PL-LEAST AREA 2-ORBIFOLDS AND ITS APPLICATIONS TO 3-ORBIFOLDS

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Introduction

In [MY1], Meeks and Yau showed the equivariant versions of Dehn's lemma, the Loop theorem, and the Sphere theorem which are fundamental theorems in the theory of 3-manifolds. They used minimal surfaces, which often appear in Differential Geometry, and the least area surface in some class. Then they proved that the least area surface is equivariant under group actions. Their 'minimal and least area surface technique' is quite useful for various equivariant arguments. For example, Meeks and Yau [MY3] showed that the irreducibility can be lifted to the covering spaces of 3-manifolds. After several years, Jaco and Rubinstein [JR] also showed the equivariant theorems by a combinatorial method. By 'combinatorial' we mean that they used PL-least area surfaces instead of analytic ones.

In this paper we define the PL-area of a map $f : F \to M$, where $M$ is a 3-orbifold with a fixed triangulation $K_M$, $F$ is a 2-orbifold, and $f$ is a 'normal' map with respect to the triangulation $K_M$. The PL-area of $f$ is defined to be the triple $(S(f) = \# \sum F, w(f) = \# (f(F) \cap K_M^{(1)}), \ell(f) = \sum_{e \in \Sigma} \text{length}(f(F) \cap e))$. We order the PL-areas lexicographically. Then we investigate PL-least area 2-orbifolds closely and show Theorems 4.3 and 5.3.

We summarize the contents of this paper. In Section 0, we prepare some basic facts about 3-orbifolds, such as a simplicial division, an orbifold covering, an orbifold fundamental group, a bad orbifold and so on.

In Section 1, we describe terms and basic properties of normal 2-orbifolds, especially the combinatorial area and mean curvature. In addition, we state the existence and uniqueness of a minimal 2-orbifold (cf. [JR]).

In Section 2, we introduce the notion of a semi-general position (S.G.P.) 2-orbifold, which comes out in a surgery of two intersecting normal 2-orbifolds. Then we show that there exists a normalization of an S.G.P. 2-orbifold decreasing its area.
if it belongs to a class which is invariant under the D-modifications. Examples of D-invariant classes are given there.

We examine intersections of minimal 2-orbifolds $F$ and $G$ in Section 3. Examining intersections of two objects, a general position approximation is usually used in Topology, but we cannot use it since $F$ and $G$ are minimal. Thus, $F \cap G$ is a 2-complex. It is the purpose in this section to yield a condition that this 2-complex has an innermost curve 'evenly' in $F$ (and $G$).

Theorem 4.3 is one of the main results of this paper, which insists that any two least area 2-orbifolds in a class $\Omega$ do not intersect each other. Section 4 is devoted to the proof of it.

**Theorem 4.3.** Let $M$ be a 3-orbifold and $\Omega$ be a surgery-invariant class of 2-suborbifolds in $M$. Let $K_M$ be a simplicial division of $M$ which is equipped with a P.H. metric and is sufficiently refined. Let $\mathcal{F}_\Omega = \{ F \in \Omega \mid F$ is an S.G.P. 2-orbifold w.r.t. $K_M \}$. Suppose $\mathcal{F}_\Omega$ satisfies the following:

1. for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
2. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $f_0 : F_0 \to M \in N(F)$ is an orbi-embedding;
3. for any normal 2-orbifolds $F, G \in \mathcal{F}_\Omega$, the 2-complex which is the intersection of the underlying spaces of the minimal 2-orbifolds $F_0, G_0 \in N(F), N(G)$ evenly includes innermost curves of $(|F_0|, |F_0| \cap |G_0|)$ and $(|G_0|, |F_0| \cap |G_0|)$.

Suppose $F_1, F_2 \in \mathcal{F}_\Omega$ are the least area 2-orbifolds of $\mathcal{F}_\Omega$. Then either $F_1 \cap F_2 = \emptyset$ or $F_1 = F_2$.

In Section 5, we show Theorem 5.3, the other main result of this paper, which states that what kind of class of 2-orbifolds has a least area 2-orbifold.

**Theorem 5.3.** (The existence of the least area 2-orbifold) Let $M$ be a compact (not necessarily locally orientable) 3-orbifold and $p : \hat{M} \to M$ be a regular covering s.t. $\hat{M}$ is locally orientable. Let $\Omega$ be a class of 2-suborbifolds of $\hat{M}$. Let $\mathcal{F}_\Omega = \{ F \in \Omega \mid F$ is an S.G.P. 2-orbifold w.r.t. $K_{\hat{M}} \}$. Suppose $\mathcal{F}_\Omega$ satisfies the following:

1. for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
2. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, $F$ is not a linking 2-orbifold;
3. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $F_0 \in N(F)$ is an orbi-embedding.

If $\Omega \neq \emptyset$, then there is a least area 2-orbifold of $\mathcal{F}_\Omega$. 
In the concluding section, Theorem 6.1 is derived from combining Theorems 4.3 and 5.3.

**Theorem 6.1.** Let $M$ be a compact (not necessarily locally orientable) 3-orbifold and $p : \tilde{M} \to M$ be a regular covering s.t. $\tilde{M}$ is locally orientable. Let $\Omega$ be a surgery-invariant class of 2-suborbifolds of $\tilde{M}$. Put $\mathcal{F}_\Omega = \{ F \in \Omega \mid F$ is an S.G.P. 2-orbifold w.r.t. $K_{\tilde{M}} \}$. Suppose $\mathcal{F}_\Omega$ satisfies the following:

1. for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
2. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, $F$ is not a linking 2-orbifold;
3. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $f_0 : F_0 \to M \in N(F)$ is an orbi-embedding;
4. for any normal 2-orbifolds $F, G \in \mathcal{F}_\Omega$, the 2-complex which is the intersection of the underlying spaces of the minimal 2-orbifolds $F_0, G_0 \in N(F), N(G)$ evenly includes innermost curves of $(|F_0|, |F_0| \cap |G_0|)$ and $(|G_0|, |F_0| \cap |G_0|)$, respectively.

