ON RIGIDITY OF PFAFFIAN SYSTEMS COMING FROM OKUBO SYSTEMS

Yoshishige HARAOKA and Toshiaki YOKOYAMA

(Received 25 October 1999 and revised 30 May 2000)

0. Introduction

Let n be an integer, A an $n \times n$ constant matrix satisfying

$$(A - \rho_1 I_n)(A - \rho_2 I_n) = O \quad (0.1)$$

for some complex numbers ρ_1, ρ_2 ($\rho_1 \neq \rho_2$), and B an $n \times n$ constant diagonal matrix of the form

$$B = \begin{pmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_p I_{n_p} \end{pmatrix}, \quad (0.2)$$

where $\lambda_1, \lambda_2, \dots, \lambda_p$ are mutually distinct complex numbers, and

$$\bar{n} := (n_1, n_2, \dots, n_p)$$

is a partition of n . In this paper we study the rigidity of the local system defined by the Pfaffian system

$$dZ = \Omega Z, \quad \Omega = (xI_n - B)^{-1} A dx + (A - (\rho_1 + \rho_2)I_n)(yI_n - B)^{-1} dy - A \frac{d(x-y)}{x-y}, \quad (0.3)$$

which is introduced in [Yok1] and studied in [Yok2], [T] and [H3]. The system (0.3) is regular on $X = \mathbf{P}^1 \times \mathbf{P}^1 \setminus (S \cup \{(x, y); y = \infty\})$, has regular singularity along $S \cup \{(x, y); y = \infty\}$, where

$$S = \bigcup_{k=1}^p (\{(x, y); x = \lambda_k\} \cup \{(x, y); y = \lambda_k\}) \cup \{(x, y); x = y\}, \quad (0.4)$$

and so it defines a local system of rank n on X .

Let S' be a non-empty finite subset of $\mathbf{P}^1$. In [K] Katz defines that a local system $\mathcal{F}'$ on $X' = \mathbf{P}^1 \setminus S'$ is *rigid* if for every local system $\mathcal{G}'$ on X' such that $\mathcal{F}'$ and $\mathcal{G}'$ have isomorphic local monodromy at each point in $\mathbf{P}^1 \setminus X'$, $\mathcal{G}'$ is isomorphic to $\mathcal{F}'$. After Katz we say that a local system $\mathcal{F}$ on X is *rigid* if for every local system $\mathcal{G}$ on X such that $\mathcal{F}$ and $\mathcal{G}$ have isomorphic local monodromy around every point in $\mathbf{P}^1 \times \mathbf{P}^1 \setminus X$, $\mathcal{G}$ is isomorphic to $\mathcal{F}$. Our main result is that the local system defined by the Pfaffian system (0.3) is rigid if and only if the local system defined by the system

$$(xI_n - B) \frac{dY}{dx} = AY \quad (0.5)$$

of ordinary differential equations is rigid. We note that the system (0.5) is called a system of Okubo normal form, which is introduced and studied by Okubo [O]. In particular, there is an algorithm to determine the rigidity of the local system defined by (0.5), which we shall explain at the end of Section 1.

We collect fundamental results on the monodromy representation of (0.3) in Section 1. In Section 2 we study relations among circuit matrices for the system (0.3) which come from relations of generators of $\pi_1(X)$. Using these results, we prove our main theorem in Section 3.

1. Monodromy representation of the system (0.3)

In this section we fix the notation, quote several results from [Yok1] and give further results necessary for our purpose.

From (0.1) it follows that

$$A \sim \begin{pmatrix} \rho_1 I_{m_1} & \\ & \rho_2 I_{m_2} \end{pmatrix}, \quad (1.1)$$

where

$$\bar{m} := (m_1, m_2)$$

is a partition of n , and $\sim$ implies that the matrices of both sides are similar. We partition the matrix A into $\bar{n} \times \bar{n}$ -blocks:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{pmatrix}, \quad A_{ij} \in \text{Mat}(n_i, n_j; \mathbb{C}),$$

where $\bar{n}$ is the partition which describes the multiplicities of the eigenvalues of B . Let $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n_i}$ be the eigenvalues of the diagonal block A_{ii} for $i = 1, \dots, p$. We assume that, for each $i = 1, 2, \dots, p$, the eigenvalues $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n_i}$ are distinct.

The system (0.3) is completely integrable by virtue of (0.1) and (0.2), with singularities along S and possibly at infinity, where S is given by (0.4).

In order to study the monodromy representation of the system (0.3), we describe the fundamental group of $\mathbb{C}^2 \setminus S$. Take a base point $(x_0, y_0) \in \mathbb{C}^2 \setminus S$. The loops μ_k, ν_k ($0 \leq k \leq p$) in $\mathbb{C}^2 \setminus S$ are specified in [Yok1, Section 3]. We do not repeat the precise definition, and we only notice that μ_k (respectively ν_k) is a loop in the plane $y = y_0$ (respectively $x = x_0$) which encircles $x = y_0$ (respectively $y = x_0$) for $k = 0$ and $x = \lambda_k$ (respectively $y = \lambda_k$) for $1 \leq k \leq p$ once in the positive direction. For a loop μ in $\mathbb{C}^2 \setminus S$ we denote by $[\mu]$ the homotopy class of μ . Then we have the following.

PROPOSITION 1.1.

- (i) $\mu_0 \mu_p \mu_{p-1} \cdots \mu_1$ (respectively $\nu_0 \nu_1 \nu_2 \cdots \nu_p$) is homotopic to a loop on $y = y_0$ (respectively $x = x_0$) encircling $x = \infty$ (respectively $y = \infty$) once in the negative direction.
- (ii) $[\mu_0] = [\nu_0]$.

