ON THE 3-COMPONENT OF $SU(n)$ AS A FRAMED MANIFOLD

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1. Introduction

Let G be a compact connected Lie group of dimension d . Then G with its left invariant framing L defines via the Thom–Pontrijagin construction an element $[G, L]$ in the stable homotopy group π_d^S of spheres. When G is positive dimensional, there is a problem of estimating the order of $[G, L]$. Concerning this there is the following elegant result:

$$72[G, L] = 0, \tag{1}$$

which is proved in [7].

In the present work we shall consider the odd component of $[SU(n), L]$. The result is the following.

THEOREM. *The odd component of $[SU(n), L]$ vanishes for $n \geq 3$.*

Combining this with (1) we have the following.

COROLLARY. $8[SU(n), L] = 0$ for $n \geq 3$.

Here we collect the other known information about $SU(n)$ from (1). In [7, Table 1] we can find concrete examples of which $[SU(3), L] = \bar{v} \in \pi_8^S = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cdot \bar{v}$ [8, 9], $[SU(4), L] = \eta\kappa \in \pi_{15}^S = \mathbb{Z}/480 \oplus \mathbb{Z}/2 \cdot \eta\kappa$ [2, 5] and $[SU(5), L] = \eta^*\sigma\eta$ or $0 \in \pi_{24}^S = \mathbb{Z}/6 \oplus \mathbb{Z}/2 \cdot \eta^*\sigma\eta$ [2] with the notations of Toda. We have such results of general nature as $v[SU(n), L] = 0$ for $n \equiv 0$ or $5 \pmod{8}$ and $\eta[SU(2n), L] = 0$ [2]. Furthermore we have

$$2[SU(n), L] = 0, \quad \text{for } n \equiv 0, \text{ or } 3 \pmod{4} \tag{2}$$

[6]. This can be easily shown by making use of the involutive automorphism of $SU(n)$ associated with the symmetric pair $(SU(n), SO(n))$ with the fixed point subgroup $SO(n)$ of odd codimension. From the corollary and (2) we may conjecture that at least $2[SU(n), L] = 0$ for $n \geq 3$.

2. Proof of the theorem

The remainder of this paper is devoted to the proof of the theorem. Let G be as above and ρ an orthogonal representation of G of dimension k . Denote by V the representation space of ρ and by $S(V)$ its unit sphere. Then the action of G on $S(V)$, $G \times S(V) \rightarrow S(V)$, determines a map $J(\rho) : G \wedge S^k \rightarrow S^k$ via the Hopf construction. We may view $J(\rho)$ as an element of the reduced zero-dimensional stable cohomotopy group $\pi_S^0(G)$ of G . It is well known that there is a natural map $J : \widetilde{KO}^{-1}(X) \rightarrow \pi_S^0(X)$ satisfying

$$J(x + y) = J(x) + J(y) + J(x)J(y) \quad \text{and} \quad J(0) = 0. \quad (3)$$

Regarding the composition $G \rightarrow O(k) \rightarrow O$ of ρ and the canonical inclusion $O(k) \subset O$ as an element of $\widetilde{KO}^{-1}(G) = [G, O]$, the $J(\rho)$ mentioned above agrees with the image of this element under the map J where O denotes the orthogonal group of infinite degree.

We make one more preparation to mention a result of Becker and Schultz [2] which is a main tool to our proof of the theorem. Let U be such a neighbourhood of the unit element of G as is homeomorphic to an open ball of dimension d . Then $G/G - U = S^d$ and the cofibre sequence $G - U \xrightarrow{i} G \xrightarrow{j} S^d$ gives rise to a short exact sequence

$$0 \rightarrow \pi_S^0 = \pi_S^0(S^d) \xrightarrow{j^*} \pi_S^0(G) \xrightarrow{i^*} \pi_S^0(G - U) \rightarrow 0$$

where i and j denote the inclusion and the canonical collapsing map, respectively. In fact, the left-inverse $\pi_S^0(G) \rightarrow \pi_S^0(S^d)$ to j^* is given as follows. Denote by $[G]$ the fundamental class of the framed manifold (G, L) in $\pi_d^S(G)$ and by $\langle \cdot, \cdot \rangle_\pi$ the Kronecker product $\pi_S^0(G) \times \pi_d^S(G) \rightarrow \pi_S^0(S^d)$. Then we find that the assignment $x \mapsto \langle x, [G] \rangle_\pi$ defines a left-inverse to j^* . We identify an element of π_d^S with its image by j^* and consider π_d^S a subgroup of $\pi_S^0(G)$ below.