If $\Omega \neq \emptyset$, then there is an element $F_0 \in \Omega$ which is equivariant under $\text{Aut}(\tilde{M}, p)$.

As applications of it, we show the following corollaries:

i. loop theorem for orbifolds (Corollary 6.4);
ii. Dehn’s lemma for orbifolds (Corollary 6.5);
iii. sphere theorem for orbifolds (Corollary 6.7);
iv. lifting of abadness of orbifolds (Corollary 6.9);
v. lifting of irreducibility of orbifolds (Corollary 6.13).

These theorems are proved by simply applying Theorem 6.1 when the 3-orbifold $M$ is compact. When $M$ is non-compact, topological arguments reduce them to the compact cases.

The theorems addressed in this paper are used in [TY1, TY2, TY3]. In [TY1] the authors extend the Waldhausen’s classification theorem for 3-manifolds to that for good (but not necessarily finitely good) 3-orbifolds, and in [TY2, TY3] find a 2-suborbifold that realizes a given $\pi_1$-decomposition.

0. Preliminaries

We summarize backgrounds for orbifolds. For other basic facts about orbifolds, see [Th, BS, Ta]. By an $n$-dimensional orbifold $M$, we shall mean a Hausdorff space $X$ together with a system $\mathcal{O} = \{ (U_i, \{ \varphi_i \}, \{ \tilde{U}_i \}, \{ G_i \}, \{ \tilde{\varphi}_{ij} \}, \{ \eta_{ij} \}) \}$ which satisfies the following conditions:

1. $\{ U_i \}$ is an open cover of $X$.
(2) \( \{U_i\} \) is locally finite and closed under finite intersections;
(3) for each \( U_i \), there exist a finite group acting smoothly and effectively on a connected open subset \( \tilde{U}_i \) of \( \mathbb{R}_+^n \) and a homeomorphism \( \varphi_i : \tilde{U}_i / G_i \cong U_i \);
(4) if \( U_i \subset U_j \), there exists a monomorphism \( \eta_{ij} : G_i \to G_j \) and a smooth embedding \( \tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j \) s.t. for \( \sigma \in G_i, x \in \tilde{U}_i, \tilde{\varphi}_{ij}(\sigma x) = \eta_{ij}(\sigma) \tilde{\varphi}_{ij}(x) \) and the following diagram commutes, where \( \varphi_{ij} \) is induced by the monomorphism and the embedding, and \( r_i \)'s are the natural projections.

\[
\begin{array}{c}
\tilde{U}_i \\
\downarrow{r_i} \\
\tilde{U}_i / G_i \\
\downarrow{\varphi_{ij}} \\
U_i
\end{array} \quad \begin{array}{c}
\tilde{\varphi}_{ij} \\
\downarrow{r_j} \\
\tilde{U}_j / G_j \\
\downarrow{\varphi_j} \\
U_j
\end{array}
\]

Each \( \varphi_i \circ r_i : \tilde{U}_i \to U_i \) is called a local chart of the orbifold \( M \). \( \mathcal{O} \) is called the atlas of the orbifold \( M \). We call \( X \) the underlying space of the orbifold \( M \), and denote it by the symbol \( |M| \). Note that the underlying space of a 2-orbifold is a topological 2-manifold. \( M \) is locally orientable if each \( G_i \) acts as a group of orientation preserving maps. Note that the underlying space of a locally orientable 3-orbifold is a topological 3-manifold. \( M \) is orientable if \( M \) is locally orientable and each embedding \( \tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j \) preserves the orientation.

Let \( x \) be any point of \(|M|\) and \( \varphi \circ r : \tilde{U} \to U \) a local chart of \( M \) containing \( x \). The local group at \( x \), denoted by \( G_x \), is the isotropy group of any point in \( \tilde{U} \) corresponding to \( x \). This is well defined up to isomorphisms. The set \( \{ x \in |M| | G_x \neq \text{id} \} \) is called the singular set of \( M \) and denote it by the symbol \( \Sigma M \). In the case \( \Sigma M = \phi \), it is easy to see that \( M \) is a usual smooth manifold.

Let \( Y \) be a subspace of \(|M|\). We define the restriction of \( \mathcal{O} \) to \( Y \), denoted by \( \mathcal{O} \mid Y \), by \( \mathcal{O} \mid Y = \{(Y_i), \{\varphi_i \mid Y_i\}, \{\tilde{Y}_i\}, \{H_i\}, \{\tilde{\varphi}_{ij} \mid \tilde{Y}_i\}, \{\eta_{ij} \mid H_i\}) \), where \( Y_i = Y \cap U_i, \tilde{Y}_i = (\text{a component of the inverse image of } Y_i \text{ under the quotient map } \tilde{U}_i \to U_i) \), and \( H_i \) is the group consists of the restrictions of the action of \( G_i \) to \( \tilde{Y}_i \). By a subspace of \( M \), we shall mean the subspace \( Y \) together with \( \mathcal{O} \mid Y \). We also call \( Y \) the underlying space of the subspace.