Applying the results of Randell [R], Salvetti [S] and Arvola [A, OT] on the presentation for the fundamental group of the complement of an arrangement of lines in $\mathbb{C}^2$, we obtain the following proposition. For words u, v in a free group we set $u^v = v^{-1}uv$.

PROPOSITION 1.2. *The fundamental group of $\mathbb{C}^2 \setminus S$ is generated by $[\mu_0], [\mu_1], \dots, [\mu_p], [\nu_1], [\nu_2], \dots, [\nu_p]$, and*

$$[\mu_i][\nu_j] = [\nu_j][\mu_i] \quad (1 \leq i, j \leq p, i \neq j), \quad (1.2)$$

$$\begin{aligned} [\nu_k][\mu_0]^{[\mu_p][\mu_{p-1}] \cdots [\mu_{k+1}]}[\mu_k] &= [\mu_0]^{[\mu_p][\mu_{p-1}] \cdots [\mu_{k+1}]}[\mu_k][\nu_k] \\ &= [\mu_k][\nu_k][\mu_0]^{[\mu_p][\mu_{p-1}] \cdots [\mu_{k+1}]} \quad (1 \leq k \leq p) \end{aligned} \quad (1.3)$$

form fundamental relations.

We assume

$$\alpha_{k,h}, \alpha_{k,h} - \alpha_{k,h'} \notin \mathbb{Z} \quad (1 \leq k \leq p, 1 \leq h, h' \leq n_k, h \neq h'), \quad (1.4)$$

$$\rho_1, \rho_2 \notin \mathbb{Z}_{<0}, \quad \rho_1 - \rho_2 \notin \mathbb{Z}, \quad (1.5)$$

and set

$$e_{k,h} = \exp(2\pi\sqrt{-1}\alpha_{k,h}) \quad (1.6)$$

for $1 \leq k \leq p$ and $1 \leq h \leq n_k$. Then there exists a fundamental solution $Z(x, y)$ of (0.3) such that, for $1 \leq i \leq p$, the circuit matrix M_i (respectively N_i) for the loop μ_i (respectively ν_i) with respect to $Z(x, y)$ has the form

$$M_i = \begin{pmatrix} I_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{n_{i-1}} & & & & \\ M_{i1} & \cdots & M_{i,i-1} & M_{ii} & M_{i,i+1} & \cdots & M_{ip} \\ & & & I_{n_{i+1}} & & & \\ & & & & \ddots & & \\ & & & & & & I_{n_p} \end{pmatrix} \quad (1.7)$$

with

$$M_{ii} = \begin{pmatrix} e_{i,1} & & & \\ & e_{i,2} & & \\ & & \ddots & \\ & & & e_{i,n_i} \end{pmatrix}, \quad (1.8)$$

respectively

$$N_i = \begin{pmatrix} I_{n_1} & & N_{1i} & & \\ & \ddots & \vdots & & \\ & & I_{n_{i-1}} & N_{i-1,i} & \\ & & & N_{ii} & \\ & & & N_{i+1,i} & I_{n_{i+1}} \\ & & & \vdots & \\ & & & N_{pi} & \\ & & & & \ddots & \\ & & & & & I_{n_p} \end{pmatrix}, \quad (1.9)$$

where we have partitioned M_i and N_i into $\bar{n} \times \bar{n}$ -blocks.

We denote by $[C]_{ij}$ the (i, j) -block of an $n \times n$ -matrix C with respect to the partition $\bar{n}$. By (1.7) and (1.9) we have

$$[M_1 M_2 \cdots M_k]_{kj} = M_{kj}, \quad (1.10)$$

$$[N_k N_{k-1} \cdots N_1]_{ik} = N_{ik} \quad (1.11)$$

for $i, j, k = 1, 2, \dots, p$.

We denote by M_0 the circuit matrix for the loop μ_0 with respect to the fundamental solution $Z(x, y)$, and set

$$f_1 = \exp(-2\pi\sqrt{-1}\rho_1), \quad f_2 = \exp(-2\pi\sqrt{-1}\rho_2). \quad (1.12)$$

Then the circuit matrices M_i 's and N_i 's satisfy

$$M_1 M_2 \cdots M_p M_0 = I_n, \quad (1.13)$$

$$N_p N_{p-1} \cdots N_1 M_0 = f_1 f_2 I_n, \quad (1.14)$$

$$N_p N_{p-1} \cdots N_1 = f_1 f_2 M_1 M_2 \cdots M_p, \quad (1.15)$$

$$M_0 \sim \begin{pmatrix} f_1 I_{m_1} & \\ & f_2 I_{m_2} \end{pmatrix}. \quad (1.16)$$

In fact, with the help of Proposition 1.1, the local analysis at $x = \infty$, $y = \infty$ and $x = y$ yields (1.13), (1.14) and (1.16), respectively. The equality (1.15) directly follows from (1.13) and (1.14).

The blocks N_{ij} ($1 \leq i, j \leq p$) are determined by f_1 , f_2 and $M_{k\ell}$ ($1 \leq k, \ell \leq p$) as follows. For the diagonal blocks we have

$$N_{kk} = f_1 f_2 M_{kk} \quad (1.17)$$

for $1 \leq k \leq p$. For the other blocks we have the following.