Now we take for G the classical groups and for ρ the realification of the identity map representations of them. Write as $\rho = \rho_n^R, \rho_n^C$ or ρ_n^H according to $G = SO(n), SU(n)$ or $Sp(n)$. From [2, Theorem (5.3)] we know that

$$\begin{aligned} [SO(2n), L] &= J(\rho_{2n}^R)^n (2 - J(\rho_{2n}^R))^{n-1}, \\ [SO(2n+1), L] &= J(\rho_{2n+1}^R)^n (2 - J(\rho_{2n+1}^R))^n, \\ [SU(n), L] &= J(\rho_n^C)^{n-1} \end{aligned}$$

and

$$[Sp(n), L] = J(\rho_n^H)^n. \quad (4)$$

Note that we have

$$J(\rho_n^H)^{n+1} = 0 \quad (5)$$

by observing the short exact sequence above together with the last formula of (4). Because $2[SO(2n+2), L] = 0$ for $n \geq 1$ [2, 4], we also have $J(\rho_{2n+2}^R)_{(\text{odd})}^{n+1} = 0$ by the first formula of (4) and the fact that $J(\rho_{2n+2}^R)$ is a nilpotent element. So by taking its restriction to $SO(2n+1)$ we obtain

$$J(\rho_{2n+1}^R)_{(\text{odd})}^{n+1} = 0 \quad (6)$$

where the subscript ‘(odd)’ attached to the element indicates its odd component.

Here we consider the symmetric pair

$$(G, K) = (SU(2n+1), SO(2n+1)) \quad \text{or} \quad (SU(2n), Sp(n)).$$

Denoting by σ its involutive automorphism of G leaving K fixed, we have a map

$$q : G/K \rightarrow G$$

given by

$$q(gK) = g\sigma(g)^{-1}$$

and using this we get a map

$$\alpha : G/K \times K \rightarrow G$$

which is defined by

$$\alpha(gK, k) = q(gK)i(k)$$

where i is the inclusion of K in G . By [3] we know that the map α is a p -equivalence for any odd prime p . Hence

$$\alpha^* : \pi_S^0(G) \otimes \mathbb{Z}(\frac{1}{2}) \cong \pi_S^0(G/K \times K) \otimes \mathbb{Z}(\frac{1}{2})$$

follows where $\mathbb{Z}(\frac{1}{2})$ denotes the subring of the rationals generated by $\frac{1}{2}$.

This fact is another key to prove our theorem. In [1] we also have a different proof for this, which is achieved by showing that the Brouwer degree (precisely local degrees) of the composition $\pi q : G/K \rightarrow G/K$ is a power of two where π denotes the projection of G onto G/K . From this it follows that $(\pi q)^* : H^*(G/K, \mathbb{Z}(\frac{1}{2})) \rightarrow H^*(G/K, \mathbb{Z}(\frac{1}{2}))$ is a bijection [1, (1.3)]. Therefore we also see that

$$(\pi \times 1)^* : \pi_S^0(G/K \times K) \otimes \mathbb{Z}(\frac{1}{2}) \rightarrow \pi_S^0(G \times K) \otimes \mathbb{Z}(\frac{1}{2})$$

is a monomorphism.

Proof of Theorem. First, let us consider the case where $(G, K) = (SU(2n+1), SO(2n+1))$. Assume that the cohomotopy groups $\pi_S^0(X)$ are tensored with $\mathbb{Z}(\frac{1}{2})$ and write $\rho = \rho_{2n+1}^C$ for short.

Analyse $J(\rho)$ thorough this isomorphism α^* . Then from (3) we see

$$\begin{aligned}\alpha^* J(\rho) &= J(p_1^* q^* \rho \oplus p_2^* i^* \rho) \\ &= p_1^* q^* J(\rho) + p_2^* i^* J(\rho) + p_1^* q^* J(\rho) p_2^* i^* J(\rho) \\ &= p_1^* q^* J(\rho) + p_2^* J(\rho_{2n+1}^R) + p_1^* q^* J(\rho) p_2^* J(\rho_{2n+1}^R)\end{aligned}$$

where p_1 and p_2 denote the projections from $G/K \times K$ onto the first and second factors respectively.

From the facts that σ is complex conjugation and $J(0) = 0$ by (3) it now follows that

$$(\pi \times 1)^* p_1^* q^* J(\rho) = 0.$$

Therefore we see that $p_1^* q^* J(\rho) = 0$ because $(\pi \times 1)^*$ is a monomorphism. Consequently we obtain

$$\alpha^* J(\rho) = p_2^* J(\rho_{2n+1}^R)$$

by the above formula.

From (6) it follows that $\alpha^* J(\rho)^{n+1} = 0$. Thus we have $J(\rho)^{n+1} = 0$ because α^* is injective. So $J(\rho)^{2n} = 0$ for $n \geq 1$. This and the third formula of (4) imply immediately that $[SU(2n+1), L]_{(\text{odd})} = 0$ for $n \geq 1$.

The proof for the case where $(G, K) = (SU(2n), Sp(n))$ is quite parallel to that of the first case, but in this case we take complex conjugation followed by an inner automorphism for σ instead of mere complex conjugation.

Analogously we have $J(\rho_{2n}^C)^{n+1} = 0$ by (5) and so $J(\rho_{2n}^C)^{2n-1} = 0$ for $n \geq 2$. From the fourth formula of (4) it therefore follows that $[SU(2n), L]_{(\text{odd})} = 0$ for $n \geq 2$. This completes the proof of the theorem. $\square$

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