A subspace \( Q \) of \( M \) is called an \( m \)-dimensional orbifold of \( M \), if \( \mathcal{O} \mid Y \) gives an \( m \)-dimensional orbifold structure to \( Y \), where \( Y \) is the underlying space of \( Q \).

We call a point \( x \in |M| \) a boundary point of \( M \) if for some local chart \( \varphi \circ r : \tilde{U} \to U \) containing \( x \), \( x \) corresponds to a point in \( \tilde{U} \cap \mathbb{R}^{n-1} \). By the symbol \( \partial M \),
we shall mean the subspace whose underlying space consists of all boundary points of $M$ and structure is given by the restriction of $\partial$ to it. By the symbol $\text{Int}(M)$, we shall mean the subspace whose underlying space is $|M| - |\partial M|$ and the structure is given by $\partial(|M| - |\partial M|)$. $\partial M$ and $\text{Int}(M)$ are clearly suborbifolds of $M$. $\partial M$ (respectively $\text{Int}(M)$) is called the boundary (orbifold) (respectively interior (orbifold)) of $M$. Note that $|\partial M| \subset |\partial M|$ and the equality holds if $M$ is locally orientable.

We define the connectedness and compactness of (a subspace of) an orbifold by those of its underlying space. We also define the inclusion relation between the subspaces of an orbifold by those of their underlying spaces.

A stratum of $M$ is a maximal connected component on which the orders of the local groups are constant. We use the symbol $\Sigma^{(k)}M$ to denote the $k$-dimensional strata.

A simplicial division $K_M$ of $|M|$ is a simplicial division of $M$ if the $k$-dimensional strata of $M$ is included in the polyhedron consists of $\bigcup\{e \mid e \in K_M^{(k)}\}$ and for each $n$-simplex $e$, we have either $e \cap \Sigma M = \phi$ or there is a unique orientable proper face $e'$ of $e$ s.t. $e \cap \Sigma M = e'$.

Two orbifolds $M$ and $N$ are isomorphic, if there is a homeomorphism $h : |M| \to |N|$ and for each point $x \in |M|$, there is an isomorphism $\eta_x$ from the local group $G_x$ of $x$ to the local group $G'_{h(x)}$ of $h(x)$ and a diffeomorphism $\tilde{h}_x : \tilde{U}_x \to \tilde{U}'_{h(x)}$ s.t. for any $x \in G_x$ and any $z \in \tilde{U}_x$, $\tilde{h}_x(\sigma z) = \eta_x(\sigma)\tilde{h}_x(z)$, where $\tilde{U}_x \to \tilde{U}_x/G_x \cong U_x$ and $\tilde{U}'_{h(x)} \to \tilde{U}'_{h(x)}/G'_{h(x)} \cong U'_{h(x)}$ are the local charts. When $\Sigma M = \Sigma N = \phi$, it is easy to see that an isomorphism $h : M \to N$ means a usual diffeomorphism.

An orbifold $\tilde{M}$ is called a covering orbifold of an orbifold $M$, if there is a continuous map $p : |\tilde{M}| \to |M|$ which satisfies the following conditions.

1. $p$ is onto.
2. Each point $x \in |M|$ has a local chart of $M$ of the form $\tilde{U}_x \to \tilde{U}_x/G_x \cong U_x$ s.t. each point $\tilde{x} \in p^{-1}(x)$ has a local chart of $\tilde{M}$ of the form $\tilde{U}_x \to \tilde{U}_x/G_{x,i} \cong V_{x,i}$ and the following diagram commutes, where $V_{x,i}$ is the component of $p^{-1}(U_x)$ including $\tilde{x}$, $G_{x,i}$ is some subgroup of $G_x$, and $q$ is the natural projection:

$$
\begin{array}{ccc}
\tilde{U}_x/G_{x,i} \cong V_{x,i} & \xrightarrow{q} & \tilde{U}_x \\
\downarrow & & \downarrow p \\
\tilde{U}_x & \to & \tilde{U}_x/G_x \cong U_x.
\end{array}
$$
We call the pair \((\tilde{M}, p)\) an orbifold covering and \(p\) is called a covering (projection) denoted by \(p : \tilde{M} \to M\). Note that a local chart \(\varphi \circ r : \tilde{U} \to U\) is a simple example of a covering.

An isomorphism \(h : \tilde{M} \to \tilde{M}\) for which \(p \circ h = h\) is called a deck transformation of the covering \(p : \tilde{M} \to M\). By the symbol \(\text{Aut}(\tilde{M}, p)\), we mean the group which consists of all deck transformations.

We call a covering \(p : \tilde{M} \to M\) the universal covering if for any other covering \(p' : M' \to M\), there is a covering \(q : \tilde{M} \to M'\) s.t. \(p' \circ q = p\). The orbifold fundamental group of \(M\), denoted by the symbol \(\pi_1(M)\), is defined to be \(\text{Aut}(M, p)\).

The usual proof of the existence and uniqueness of the universal covering can be adapted to show that any orbifold has a (unique) universal covering (see [Th]).

A covering \(p : \tilde{M} \to M\) is called a regular covering if for any two preimage \(\tilde{x}\) and \(\tilde{x}'\) of a point \(x \in |M| - \Sigma M\), there is a deck transformation taking \(\tilde{x}\) to \(\tilde{x}'\).

A covering \(p : \tilde{M} \to M\) is called a manifold covering if \(\Sigma \tilde{M} = \emptyset\). An orbifold \(M\) is good if the universal covering of \(M\) is a manifold covering and bad otherwise.

For a good orbifold, the fundamental group of it is also defined by using paths [Ta, p. 161]. Various facts similar to those of the usual covering theory stand for orbifold coverings of good orbifolds (see [Th] and [Ta, 2.4, 2.5, 2.6, 2.8 and 2.9]).