PROPOSITION 1.3. *The blocks N_{ij} ($1 \leq i, j \leq p$) can be written polynomially in terms of f_1 , f_2 and the blocks $M_{k\ell}$ ($1 \leq k, \ell \leq p$) and M_{kk}^{-1} ($1 \leq k \leq p$).*

Proof. Put $k = p$ in (1.11) and use (1.15) to obtain

$$N_{ip} = [N_p N_{p-1} \cdots N_1]_{ip} = [f_1 f_2 M_1 M_2 \cdots M_p]_{ip}$$

for $1 \leq i \leq p$. By noting (1.9), this shows that N_p is determined by the blocks $M_{k\ell}$. Again using (1.11) and (1.15), we get

$$N_{ik} = [(N_p N_{p-1} \cdots N_{k+1})^{-1} f_1 f_2 M_1 M_2 \cdots M_p]_{ik}$$

for $1 \leq i, k \leq p$, which shows that the blocks in N_k are recursively determined by the blocks in $N_p^{-1}, \dots, N_{k+1}^{-1}$ and the blocks $M_{j\ell}$. Noting

$$N_i^{-1} = \begin{pmatrix} I_{n_1} & & & -N_{1i} N_{ii}^{-1} & & \\ & \ddots & & \vdots & & \\ & & I_{n_{i-1}} & -N_{i-1,i} N_{ii}^{-1} & & \\ & & & N_{ii}^{-1} & & \\ & & & -N_{i+1,i} N_{ii}^{-1} & I_{n_{i+1}} & \\ & & & \vdots & & \ddots \\ & & & -N_{pi} N_{ii}^{-1} & & & I_{n_p} \end{pmatrix}$$

and the relation (1.17), we obtain the assertion of the proposition. $\square$

Tracing the argument in the above proof, we obtain the following.

COROLLARY 1.4. *For each $1 \leq i, j \leq p$ take any $n_i \times n_j$ -matrix M_{ij} satisfying $\det M_{ii} \neq 0$ ($1 \leq i \leq p$), and define M_i ($1 \leq i \leq p$) by (1.7). Define N_{ij} ($1 \leq i, j \leq p$) by (1.17) and the manner given in the proof of Proposition 1.3, and define N_i ($1 \leq i \leq p$) by (1.9). Then $N_1, N_2, \dots, N_p$ satisfy (1.15).*

The matrices $M_0, M_1, \dots, M_p$ are also circuit matrices of the system (0.5). The following proposition holds.

PROPOSITION 1.5. *There exists a fundamental solution $Y(x)$ of (0.5) such that $M_0, M_1, \dots, M_p$ become the circuit matrices with respect to $Y(x)$.*

Proof. If we set $W(x, y) = y^{\rho_1 + \rho_2} Z(x, y)$, W is holomorphic at $y = \infty$ by virtue of (0.3). We can take a path α starting at (x_0, ∞) and ending at (x_0, y_0) such that, for $1 \leq k \leq p$, $\alpha \cdot \mu_k \cdot \alpha^{-1}$ is homotopic to a loop μ'_k on the plane $y = \infty$ which encircles $x = \lambda_k$ once in the positive direction, and that the loop $\mu'_0 := \mu'_p \mu'_{p-1} \cdots \mu'_1$ is homotopic to a loop on $y = \infty$ which encircles $x = \infty$ once in the negative direction. Let $Y(x)$ be the section on $y = \infty$ of the analytic continuation of $W(x, y)$ along the path α . Then we see that $Y(x)$ becomes a fundamental solution of (0.5), and that, for $0 \leq k \leq p$, the circuit matrix for μ'_k with respect to $Y(x)$ coincides with M_k . Thus the M_k 's become the circuit matrices with respect to $Y(x)$. $\square$

We set

$$a(n; \bar{n}, \bar{m}) := n^2 - n + 2 - \sum_{k=1}^p n_k^2 - \sum_{i=1}^2 m_i^2.$$

Under conditions (1.4) and (1.5), we can write

$$\begin{aligned} a(n; \bar{n}, \bar{m}) &= (p-1)n^2 + 2 - \sum_{k=0}^p \dim Z(M_k) \\ &= 2 - \left\{ (2 - (p+1))n^2 + \sum_{k=0}^p \dim Z(M_k) \right\}, \end{aligned}$$

where $Z(M)$ denotes the centralizer of a matrix M . It is known that, if the local system $\mathcal{H}$ defined by (0.5) is irreducible, then we have $a(n; \bar{n}, \bar{m}) \geq 0$, and in this case $\mathcal{H}$ is rigid if and only if $a(n; \bar{n}, \bar{m}) = 0$ $[\mathbf{O}, \mathbf{K}]$. Thus we can determine the rigidity of $\mathcal{H}$ by using the partitions $\bar{n}$ and $\bar{m}$.

2. Relations among the circuit matrices

The following is a direct consequence of Proposition 1.2.

PROPOSITION 2.1. *The relations among the circuit matrices M_i 's and N_i 's derived from the relations among the loops μ_i 's and v_i 's are generated by the relations*

$$M_i N_j = N_j M_i \quad (1 \leq i, j \leq p, i \neq j), \quad (2.1)$$

$$\begin{aligned} & M_k M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} N_k \\ &= N_k M_k M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} \\ &= M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} N_k M_k \quad (1 \leq k \leq p). \end{aligned} \quad (2.2)$$

In this section we show that the relations in Proposition 2.1 follow from the relations (1.7), (1.9), (1.13), (1.15) and (1.16). In other words, if we determine the matrices M_i 's and N_j 's so as to satisfy the relations (1.7), (1.9), (1.13), (1.15) and (1.16), then the relations in Proposition 2.1 will be automatically satisfied.

First we derive the relation (2.1).

