} \times \mathbb{C}^* \).

**Proof.** (i) It is obvious that \( (\hat{Y} \cdot \hat{l}) = 0 \). We note that \( \hat{l} \) is a \((-1)\)-curve in \( M \) by Proposition 3.1(vi). Since \( \hat{\Gamma} \sim d\hat{Y} + \sum b_i E_i \) with \( 0 < b_i \in \mathbb{Z} \) (\( \forall i \)) by Proposition 2.5(ii), we obtain

\[ 1 = (\Gamma \cdot l) = (\hat{\Gamma} \cdot \hat{l}) = d(\hat{Y} \cdot \hat{l}) + \sum b_i (E_i \cdot \hat{l}). \]

By the relations \( d \geq 4 \), \( (\hat{Y} \cdot \hat{l}) = 0 \), \( (E_i \cdot \hat{l}) \geq 0 \) and \( b_i > 0 \) (\( \forall i \)), there exists a unique integer \( i_1 \) such that \( b_{i_1} = (E_{i_1} \cdot \hat{l}) = 1 \) and \( (E_i \cdot \hat{l}) = 0 \) for each \( i \neq i_1 \), and hence \( (E \cdot \hat{l}) = 1 \).

(ii) We note that \( M_\alpha := M \setminus (\hat{Y} \cup E) \) is birational to \( \mathbb{C}^2 \) and \( \hat{l} \cap M_\alpha \) is birational to \( \mathbb{C}^1 \). Then, by the Abhyankar–Moh–Suzuki theorem [AM, Su1] (cf. [Su3]), there exists a polynomial automorphism of \( \mathbb{C}^2 \) which transforms \( \hat{l} \cap M_\alpha \) onto the coordinate axis. Hence \( M \setminus (\hat{Y} \cup E \cup \hat{l}) \) is birational to \( \mathbb{C} \times \mathbb{C}^* \).

**Proposition 3.3.** \( Y \) is a unique line in \( X \) through \( x_1 \).

**Proof.** We assume that there exists a line \( l \) (\( \neq Y \)) in \( X \) through \( x_1 \). Then, by Lemma 3.2, we note that \( \hat{l} \) is a \((-1)\)-curve in \( M \) with \( (\hat{Y} \cdot \hat{l}) = 0 \) and there exists a unique integer \( i_1 \) such that \( (E_{i_1} \cdot \hat{l}) = 1 \) and \( (E_i \cdot \hat{l}) = 0 \) for each \( i \neq i_1 \). We also note that \( (M, \hat{Y} \cup E \cup \hat{l}) \) is a nonsingular compactification of \( \mathbb{C} \times \mathbb{C}^* \).
Now we use the same notation as in Section 2 such as $M^*$, $A^*$, $\tau$, $\tau_i$, $M^{(i)}$, $A^{(i)}$, $P_i$, $B_i$. Let us denote by $\tau = \tau_1 \circ \cdots \circ \tau_m : M^{(m)} = M \to \cdots \to M^{(0)} = M^*$ the blowing-downs on $A$ to a minimal normal compactification $(M^*, A^*)$ of $\mathbb{C}^2$.

By Corollary 2.10, there exists an integer $2 \leq j_1 \leq 3$ such that:

1. the weighted dual graph of $A^{(j_1)}$ is

$$
\begin{array}{c}
 b' & 0 & 0 & b' & 0 \\
 0 & 0 & 0 & 0 & 0
\end{array}$$

$(b' \geq 2)$;

2. for any $j \geq j_1$, the weighted dual graph of $A^{(j)}$ is a tree of a unique $(-1)$-curve $B_j$ and smooth rational curves with self-intersection numbers less than or equal to $-2$.

We put $l^{(j)} := (\tau_{j+1} \circ \cdots \circ \tau_m)(\hat{l})$ and $E^{(j)}_{i_1} := (\tau_{j+1} \circ \cdots \circ \tau_m)(E_{i_1})$ for $j \geq 0$. Then we have the two cases whether $l^{(j_1)}$ meets $B_{j_1}$ or not.

(i) We assume that $l^{(j_1)}$ meets $B_{j_1}$. Since $A \cup \hat{l}$ is a curve of normal crossing type with $(\tilde{Y}, \hat{l}) = 0$ and by Lemma 2.8(i), there exists an integer $j_2$ such that $j_1 \leq j_2 < m$ and

(a) $A^{(j_2)} \cup l^{(j_2)}$ is of normal crossing type,
(b) $l^{(j_2)}$ and $B_{j_2}$ meet transversally at one point,
(c) the center $P_{j_2} \in B_{j_2}$ of $\tau_{j_2+1}$ does not coincide with $l^{(j_2)} \cap B_{j_2}$.