Let \(M\) and \(N\) be orbifolds. Let \(\{\varphi_i \circ r_i : \tilde{U}_i \to U_i\}_{i \in I}\) and \(\{\psi_j \circ s_j : \tilde{V}_j \to V_j\}_{j \in J}\) be the local charts of \(M\) and \(N\), respectively. By an orbi-map \(f : M \to N\), we shall mean a pair of a map \(\tilde{f} : |M| \to |N|\) and a system of maps \(\{\tilde{f}_{iv} : \tilde{U}_i \to \tilde{V}_v\}_{i \in I, v \in J}\) (where \(J_v = \{v \in J \mid \tilde{f}(U_i) \subset V_v\}\)), which satisfies the following conditions:

1. For all \(i \in I\) and \(v \in J_i\), \(\varphi_i \circ r_i \circ (\psi_v \circ s_v) \circ \tilde{f}_{iv} = \tilde{f}_i \circ r_i \circ s_v\).
2. For all \(i \in I\) and \(v \in J_i\), and for all \(A \in \text{Aut}(\tilde{U}_i, \varphi_i \circ r_i)\), there is an element \(\tau_A \in \text{Aut}(\tilde{V}_v, \psi_v \circ s_v)\) s.t. \(\tilde{f}_{iv} \circ \tau_A = \tau_A \circ \tilde{f}_{iv}\).
3. If \(U_i \subset U_j\), \(\tilde{f}(U_i) \subset V_v\), \(\tilde{f}(U_j) \subset V_{v'}\), and \(V_v \subset V_{v'}\), then \(\tilde{f}_{ij} \circ \tilde{f}_{iv} = \tilde{f}_{ij} \circ \tilde{f}_{iv}\).
4. There is a point \(x \in |M| - \Sigma M\) such that \(\tilde{f}(x) \in |N| - \Sigma N\).

We call \(\tilde{f}\) and \(\{\tilde{f}_{iv}\}\) the underlying map and the structure maps of the orbi-map \(f\), respectively.

Two orbi-maps \(f = (\tilde{f}, \{\tilde{f}_{iv}\})\) and \(g = (\tilde{g}, \{\tilde{g}_{iv}\})\) are equivalent if \(\tilde{f} = \tilde{g}\) and for all \(i \in I\) and \(v \in J_i\), there is an element \(\gamma_{iv} \in \text{Aut}(\tilde{V}_v, \psi_v \circ s_v)\) s.t. \(\tilde{g}_{iv} = \gamma_{iv} \circ \tilde{f}_{iv}\). It is easy to see that isomorphisms and coverings are given orbi-map structures uniquely up to equivalence. Hence, isomorphisms and coverings are also called orbi-isomorphisms and orbi-coverings, respectively.

In [Ta], the author defines an orbi-map between good orbifolds. Let us call it an ‘old’ orbi-map and the above a ‘new’ orbi-map in this paragraph. It is easy to see that an ‘old’ orbi-map uniquely (up to equivalence) induces a ‘new’ orbi-map, and on
the other hand, a ‘new’ orbi-map is uniquely (up to equivalence) extended to an ‘old’ orbi-map that induces the original ‘new’ orbi-map.

Let \( f : A \to M \) be an orbi-map. By the symbol \( f(A) \), we mean the subspace of \( M \) whose underlying space is \( \overline{f(|A|)} \).

An orbi-map \( f : F \to M \) is called an orbi-embedding, if \( f(F) \) is a suborbifold of \( M \) and \( f : F \to f(F) \) is an orbi-isomorphism. An orbi-map \( f : F \to M \) is called an orbi-immersion, if \( f \) is locally an orbi-embedding.

A suborbifold \( V \) is called the regular neighborhood of a point \( P \) in \( M \), if \( V \) is orbi-isomorphic to such an orbifold \( V' \) that \( V' \) is included in a local chart \( \varphi \circ r : \tilde{U} \to U \) and a component of \( (\varphi \circ r)^{-1}(V') \) is the regular neighborhood of \( (\varphi \circ r)^{-1}(P) \). A suborbifold \( N \) is called the regular neighborhood of a subspace \( Q \) in \( M \), if \( |N| \) is the regular neighborhood of \( |Q| \) in \( |M| \) and \( N = \bigcup_{P \in Q} V_P \), where \( V_P \) is the regular neighborhood of \( P \) in \( M \).

A bad 2-orbifold \( S \) is called a bad sphere. If \( S \) is an orientable bad sphere, then \( |S| \) is the 2-sphere and either \( \Sigma S = \{ \text{one point} \} \) or \( \Sigma S \) consists of two points whose local groups are not isomorphic to each other. A 3-orbifold \( M \) is bad if there are no bad subspheres in \( M \). If \( S \) is a non-orientable bad sphere, then \( |S| \) is the 2-disc, \( S \) has a mirror boundary which includes \( \Sigma^{(0)} S \), and \( \Sigma^{(0)} S = \{ \text{one point} \} \) or \( \Sigma^{(0)} S \) consists of two points whose local groups are not isomorphic to each other.

A 2-orbifold which is orbi-isomorphic to \( D^2/G \) is called a discal 2-orbifold where \( D^2 \) is the 2-disc and \( G \) is a finite subgroup of \( O(2) \).

A 2-orbifold which is orbi-isomorphic to \( S^2/G \) is called a spherical 2-orbifold where \( S^2 \) is the 2-sphere and \( G \) is a finite subgroup of \( O(3) \).