PROPOSITION 2.2. *The relation (2.1) follows from (1.7), (1.9), (1.13), (1.15) and (1.16).*

We set

$$\begin{aligned} M &= \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1p} \\ M_{21} & M_{22} & \cdots & M_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ M_{p1} & M_{p2} & \cdots & M_{pp} \end{pmatrix}, & N &= \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1p} \\ N_{21} & N_{22} & \cdots & N_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ N_{p1} & N_{p2} & \cdots & N_{pp} \end{pmatrix}, \\ L &= \begin{pmatrix} M_{11} & & & \\ M_{21} & M_{22} & & \\ \vdots & \vdots & \ddots & \\ M_{p1} & M_{p2} & \cdots & M_{pp} \end{pmatrix}, & U &= \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1p} \\ & N_{22} & \cdots & N_{2p} \\ & & \ddots & \vdots \\ & & & N_{pp} \end{pmatrix}. \end{aligned}$$

Then in a similar manner to [O, Chapter 3, Lemma 1] we obtain the following.

LEMMA 2.3. *We have*

$$M_1 M_2 \cdots M_p = (I_n + L - M)^{-1} L, \quad (2.3)$$

$$N_p N_{p-1} \cdots N_1 = U (I_n + U - N)^{-1}. \quad (2.4)$$

Proof of Proposition 2.2. First we note that, from (1.13) and (1.16), we obtain

$$(M_1 M_2 \cdots M_p - f_1^{-1} I_n)(M_1 M_2 \cdots M_p - f_2^{-1} I_n) = O. \quad (2.5)$$

Step 1. In this step we prove $M_i N_p = N_p M_i$ for $i \neq p$. By a direct calculation it is easy to see that

$$[M_i N_p - N_p M_i]_{kj} = O$$

holds for all (k, j) except (i, p) . For the (i, p) -block, using (1.11) with $k = p$, (1.15) and the fact that $[I_n]_{ip} = O$ ($i \neq p$), we have

$$\begin{aligned} [M_i N_p - N_p M_i]_{ip} &= \sum_{j=1}^p M_{ij} N_{jp} - M_{ip} - N_{ip} \\ &= \sum_{j=1}^p M_{ij} [N_p N_{p-1} \cdots N_1]_{jp} - M_{ip} - [N_p N_{p-1} \cdots N_1]_{ip} \\ &= \sum_{j=1}^p M_{ij} [f_1 f_2 M_1 M_2 \cdots M_p]_{jp} - M_{ip} - [f_1 f_2 M_1 M_2 \cdots M_p]_{ip} \\ &= [M \cdot f_1 f_2 M_1 M_2 \cdots M_p]_{ip} - M_{ip} - [f_1 f_2 M_1 M_2 \cdots M_p]_{ip} \\ &= [M \cdot f_1 f_2 M_1 M_2 \cdots M_p - M - f_1 f_2 M_1 M_2 \cdots M_p + I_n]_{ip} \\ &= [(M - I_n)(f_1 f_2 M_1 M_2 \cdots M_p - I_n)]_{ip}. \end{aligned}$$

From (2.3) we obtain

$$L = (I_n + L - M)M_1 M_2 \cdots M_p,$$

and hence

$$\begin{aligned} M - I_n &= L - (I_n + L - M) \\ &= (I_n + L - M)M_1 M_2 \cdots M_p - (I_n + L - M) \\ &= (I_n + L - M)(M_1 M_2 \cdots M_p - I_n). \end{aligned}$$

Multiplying this by $f_1 f_2 M_1 M_2 \cdots M_p - I_n$ from the right, we obtain

$$\begin{aligned} &(M - I_n)(f_1 f_2 M_1 M_2 \cdots M_p - I_n) \\ &= (I_n + L - M)(M_1 M_2 \cdots M_p - I_n)(f_1 f_2 M_1 M_2 \cdots M_p - I_n) \\ &= (I_n + L - M)\{f_1 f_2 (M_1 M_2 \cdots M_p)^2 - (f_1 f_2 + 1)M_1 M_2 \cdots M_p + I_n\}. \end{aligned}$$

Since the relation

$$(M_1 M_2 \cdots M_p)^2 = (f_1^{-1} + f_2^{-1})M_1 M_2 \cdots M_p - f_1^{-1} f_2^{-1} I_n$$

holds by (2.5), we have

$$\begin{aligned}
 & (M - I_n)(f_1 f_2 M_1 M_2 \cdots M_p - I_n) \\
 &= (I_n + L - M)\{(f_1 + f_2)M_1 M_2 \cdots M_p - (f_1 f_2 + 1)M_1 M_2 \cdots M_p\} \\
 &= -(f_1 - 1)(f_2 - 1)(I_n + L - M)M_1 M_2 \cdots M_p \\
 &= -(f_1 - 1)(f_2 - 1)L.
 \end{aligned}$$

Therefore we have

$$[M_i N_p - N_p M_i]_{ip} = [-(f_1 - 1)(f_2 - 1)L]_{ip} = 0$$

for $i = 1, 2, \dots, p-1$. Thus the relation $M_i N_p = N_p M_i$ holds for $i = 1, 2, \dots, p-1$.

Step 2. We next prove $M_p N_j = N_j M_p$ ($j \neq p$). It is easy to see that

$$[M_p N_j - N_j M_p]_{ik} = 0$$

holds for $(i, k) \neq (p, j)$. In the same way as Step 1, from (1.10) with $k = p$, (1.15), (2.4) and (2.5) we can obtain

$$[M_p N_j - N_j M_p]_{pj} = [-(f_1^{-1} - 1)(f_2^{-1} - 1)U]_{pj} = 0$$

for $j = 1, 2, \dots, p-1$. Therefore we have $M_p N_j = N_j M_p$ for $j = 1, 2, \dots, p-1$.