By (c), we see that $l^{(j_2)}$ is a $(-1)$-curve in $M^{(j_2)}$ and that the proper transforms of $l^{(j_2)}$ and $B_{j_2}$ in $M$ meet transversally at one point. In particular, we have that $B_{j_2} = E^{(j_2)}_{i_1}$, which is a $(-1)$-curve. Then, by noticing (2) and by blowing-down the $(-1)$-curve $l^{(j_2)}$, we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C} \times \mathbb{C}^*$, which is a tree of a unique smooth rational curve with self-intersection number equal to zero and smooth rational curves with self-intersection numbers less than or equal to $-2$. However, the weighted dual graph of this type is not found in Proposition 2.4. This is a contradiction.

(ii) We assume that $l^{(j_1)}$ does not meet $B_{j_1}$. Since the centers of $\tau_{j_1+1}, \ldots, \tau_m$ lie on $B_{j_1}$ and since $\hat{l}$ intersects $A$ transversally at one point, we have that $l^{(j_1)}$ intersects one of $A_1$, $A_2$ and $A_3$ transversally at one point, where $A_i$ is an irreducible component of $A^{(j_1)}$ as in (1). By blowing-down $B_{j_1}, A_3$ and some curves successively, we obtain the weighted dual graphs of boundaries of minimal normal compactifications of $\mathbb{C} \times \mathbb{C}^*$ as follows:

$$
\begin{array}{c}
 1 & -b' & 0 & -b' & 1 & -b' & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$(b' \geq 2)$.

However, the weighted dual graphs of these types are not found in Proposition 2.4. This is a contradiction.

$\square$
PROPOSITION 3.4. (cf. [Ts])

(i) \( \overline{E} = \overline{\Sigma} \cap \Sigma \) is a line in \( \Sigma \cong \mathbb{P}^2 \). In particular, \( \overline{E} = \overline{E_1} \) and \( \Sigma \cap \overline{E_1} = (d - 1) \overline{E_1} \).

(ii) \( \overline{x} = \overline{Y} \cap \overline{E_1} \) and the weighted dual graph of \( \overline{Y} \cup \overline{\pi^{-1}(x)} \) is \( \bullet \cdots \cdots \circ \).

(iii) \( \overline{x} \) is a rational double point of \( A_{d^2 - d - 1} \)-type and \( x = \{x_1\} \).

(iv) \( b_2(E) = d^2 - d, \ E_1^2 = -(d - 1), \ E_i^2 = -2 \ (\forall i \neq 1) \).

Proof. (i) Since we obtain the isomorphism \( \overline{\psi}|_\overline{X} : \overline{X} \setminus \overline{Y} \cong \mathbb{P}^2 \setminus \overline{Y}^* \) by Proposition 3.3, we also obtain an isomorphism \( \mathbb{P}^2 \setminus \overline{E}^* \cong \overline{X} \setminus (\overline{Y} \cup \overline{E}) \cong X \setminus \overline{Y} \cong \mathbb{C}^2 \) and hence \( (\mathbb{P}^2, \overline{E}^*) \) is a compactification of \( \mathbb{C}^2 \). Thus \( \overline{E}^* \) is a line in \( \mathbb{P}^2 \) and hence \( \overline{E} \) is a line in \( \Sigma \cong \mathbb{P}^2 \).

(ii) If \( \overline{x} \cap \Sigma = \emptyset \), then \( \pi^{-1}(x_1) \) is a smooth rational curve with self-intersection number less than or equal to \(-2\). Thus we get \( p_x(x_1) = 0 \), which is a contradiction. Hence we obtain \( \overline{x} \cap \Sigma \neq \emptyset \). By the isomorphism \( \overline{\psi}|_\overline{X} : \overline{X} \setminus \overline{Y} \cong \mathbb{P}^2 \setminus \overline{Y}^* \), we have that \( \overline{x} \subset \overline{Y} \) and \( \overline{x} \cap \Sigma = \overline{Y} \cap \overline{E_1} \). Since \( \overline{x} \subset \overline{Y} \) and by Proposition 3.1(vi), we have that \( \overline{x} \) is one rational double point of \( A_n \)-type and \( \overline{x} = \overline{Y} \cap \overline{E_1} \), and we get the weighted dual graph of \( \overline{Y} \cup \overline{\pi^{-1}(x)} \) as required.

(iii) Since \( \overline{x} = \overline{Y} \cap \overline{E_1} \), we obtain that \( \overline{x} \subset \overline{E_1} \subset \Sigma \) and hence \( x \) is one point \( x_1 \). Now we note that \( K_M \sim \pi^* K_X \) by [Ar], \( K_X \sim (d - 4) \Gamma - (d - 3) \Sigma|_X \) and \( b_2(E) = \# \text{Exc} \pi = \# \text{Exc} \pi + 1 = \# \text{Exc}(\overline{\psi}|_X \circ \overline{\pi}) \), where we denote by \( \# \text{C} \) the number of irreducible components of a curve \( C \). By the projection formula and the equalities \( (\Gamma^2)_X = d, (\Gamma \cdot \Sigma|_X)_X = 0 \) and \( (\Sigma^2|_X)_X = -(d - 1) \), we obtain

\[
K_M^2 = \# \text{Exc}(\overline{\psi}|_X \circ \overline{\pi}) = K_M^2 = K_X^2 = -d^2 + d + 9.
\]

Hence we have that \( b_2(E) = \# \text{Exc}(\overline{\psi}|_X \circ \overline{\pi}) = d^2 - d \) and \( \overline{x} \) is a rational double point of \( A_{d^2 - d - 1} \)-type.