A 3-orbifold which is orbi-isomorphic to \( B^3/G \) is called a ballic 3-orbifold where \( B^3 \) is the 3-ball and \( G \) is a finite subgroup of \( O(3) \).

A spherical 2-suborbifold \( S \) in a 3-orbifold \( M \) is incompressible if there is no ballic 3-suborbifold \( B \) of \( M \) s.t. \( \partial B = S \). A 3-orbifold \( M \) is irreducible if there are no incompressible spherical 2-suborbifolds in \( M \).

1. Normal 2-orbifolds

Let \( M \) be a 3-orbifold, \( K_M \) a triangulation of \( M \), and \( F \) a 2-orbifold.

Definition 1.1. An orbi-immersion \( f : (F, \partial F) \to (M, \partial M) \) is called a normal 2-orbifold w.r.t. \( K_M \) if there is a cell division \( K_F \) of \( F \) with triangles and rectangles s.t. for each 2-cell \( e \in K_F \), there is a 3-simplex \( \Delta \in K_M \) s.t.

\[
(1) \quad \overline{f|e} \text{ is an embedding from } e \text{ to } \Delta.
\]
(2) \( \overline{f}(\text{Int } e^{(i)}) \subset \text{Int } \Delta^{(i+1)}, i = 0, 1, 2, \)

(3) 1-faces of \( e \) are mapped into mutually different 2-faces of \( \Delta \) (see Figure 1.1),

(4) with an appropriate complete Riemannian metric for \( \Delta \), \( \overline{f}(e) \) is the least area
disc spanned by \( \overline{f}(\partial e) \).

That is, a normal 2-orbifold \( f \) is an orbi-map whose underlying map is a normal
surface. We use the terminology a normal 2-orbifold \( F \) w.r.t. \( K_M \) if \( f : F \to M \) is an
orbi-embedding and \( f \) is normal w.r.t. \( K_M \). We call \( K_F \) (respectively \( e \in K_F^{(2)} \)) the
normal division (respectively a cell) induced by \( f \).

Note that, by (1), (2), and transversality, it stands that \( f(\text{Int } F) \subset \text{Int } M \). Hence
the normal 2-orbifold is so-called ‘properly immersed’.

Remark 1.2. Let \( \Delta \) be a 3-simplex of \( K_M \), and \( e_1, e_2 \) be 2-cells of a normal 2-orbifold
\( f : F \to M \) w.r.t. \( K_M \) which are mapped into \( \Delta \). Since both \( \overline{f}(\partial e_1) \) and \( \overline{f}(\partial e_2) \) span
the least area discs \( \overline{f}(e_1) \) and \( \overline{f}(e_2) \) respectively, the following stand:

(1) if \( \overline{f}(\partial e_1) \cap \overline{f}(\partial e_2) \neq \emptyset \), then \( \overline{f}(e_1) \cap \overline{f}(e_2) = \emptyset \); 

(2) if \( \overline{f}(\partial e_1) = \overline{f}(\partial e_2) \), then \( \overline{f}(e_1) = \overline{f}(e_2) \); 

(3) if \( \overline{f}(\partial e_1) \neq \overline{f}(\partial e_2) \) and \( \overline{f}(\partial e_1) \) is contained in the closure of one component of
\( \partial \Delta - \overline{f}(\partial e_2) \), then \( \overline{f}(e_1) \cap \overline{f}(e_2) = \overline{f}(\partial e_1) \cap \overline{f}(\partial e_2) \) (see Figure 1.2); 

(4) if \( \overline{f}(\partial e_1) \) intersects \( \overline{f}(\partial e_2) \) transversely in two points \( u \) and \( v \), then \( \overline{f}(e_1) \cap \overline{f}(e_2) \)
is a simple arc in \( \overline{f}(e_1) \) (and \( \overline{f}(e_2) \)) joining \( u \) and \( v \) (see Figure 1.3).

From now on, we consider a normal 2-orbifold w.r.t. a fixed simplicial division of
a 3-orbifold so that we do not write ‘w.r.t. \( K_M \)’ unless it is an absolute necessity.

Definition 1.3. We call the number of vertices of \( K_F \) (respectively \( \#(\Sigma^{(0)}F) \)) the
weight (respectively singular weight) of a normal 2-orbifold \( f : F \to M \), denoted
by \( w(f) \) (respectively \( s(f) \)).
Definition 1.4. Let $K$ be a simplicial 2-complex and $\mathbf{P}$ a Poincaré disc. A map $\varphi : K \rightarrow \mathbf{P}$ is a piecewise hyperbolic atlas if for each 2-simplex $e \in K$, there is a geodesic ideal triangle $\Delta$ in $\mathbf{P}$ s.t. $\varphi : (e - e^{(0)}) : (e - e^{(0)}) \rightarrow \Delta$ is a homeomorphism and $\varphi$ maps the vertices of $e$ to the ideal points of $\Delta$. Let $g_i$ be the induced metric on $e_i \in K^{(2)}$ by $\varphi \mid e_i$. Suppose that $g_i(x) = g_j(x)$ for $\forall x \in e_i \cap e_j$ and each pair $e_i$, $e_j \in K^{(2)}$. Then we define $g(x) = g_i(x)$ for $x \in e_i$ where $e_i \in K^{(2)}$, and call $g$ a piecewise hyperbolic metric (P.H. metric) on $K^{(2)}$.

We assume that there is a P.H. metric $g$ on $K^{(2)}_M$.