Step 3. We next prove $M_i N_{p-1} = N_{p-1} M_i$ for $i \neq p-1$ and $M_{p-1} N_j = N_j M_{p-1}$ for $j \neq p-1$. Multiplying (1.15) by N_p^{-1} from the left and by M_p^{-1} from the right, we obtain

$$N_{p-1} N_{p-2} \cdots N_1 M_p^{-1} = f_1 f_2 \cdot N_p^{-1} M_1 M_2 \cdots M_{p-1}.$$

Applying the results of Steps 1 and 2, we can obtain

$$M_p^{-1} N_{p-1} N_{p-2} \cdots N_1 = f_1 f_2 \cdot M_1 M_2 \cdots M_{p-1} N_p^{-1}.$$

Multiplying this by M_p from the left and by N_p from the right, we obtain

$$N_{p-1} N_{p-2} \cdots N_1 N_p = f_1 f_2 \cdot M_p M_1 M_2 \cdots M_{p-1}. \quad (2.6)$$

Using the matrix

$$Q = \begin{pmatrix} I_{n_1} & & \\ & \ddots & \\ & & I_{n_{p-1}} \\ I_{n_p} & & \end{pmatrix},$$

we define

$$\tilde{M}_1 = Q^{-1} M_p Q = \begin{pmatrix} M_{pp} & M_{p1} & \cdots & M_{p,p-1} \\ & I_{n_1} & & \\ & & \ddots & \\ & & & I_{n_{p-1}} \end{pmatrix},$$

$$\tilde{M}_i = Q^{-1} M_{i-1} Q = \begin{pmatrix} I_{n_p} & & & & & \\ & I_{n_1} & & & & \\ & & \ddots & & & \\ M_{i-1,p} & M_{i-1,1} & \cdots & M_{i-1,i-1} & \cdots & M_{i-1,p-1} \\ & & & & \ddots & \\ & & & & & I_{n_{p-1}} \end{pmatrix}$$

for $i = 2, 3, \dots, p$, and

$$\tilde{N}_1 = Q^{-1} N_p Q = \begin{pmatrix} N_{pp} & & & \\ N_{1p} & I_{n_1} & & \\ \vdots & & \ddots & \\ N_{p-1,p} & & & I_{n_{p-1}} \end{pmatrix},$$

$$\tilde{N}_j = Q^{-1} N_{j-1} Q = \begin{pmatrix} I_{n_p} & & N_{p,j-1} \\ & I_{n_1} & N_{1,j-1} \\ & & \ddots & \vdots \\ & & & N_{j-1,j-1} \\ & & & \vdots \\ & & & N_{p-1,j-1} & & I_{n_{p-1}} \end{pmatrix}$$

for $j = 2, 3, \dots, p$. These matrices satisfy

$$\begin{aligned} \tilde{M}_1 \tilde{M}_2 \cdots \tilde{M}_p &= Q^{-1} M_p M_1 M_2 \cdots M_{p-1} Q, \\ \tilde{N}_p \tilde{N}_{p-1} \cdots \tilde{N}_1 &= Q^{-1} N_{p-1} N_{p-2} \cdots N_1 N_p Q, \end{aligned}$$

and hence

$$\tilde{N}_p \tilde{N}_{p-1} \cdots \tilde{N}_1 = f_1 f_2 \tilde{M}_1 \tilde{M}_2 \cdots \tilde{M}_p$$

follows from (2.6). Moreover, we have

$$\tilde{M}_1 \tilde{M}_2 \cdots \tilde{M}_p = Q^{-1} M_p (M_1 M_2 \cdots M_{p-1} M_p) M_p^{-1} Q \sim \begin{pmatrix} f_1^{-1} I_{m_1} & \\ & f_2^{-1} I_{m_2} \end{pmatrix}.$$

Therefore we can apply the results of Steps 1 and 2 to the $\tilde{M}_i$'s and the $\tilde{N}_j$'s to obtain

$$\tilde{M}_i \tilde{N}_p = \tilde{N}_p \tilde{M}_i \quad (i \neq p)$$

and

$$\tilde{M}_p \tilde{N}_j = \tilde{N}_j \tilde{M}_p \quad (j \neq p).$$

These relations are equivalent to

$$M_i N_{p-1} = N_{p-1} M_i \quad (i \neq p-1)$$

and

$$M_{p-1} N_j = N_j M_{p-1} \quad (j \neq p-1),$$

respectively.

Step 4. Inductively repeating the consideration above, for $k = p-2, p-3, \dots, 1$ we can prove

$$M_i N_k = N_k M_i \quad \text{and} \quad M_k N_j = N_j M_k$$

for $i \neq k$ and $j \neq k$, respectively. Thus the proof of Proposition 2.2 is complete. $\square$

PROPOSITION 2.4. *The second relation (2.2) in Proposition 2.1 follows from (1.13), (1.15) and (2.1).*

Proof. By using (1.13) we have

$$M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} = M_k^{-1} \cdots M_1^{-1} M_p^{-1} \cdots M_{k+1}^{-1}, \quad (2.7)$$

and hence

$$M_k M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} = M_{k-1}^{-1} \cdots M_1^{-1} M_p^{-1} \cdots M_{k+1}^{-1}.$$

Since M_k and M_0 do not appear in the last member, the equality of the first and the second members of (2.2) holds by virtue of the commutativity (2.1).