(iv) By the projection formula and the adjunction formula for \( E_1 \cong \mathbb{P}^1 \), we have that \( (K_M \cdot E_1)_M = (K_X \cdot E_1)_X = d - 3 \) and \( E_1^2 = -(d - 1) \). Since \( \overline{x} \) is a rational double point, we obtain \( E_i^2 = -2 \) for each \( i \neq 1 \).

Now we put \( \phi := \overline{\psi}|_X \circ \overline{\pi} \). Since we have \#Exc \( \phi = b_2(E) = d^2 - d \) by Proposition 3.4(iv), we note that \( \phi \) is a composite of \( (d^2 - d) \) blowing-ups. We also note that \( L := \phi(A) = \phi(E_1) \) is a line in \( \mathbb{P}^2 \) and \( P := \phi(\overline{Y} \cup \cup_{i=2}^{d^2 - d} E_i) \) is a point of \( L \).
Normal hypersurfaces

PROPOSITION 3.5. (cf. [Ts]) The weighted dual graph of $A$ is

![Diagram](image)

Proof. By Proposition 3.4(ii) and (iv), we have the following:

1. The weighted dual graph of $\phi^{-1}(P) = \hat{Y} \cup (\bigcup_{i=2}^{d^2-d} E_i) = \hat{Y} \cup \varpi^{-1}(\bar{x})$ is a linear tree $\bullet \to \cdots \to \bullet$.
2. $(E_1^2) = -(d-1)$, $(L^2) = (\phi(E_1)^2) = 1$.

By noticing (1) and (2), we blow-up $\mathbb{P}^2$ at $(d^2 - d)$ times on infinitely near points of $P \in L$. Indeed, we first blow-up at $P$ and blow-up at $(d-1)$ times on the intersection points of the new $(-1)$-curves and the proper transforms of $L$. Moreover, we blow-up at $(d^2 - 2d)$ times on general points of the new $(-1)$-curves. Then we get the weighted dual graph of $A = \hat{Y} \cup E = \phi^{-1}(L)$ as required.

By the shape of the weighted dual graph of $A$ and the equality $(\hat{G} \cdot \hat{Y}) = (\Gamma \cdot Y) = 1$, the image $G := \phi(\hat{G})$ is a smooth plane curve of degree $d$ with $L \cap G = \{P\}$ and hence $L \cdot G = dP$. The birational map $\Phi := \pi \circ \phi^{-1} : \mathbb{P}^2 \to X$, which has a point of indeterminacy at $P$, gives an isomorphism $\mathbb{P}^2 \setminus L \cong X \setminus Y$. We easily see that the commutative diagram in Figure 3 gives a resolution of indeterminacy of $\Phi$.

![Diagram](image)

**Figure 3.**

Let $(w_0 : w_1 : w_2)$ be a homogeneous coordinate system of $\mathbb{P}^2$. We may assume that $L = \{w_2 = 0\}$ and $P = (0 : 1 : 0)$. Then we easily see that the birational map
\( \Phi_d := \Phi \) is given by the defining equation

\[
\Phi_d : \begin{cases}
z_0 &= w_0 w_2^{d-1} \\
z_1 &= w_1 w_2^{d-1} \\
z_2 &= w_2^d \\
z_3 &= \sum_{0 \leq i+j \leq d} c_{ij} w_0^i w_1^j w_2^{d-i-j},
\end{cases}
\]

where \( c_{ij} \in \mathbb{C} \) satisfy

\[
c_{0,d} = c_{1,d-1} = c_{2,d-2} = \cdots = c_{d-1,1} = 0, \\
c_{d,0} = c_{0,d-1} = 1,
\]

and that \((X, Y) \cong (V_d, \Delta_d)\) (up to automorphisms of \( \mathbb{P}^3 \)), where \( V_d \) and \( \Delta_d \) are defined as follows:

\[
V_d := \left\{ (z_0 : z_1 : z_2 : z_3) \in \mathbb{P}^3 ; z_2^{d-1} z_3 = \sum_{0 \leq i+j \leq d} c_{ij} z_0^i z_1^j z_2^{d-i-j} \right\},
\]

\[
\Delta_d := \{ z_0 = z_2 = 0 \}.
\]

Thus we obtain Theorem 1.7.

Acknowledgements. The author would like to express his hearty gratitude to Professor Mikio Furushima for his invaluable advice and helpful discussions. Furthermore, the author is grateful to the referee for his suggestions and corrections.

REFERENCES


Normal hypersurfaces


Tomoaki Ohta
Graduate School of Mathematics
Kyushu University
Hakozaki 6-10-1
Higashi-ku
Fukuoka, 812-8581
Japan
(E-mail: ohta@math.kyushu-u.ac.jp)