Definition 1.5. We define the length of a normal 2-orbifold $f : F \rightarrow M$ w.r.t. a P.H. metric $g$, denoted by $\ell(f)$, by the following equation (see Figure 1.4):

$$
\ell(f) = \sum_{I_i \in K^{(1)}_F} (\text{the length of } f(I_i) \text{ w.r.t. } g).
$$
Definition 1.6. We define the area of a normal 2-orbifold $f$, denoted by $\text{Area}(f)$, by $(s(f), w(f), \ell(f))$ ordered lexicographically.

Definition 1.7.

(i) An orbi-map $\xi : F \times [0, 1] \to M$ is a normal orbi-homotopy if for any $t \in [0, 1]$, $\xi_t = \xi(t)(F \times \{t\})$ is a normal 2-orbifold.

(ii) Two normal 2-orbifolds $f, f' : F \to M$ are normally orbi-homotopic if there exists a normal orbi-homotopy $\xi : F \times [0, 1] \to M$ s.t. $\xi_0 = f$ and $\xi_1 = f'$.

Definition 1.8. A normal orbi-homotopy class of a normal 2-orbifold $f : F \to M$, denoted by $N(f)$, is defined as follows:

$$N(f) = \{ f' \mid f' \text{ is a normal 2-orbifold} : F \to M, f' \text{ and } f \text{ are normally orbi-homotopic} \}.$$ 

Note that $s(f') = s(f'')$ and $w(f') = w(f'')$ for any $f', f'' \in N(f)$. When $f$ is orbi-embedding, we also use the symbol $N(F)$ instead of $N(f)$. Note that all elements of $N(F)$ are not necessarily orbi-embedding.

Definition 1.9. $f_0 \in N(f)$ is a minimal 2-orbifold in $N(f)$ if for any $f' \in N(f)$, $\ell(f_0) \leq \ell(f')$.

Definition 1.10. We define the mean curvature of a normal 2-orbifold $f : F \to M$ at $v_i \in K_F^{(0)}$, denoted by $H_f(v_i)$, as follows. Let $V^{(i)}$ be any one of unit tangent vectors at $f(v_i)$ tangent to the edge $e \in K_M^{(1)}$ including $f(v_i)$. Let $I_v^{(i)}$, $v = 1, 2, \ldots, k_i$, be all edges of $St(v_i, K_F^{(1)})$. Let $T_v^{(i)}$ be the unit tangent vector at $f(v_i)$ tangent to $f(I_v^{(i)})$ which is oriented to be interior of $f(I_v^{(i)})$. Define $H_f(v_i) = \langle V^{(i)}, \sum_{v=1}^{k_i} T_v^{(i)} \rangle$, where $\langle \cdot, \cdot \rangle$ means the inner product (see Figure 1.5).
**Proposition 1.11.** If \( f_0 : F \to M \in N(f) \) is a minimal 2-orbifold in \( N(f) \), then for any \( v_i \in K_F^{(0)} \), \( H_{f_0}(v_i) = 0 \) [JR].

**Definition 1.12.** A normal 2-orbifold \( f : F \to M \) is a linking 2-orbifold if there is a point \( v \in K_M^{(0)} \) s.t. \( f(F) \subset St(v, K_M) \).

**Theorem 1.13.** (The existence and uniqueness of a minimal 2-orbifold) If a normal 2-orbifold \( f : F \to M \) is not a linking 2-orbifold, then there exists one and only one minimal 2-orbifold in \( N(f) \) [JR].

**Proposition 1.14.** Suppose \( F \) is compact. If a normal 2-orbifold \( f : F \to M \) is an orbi-embedding, then the minimal 2-orbifold \( g \in N(f) \) satisfies either:

(a) \( g \) is an orbi-embedding; or

(b) \( |g(F)| \) is a non-orientable 2-manifold, \( |f(F)| \) is the boundary of the twisted 1-bundle over \( |g(F)| \) and \( \bar{g} : |F| \to |g(F)| \) is a double covering.

(See [JR, Corollary 3].)
Remark 1.15.
(1) If $|F|$ is a 2-disc, then Proposition 1.14(b) must not happen.
(2) If $|F|$ is a 2-sphere and $\pi_1(|M|) = 1$, then Proposition 1.14(b) must not happen.

Definition 1.16. Let $\mathcal{N}$ be a class of normal 2-orbifolds. $f_0 \in \mathcal{N}$ is called a least area 2-orbifold in $\mathcal{N}$ if for any $f \in \mathcal{N}$, $\text{Area}(f_0) \leq \text{Area}(f)$.

Remark 1.17. Let $\mathcal{N}$ be a class of embedded normal 2-orbifolds s.t. for any $g \in \mathcal{N}$, $N(g) \subset \mathcal{N}$. In the situations of 1.15, the least area 2-orbifold $f \in \mathcal{N}$ is minimal in $N(f)$.

2. The decreased normalization

Let $M$ be a 3-orbifold, $K_M$ a triangulation of $M$, and $F$ a 2-orbifold. Now we introduce the notion of a semi-general position (S.G.P.) 2-orbifold which appears after the surgery of embedded normal 2-orbifolds along their intersections. Throughout this section, we assume that all orbifolds are locally orientable.

Definition 2.1. An orbi-map $f : (F, \partial F) \to (M, \partial M)$ is called a semi-general position 2-orbifold (S.G.P. 2-orbifold) w.r.t. $K_M$, if $\bar{f} : |F| \to |M|$ is an embedding and the following (1)–(4) stand.