From (1.15) we obtain

$$M_{k+1} \cdots M_p M_1 \cdots M_k = (f_1 f_2)^{-1} N_k \cdots N_1 N_p \cdots N_{k+1}$$

by using (2.1) in the same manner as we obtain (2.6) in the proof of Proposition 2.2.

Using this relation we can rewrite (2.7) to obtain

$$\begin{aligned} M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} &= (M_{k+1} \cdots M_p M_1 \cdots M_k)^{-1} \\ &= ((f_1 f_2)^{-1} N_k \cdots N_1 N_p \cdots N_{k+1})^{-1} \\ &= f_1 f_2 \cdot N_{k+1}^{-1} \cdots N_p^{-1} N_1^{-1} \cdots N_k^{-1}. \end{aligned}$$

Hence we have

$$M_{k+1} \cdots M_p M_0 M_p^{-1} \cdots M_{k+1}^{-1} N_k = f_1 f_2 \cdots N_{k+1}^{-1} \cdots N_p^{-1} N_1^{-1} \cdots N_{k-1}^{-1}.$$

Since N_k and M_0 do not appear in the last member, the equality of the first and the third members of (2.2) holds by virtue of the commutativity (2.1). $\square$

To summarize Propositions 2.2 and 2.4, we have the following.

THEOREM 2.5. *The relations (2.1) and (2.2) which come from the fundamental relations (1.2) and (1.3) can be derived from the relations (1.7), (1.9), (1.13), (1.15) and (1.16).*

3. The equivalence of the rigidity

The Pfaffian system (0.3) defines a local system $\mathcal{F}$ on $\mathbf{P}^1 \times \mathbf{P}^1 \setminus (S \cup \{(x, y); y = \infty\})$, and the local system $\mathcal{F}$ is represented by the circuit matrices M_i ($0 \leq i \leq p$) and N_j ($1 \leq j \leq p$). We translate the definition of a rigid local system given in the introduction in terms of the circuit matrices.

Let $\bar{\rho}$ be an n -dimensional representation of $\pi_1(\mathbf{P}^1 \times \mathbf{P}^1 \setminus (S \cup \{(x, y); y = \infty\}))$, and set

$$\bar{\rho}(\mu_i) = \bar{M}_i \quad (0 \leq i \leq p), \quad \bar{\rho}(v_j) = \bar{N}_j \quad (1 \leq j \leq p).$$

The local system $\mathcal{F}$ is *rigid* if, for any $\bar{\rho}$ satisfying

$$\bar{M}_i \sim M_i \quad (0 \leq i \leq p), \quad \bar{N}_j \sim N_j \quad (1 \leq j \leq p), \quad \bar{N}_p \cdots \bar{N}_1 \bar{M}_0 = f_1 f_2 I_n, \quad (3.1)$$

there exists a matrix C such that

$$\bar{M}_i = C M_i C^{-1} \quad (0 \leq i \leq p), \quad \bar{N}_j = C N_j C^{-1} \quad (1 \leq j \leq p). \quad (3.2)$$

First we assume that the local system $\mathcal{H}$ defined by the system (0.5) is rigid. Take a representation $\bar{\rho}$ satisfying (3.1). Since $M_0, M_1, \dots, M_p$ can be regarded as the circuit matrices of (0.5) by Proposition 1.5, by the rigidity there exists a matrix C such that

$$\bar{M}_i = C M_i C^{-1} \quad (0 \leq i \leq p). \quad (3.3)$$

LEMMA 3.1. *If the local system $\mathcal{F}$ defined by (0.5) is irreducible, then for each $j = 1, 2, \dots, p$, $\bar{N}_j$ can be written as*

$$\bar{N}_j = C \begin{pmatrix} I_{n_1} & & & N'_{1j} & & \\ & \ddots & & \vdots & & \\ & & I_{n_{j-1}} & N'_{j-1,j} & & \\ & & & N'_{jj} & & \\ & & & N'_{j+1,j} & I_{n_{j+1}} & \\ & & & \vdots & & \ddots \\ & & & N'_{pj} & & & I_{n_p} \end{pmatrix} C^{-1} \quad (N'_{ij} \in \text{Mat}(n_i, n_j; \mathbb{C})) \quad (3.4)$$

except for the case

$$n = 2, \quad \bar{n} = \bar{m} = (1, 1), \quad e_{1,1} = e_{2,1} = -1, \quad f_1 f_2 = 1, \quad (3.5)$$

in which case $-\bar{N}_j$ ($j = 1, 2$) is also possible in addition to (3.4).

Proof. We set $N'_j = C^{-1} \bar{N}_j C$ ($1 \leq j \leq p$). Since $M_i = C^{-1} \bar{M}_i C$ ($0 \leq i \leq p$) and N'_j ($1 \leq j \leq p$) are images by the adjoint representation $\text{ad}(C^{-1})\bar{\rho}$, from Proposition 1.2 we obtain

$$M_i N'_j = N'_j M_i \quad (3.6)$$

for $1 \leq i, j \leq p, i \neq j$.