1. $\bar{f}(|F|) \cap K_M^{(0)} = \emptyset$.
2. $\bar{f}(|F|) \cap K_M^{(1)} = \text{(points)}$.
3. $\bar{f}(|F|) \cap K_M^{(2)} = \text{(segments properly embedded in a 2-simplex)}$
   $\cup \text{(simple closed curves properly embedded in a 2-simplex)}$
   $\cup \text{(points on } K_M^{(1)})$.
4. $\bar{f}(|F|) \cap K_M^{(3)} = \text{(cells properly embedded in a 3-simplex)}$
   $\cup \text{(segments properly embedded in a 2-simplex)}$
   $\cup \text{(simple closed curves properly embedded in a 2-simplex)}$
   $\cup \text{(points on } K_M^{(1)})$.

We use the terminology an S.G.P. 2-orbifold $F$ w.r.t. $K_M$ if $f : F \to M$ is an orbi-embedding and $f$ is S.G.P. w.r.t. $K_M$. Note that an embedded normal 2-orbifold is an S.G.P. 2-orbifold. Note also that a normal and S.G.P. 2-orbifold is an embedded normal 2-orbifold.

Definition 2.2. Let $f : F \to M$ be an S.G.P. 2-orbifold and $G$ a compact subset of $F$. Define $s(f|G) = \#(f(G) \cap \Sigma^{(1)} M)$ and $w(f|G) = \#(f(G) \cap K_M^{(1)})$ and define a length of $f|G$ (w.r.t. $g$), denoted by $\ell(f|G)$, by $\ell(f|G) = \sum_{e_i \in K_M^{(2)}}$ (the length of $f(G) \cap e_i$ w.r.t. $g$). Define the area of $f|G$, denoted by $\text{Area}(f|G)$, by
(s(f|G), w(f|G), d(f|G)) ordered lexicographically. Let \( \mathfrak{R} \) be a class of (possibly restrictions of) S.G.P. 2-orbifolds. \( f_0 \in \mathfrak{R} \) is a least area 2-orbifold in \( \mathfrak{R} \) if for any \( f \in \mathfrak{R} \), Area\( (f_0) \leq \) Area\( (f) \).

**Definition 2.3.** Let \( f : F \rightarrow M \) be an S.G.P. 2-orbifold. Define

\[
\alpha(f) = w(f), \quad \beta(f) = \sum_{\sigma \in K_M^{(2)}} \text{#} \text{(simple closed curves consist of the components of } f(F) \cap \text{Int } \sigma). \]

The pair \((\alpha(f), \beta(f))\) is called the complexity of \( f \). We order it lexicographically.

**Definition 2.4.** Let \( M \) be a 3-orbifold and \( F \) a 2-suborbifold properly embedded in \( M \). Let \( D_0 \) be a 2-disc (i.e. \( \Sigma D_0 = \emptyset \)) in \( M \) s.t. either \( D_0 \cap F = \partial D_0 \) and \( D_0 \cap \partial M = \phi \), or \( D_0 \cap F = \text{(an arc in } \partial D_0) \) and \( D_0 \cap \partial M = \partial D_0 - \text{Int } I \). Let \( A = (\partial D_0 \cap F) \times [0, 1] \) be the regular neighborhood of \( \partial D_0 \) in \( F \). We call \( F' = (F - \text{Int } A) \cup (D_0 \times 0) \cup (D_0 \times 1) \) the \( D \)-modification of \( F \). A class \( \Omega \) of 2-suborbifolds of \( M \) is called \( D \)-invariant class if for any \( F \in \Omega \) and any \( D \)-modification \( F' \) of \( F \), \( F' \) consists of two components and at least one component of \( F' \) belongs to \( \Omega \).

**Example 2.5.** The following classes are \( D \)-invariant.

(a) Let \( M \) be a 3-manifold and \( F \) be a 2-submanifold in \( \partial M \). \( \Omega = \{ D \mid D \) is a 2-disc properly embedded in \( M \) s.t. \( \partial D \subset F \) and \( [\partial D] \neq 1 \) in \( \pi_1 (F) \} \).

(b) Let \( M \) be a 3-manifold. \( \Omega = \{ S \mid S \) is a 2-sphere in \( M \) s.t. \( [S] \neq 0 \) in \( \pi_2 (M) \} \).

(c) Let \( M \) be a 3-orbifold. \( \Omega = \{ S \mid S \) is a bad subsphere in \( M \} \).

(d) Let \( M \) be an abad 3-orbifold. \( \Omega = \{ S \mid S \) is an incompressible spherical 2-suborbifold in \( M \} \).

The following lemma is due to [Kn] (see also [He, pp. 29–30]).

**Lemma 2.6.** Let \( M \) be a 3-orbifold and \( K_M \) a simplicial division of \( M \) which satisfies that

\[
\forall \Delta \in K_M^{(3)}, \quad \Delta \cap [\partial M] \text{ is either one face of } \Delta, \quad \text{or one edge of } \Delta, \quad \text{or one vertex of } \Delta. 
\]

(*

Let \( \Omega \) be a \( D \)-invariant class of 2-suborbifolds. Put \( \mathfrak{F}_\Omega = \{ F \in \Omega \mid F \) is an S.G.P. 2-orbifold w.r.t. \( K_M \} \). If \( F_0 \in \mathfrak{F}_\Omega \) attains the minimum of the complexity \((\alpha, \beta)\) in \( \mathfrak{F}_\Omega \), then \( F_0 \) is a normal 2-orbifold.

**Proof.**

**Claim 1.** \( |F_0| \) is general position w.r.t. \( K_M^{(1)} \) and \( K_M^{(2)} \).
Otherwise, we can immediately get an S.G.P. 2-orbifold $F' \in \mathcal{F}_\Omega$ s.t. $(\alpha(F'), \beta(F')) < (\alpha(F_0), \beta(F_0))$. It is a contradiction.