Take an index j and fix it. Set $N'_j = (H_{k\ell})_{1 \leq k, \ell \leq p}$. From (3.6) and (1.7) we obtain

$$H_{ki} = H_{ki} M_{ii} \quad (k \neq i, i \neq j). \quad (3.7)$$

By assumption (1.4) we have $e_{i,h} \neq 1$ ($1 \leq h \leq n_i$), and hence

$$H_{ki} = O \quad (k \neq i, i \neq j) \quad (3.8)$$

follows from (1.8) and (3.7). Then we have

$$N'_j = \begin{pmatrix} H_{11} & & & H_{1j} & & \\ & \ddots & & \vdots & & \\ & & & H_{jj} & & \\ & & & \vdots & & \ddots \\ & & & H_{pj} & & & H_{pp} \end{pmatrix}. \quad (3.9)$$

Put (3.9) into (3.6) and compare the (i, k) -blocks of both sides ($k \neq j$). Then we get

$$M_{ik}H_{kk} = H_{ii}M_{ik} \quad (i, k \neq j). \quad (3.10)$$

Again by (1.4) we have $e_{i,h} \neq e_{i,h'}$ ($h \neq h'$), and by using (3.10) with $i = k$ we see that H_{kk} ($k \neq j$) is diagonal. Since $N'_j \sim N_j$ and the form of N_j is specified by (1.9) and (1.17), the eigenvalues of N'_j , and hence the diagonal elements of the diagonal matrices H_{kk} ($k \neq j$) and the eigenvalues of H_{jj} , are $f_1 f_2 e_{j,h}$ ($1 \leq h \leq n_j$) of multiplicity one and 1 of multiplicity $n - n_j$. For $k \neq j$, set

$$H_{kk} = \begin{pmatrix} g_{k,1} & & & \\ & g_{k,2} & & \\ & & \ddots & \\ & & & g_{k,n_k} \end{pmatrix}.$$

We consider relation (3.10) with $i \neq k$. If an (h, h') -element of M_{ik} does not vanish, it follows that

$$g_{k,h'} = g_{i,h},$$

and hence $g_{k,h'}$ is an eigenvalue of N'_j of multiplicity at least two, which implies $g_{k,h'} = 1$. From the irreducibility it follows that, for each h' , there is at least one h such that the (h, h') -element of M_{ik} does not vanish. Thus we obtain $g_{k,h'} = 1$ for $1 \leq h' \leq n_k$, and hence $H_{kk} = I_{n_k}$ ($k \neq j$).

The above argument holds if there are at least two indices i, k distinct from j , namely if $p \geq 3$. It is easy to see that $a(n; \bar{n}, \bar{m}) < 0$ if $p = 2$ unless

$$n = 2, \quad \bar{n} = \bar{m} = (1, 1). \quad (3.11)$$

Then by the irreducibility we have $p \geq 3$ or (3.11). We study the remaining case (3.11). We can set

$$N'_1 = \begin{pmatrix} g_{1,1} & \\ H_{21} & g_{1,2} \end{pmatrix}, \quad N'_2 = \begin{pmatrix} g_{2,1} & H_{12} \\ & g_{2,2} \end{pmatrix}. \quad (3.12)$$

Moreover, we know $\{g_{1,1}, g_{1,2}\} = \{1, f_1 f_2 e_{1,1}\}$ and $\{g_{2,1}, g_{2,2}\} = \{1, f_1 f_2 e_{2,1}\}$. Put (3.12) into relations (3.6) and $N'_2 N'_1 = f_1 f_2 M_1 M_2$, which comes from the third condition in (3.1). If we assume $g_{2,2} = f_1 f_2 e_{2,1}$, it follows that $g_{2,1} = g_{1,2} = 1$ and hence we obtain (3.4) for this case:

$$N'_1 = \begin{pmatrix} f_1 f_2 e_{1,1} & \\ M_{21} e_{2,1}^{-1} & 1 \end{pmatrix}, \quad N'_2 = \begin{pmatrix} 1 & M_{12} e_{1,1}^{-1} \\ & f_1 f_2 e_{2,1} \end{pmatrix}. \quad (3.13)$$

If we assume $g_{2,2} = 1$, with the help of $e_{1,1} \neq 1$ we obtain $e_{1,1} = e_{2,1} = -1$, $f_1 f_2 = 1$ and

$$N'_1 = \begin{pmatrix} 1 & \\ M_{21} & -1 \end{pmatrix}, \quad N'_2 = \begin{pmatrix} -1 & M_{12} \\ & 1 \end{pmatrix}.$$

We see that these matrices multiplied by -1 respectively coincide with the matrices in (3.13) in this case. $\square$

We exclude case (3.5). Then by Lemma 3.1 we see that $N'_i = C^{-1} \tilde{N}_i C$ ($1 \leq i \leq p$) has the form (1.9). Since M_i ($0 \leq i \leq p$) and N'_j ($1 \leq j \leq p$) satisfy (1.13)–(1.16), where each N_j is replaced by N'_j , Proposition 1.3 also works for the N'_j 's. The proposition asserts the uniqueness of the N_j 's, and hence we have $N'_j = N_j$ ($1 \leq j \leq p$). Thus we get (3.2), which shows that the local system $\mathcal{F}$ defined by (0.3) is rigid.

Conversely, we assume that the local system $\mathcal{H}$ defined by (0.5) is irreducible and not rigid. In this case we have $a(n; \bar{n}, \bar{m}) > 0$. Then there exist $\bar{M}_0, \bar{M}_1, \dots, \bar{M}_p$ such that

$$\bar{M}_1 \bar{M}_2 \cdots \bar{M}_p \bar{M}_0 = I_n, \quad \bar{M}_i \sim M_i \quad (0 \leq i \leq p), \quad (3.14)$$

and that the local system $\bar{\mathcal{H}}$ defined by the $\bar{M}_i$'s is not isomorphic to the local system $\mathcal{H}$. We may assume that $\bar{M}_i$ ($1 \leq i \leq p$) has the form

$$\bar{M}_i = \begin{pmatrix} I_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{n_{i-1}} & & & & \\ \bar{M}_{i1} & \cdots & \bar{M}_{i,i-1} & M_{ii} & \bar{M}_{i,i+1} & \cdots & \bar{M}_{ip} \\ & & & I_{n_{i+1}} & & & \\ & & & & \ddots & & \\ & & & & & & I_{n_p} \end{pmatrix}, \quad (3.15)$$

where M_{ii} ($1 \leq i \leq p$) is given by (1.8). In fact, if we set $\bar{M}_i$ ($1 \leq i \leq p$) by (3.15), the relations $\bar{M}_i \sim M_i$ ($1 \leq i \leq p$) are obviously satisfied. There remains a single relation

$$\bar{M}_1 \bar{M}_2 \cdots \bar{M}_p \sim \begin{pmatrix} f_1^{-1} I_{m_1} & \\ & f_2^{-1} I_{m_2} \end{pmatrix} \quad (3.16)$$

for unknowns $\bar{M}_{ij}$ ($1 \leq i, j \leq p, i \neq j$). The sum of the numbers of the elements of

the $\bar{M}_{ij}$'s is

$$\sum_{i=1}^p n_i(n - n_i) = n^2 - \sum_{i=1}^p n_i^2,$$

while the number of equations coming from (3.16) is

$$m_1^2 + m_2^2 - 1$$

[O, Chapter 2, Lemma 1]. If two sets $\{\bar{M}_i^{(1)}\}_i, \{\bar{M}_i^{(2)}\}_i$ of solutions of (3.14) of the form (3.15) are conjugate, i.e. if there exists $C \in \mathrm{SL}(n; \mathbb{C})$ such that $C\bar{M}_i^{(1)}C^{-1} = \bar{M}_i^{(2)}$ ($0 \leq i \leq p$), C becomes a diagonal matrix, and such matrices form a manifold of dimension $n - 1$. Now we obtain

$$\left(n^2 - \sum_{i=1}^p n_i^2\right) - (m_1^2 + m_2^2 - 1) > n - 1$$

from $a(n; \bar{n}, \bar{m}) > 0$. This means that there exists a set of solutions $\{\bar{M}_i\}$ of the form (3.15) not conjugate to $\{M_i\}$ (cf. [O, Chapter 2, Theorem 1.2]).

Define $\bar{N}_j$ ($1 \leq j \leq p$) in the same manner as in Corollary 1.4 by using $\bar{M}_i$ ($1 \leq i \leq p$) instead of the M_i 's. Then, by the corollary and the first equation in (3.14), the $\bar{M}_i$'s and the $\bar{N}_j$'s satisfy (3.1). From Theorem 2.5 it follows that the $\bar{M}_i$'s and the $\bar{N}_j$'s define a representation $\bar{\rho}$ of $\pi_1(\mathbf{P}^1 \times \mathbf{P}^1 \setminus (S \cup \{(x, y); y = \infty\}))$. Since $\bar{\rho}$ is not conjugate to the representation defined by the M_i 's and the N_j 's, the local system $\mathcal{F}$ defined by (0.3) is not rigid.

Summing up the above, we get the main theorem.

THEOREM 3.2. *We exclude case (3.5). If the local system $\mathcal{H}$ defined by the system (0.5) is irreducible, then the local system $\mathcal{F}$ defined by the Pfaffian system (0.3) is rigid if and only if the local system $\mathcal{H}$ is rigid.*

REFERENCES

- [A] W. Arvola. The fundamental group of the complement of an arrangement of complex hyperplanes. Ph.D. Thesis, University of Wisconsin-Madison, 1990.
- [H1] Y. Haraoka. Canonical forms of differential equations free from accessory parameters. *SIAM J. Math. Anal.* **25** (1994), 1203–1226.
- [H2] Y. Haraoka. Monodromy representations of systems of differential equations free from accessory parameters. *SIAM J. Math. Anal.* **25** (1994), 1595–1621.

- [H3] Y. Haraoka. Integral representations of solutions of differential equations free from accessory parameters. Preprint.
- [IKSY] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida. From Gauss to Painlevé. Vieweg, 1991.
- [K] N. M. Katz. Rigid Local Systems. Princeton University Press, 1996.
- [O] K. Okubo. On the group of Fuchsian equations. Seminar Reports of Tokyo Metropolitan University, 1987.
- [OT] P. Orlik and H. Terao. Arrangements of Hyperplanes. Springer, 1992.
- [R] R. Randell. The fundamental group of the complement of a union of complex hyperplanes. *Invent. Math.* **69** (1982), 103–108. Correction: *Invent. Math.* **80** (1985), 467–468.
- [S] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbb{C}^N$. *Invent. Math.* **88** (1987), 603–618.
- [T] N. Takayama. Propagation of singularities of solutions of the Euler–Darboux equation and a global structure of the space of holonomic solutions I. *Funkcial. Ekvac.* **35** (1992), 343–403.
- [Yok1] T. Yokoyama. A system of total differential equations of two variables and its monodromy group. *Funkcial. Ekvac.* **35** (1992), 65–93.
- [Yok2] T. Yokoyama. Monodromy groups of systems of total differential equations of two variables. *SIAM J. Math. Anal.* **28** (1997), 1227–1247.
- [Yos] M. Yoshida. Fuchsian Differential Equations. Vieweg, 1987.

Yoshishige Haraoka
Department of Mathematics
Faculty of Science
Kumamoto University
Kumamoto 860, Japan
(E-mail: haraoka@gpo.kumamoto-u.ac.jp)

Toshiaki Yokoyama
Department of Mathematics
Chiba Institute of Technology
Narashino 275, Japan
(E-mail: yokoyama@pf.it-chiba.ac.jp)