Claim 2. For any $\Delta \in K_M^{(3)}$ with $\text{Int} \Delta \cap |F_0| \neq \phi$, each component of $|F_0| \cap \Delta$ is a properly embedded disc in $\Delta$.

Suppose that a component $P$ of $|F_0| \cap \Delta$ is not a disc. By Claim 1, $P$ is properly embedded in $\Delta$. Hence, there is an essential simple closed curve $C$ in $P \cap \text{Int} \Delta$. Take a disc $E$ in $\text{Int} \Delta$ s.t. $E \cap P = \partial E = C$. Then, we can cut open $P$ along $C$ and attach the two copies of $E$ to get S.G.P. 2-orbifolds $F_1$, $F_2$. We may assume $F_1 \in \mathcal{F}_\Omega$. Clearly, $(\alpha(F_1), \beta(F_1)) < (\alpha(F_0), \beta(F_0))$. It is a contradiction.

Claim 3. For any $e \in K_M^{(2)}$, each component of $|F_0| \cap e$ is an arc.

Suppose that a component of $|F_0| \cap e$ is a point $p$. By the definition of an S.G.P. 2-orbifold, $p$ must be on the interior of an edge of $e$. This is impossible from Claim 1.

Suppose that a component of $|F_0| \cap e$ is a simple closed curve $C$. Then, by the definition of an S.G.P. 2-orbifold and Claim 1, $C \subset \text{Int} e$. By Claim 2, there is a sub-disc $D'$ of $|F_0|$ and 3-simplex $\Delta \in K_M^{(3)}$ s.t. a face of $\Delta$ is $e$, $D'$ is properly embedded in $\Delta$, and $\partial D' = C$ (see Figure 2.1). Then, we can immediately get an S.G.P. 2-orbifold $F' \in \mathcal{F}_\Omega$ s.t. $(\alpha(F'), \beta(F')) < (\alpha(F_0), \beta(F_0))$. It is a contradiction.

Claim 4. For any $\Delta \in K_M^{(3)}$, the boundary of each component of $|F_0| \cap \Delta$ does not pass the same edge of $\Delta$ more than twice.

Suppose that there is an edge $\ell$ of $\Delta$ and a component $D'$ of $|F_0| \cap \Delta$ (by Claim 2, $D'$ is a disc) s.t. $\partial D'$ intersects $\ell$ more than two points. Let $x_1, x_2$ be the points of $\partial D' \cap \ell$ which are innermost in $\ell$. Let $\ell_1$ be the subarc of $\ell$ bounded by $x_1$ and $x_2$. 
Let \( \ell_2 \) be arbitrary arc properly embedded in \( D' \) s.t. \( \partial \ell_2 = \{x_1, x_2\} \). Since \( \Delta \) is homeomorphic to a 3-ball and \( D' \) is a disc properly embedded in \( \Delta \), \((\Delta, D', \ell_1, \ell_2)\) is homeomorphic to those in Figure 2.2. For example, they are as in Figure 2.3(1).

Suppose \( \ell \subseteq \text{Int } M \). By Claim 1, there are disc neighborhoods \( B_i \)'s of \( x_i \)'s in \( F_0 \) s.t. \( B_i \) intersects \( \ell \) transversely at \( x_i \), \( i = 1, 2 \), and \( B_1 \cap B_2 = \emptyset \). Furthermore, \( \ell_1 \cup \ell_2 \) bounds a disc in \( \Delta \). Using these facts, we can decrease \( \alpha \) by a D-modification as in Figure 2.4 (see Figure 2.3(2)). It is a contradiction.

Suppose \( \ell \subseteq \partial M \). By Claim 1, there is a regular neighborhood \( U \) of \( \ell_1 \) in \( M \) s.t. \( U \cap F_0 \) consists of two half-disc neighborhoods \( E_i \)'s of \( x_i \)'s in \( F_0 \) s.t. \( E_i \) intersects \( \ell \) transversely at \( x_i \), \( i = 1, 2 \), and \( E_1 \cap E_2 = \emptyset \). Using these facts, we can change \( F_0 \) as in Figure 2.5 to get new S.G.P. 2-orbifolds \( F_1 \) and \( F_2 \). We may assume that \( F_1 \in \mathfrak{F}_\Omega \).

It is clear that \( (\alpha(F_1), \beta(F_1)) < (\alpha(F_0), \beta(F_0)) \). It is a contradiction.
By Claims 1–4, each component of $|F_0| \cap \Delta, \Delta \in K^{(3)}_M$ must be one of those in Figure 1.1.

Throughout Propositions 2.7 and 2.9, Definition 2.8 and Remark 2.10, $M$, $K_M$, $\Omega$, and $\mathfrak{F}_\Omega$ are as same as in Lemma 2.6.

**Proposition 2.7.** Let $F \in \mathfrak{F}_\Omega$ be a normal 2-orbifold. For any normal 2-orbifold $F' \in \mathfrak{F}_\Omega$, it stands that $w(F) \leq w(F')$ if and only if $F$ attains the minimum of the complexity $(\alpha, \beta)$ in $\mathfrak{F}_\Omega$.

**Proof.** Let $(\alpha_0, \beta_0)$ be the minimum of $(\alpha, \beta)$ in $\mathfrak{F}_\Omega$. Suppose $G \in \mathfrak{F}_\Omega$ attains $(\alpha_0, \beta_0)$. By Lemma 2.6, $G$ is a normal 2-orbifold. Hence $w(F) \leq w(G)$. On the other hand, by the minimality of $(\alpha_0, \beta_0)$, $\alpha_0 \leq \alpha(F)$. Since $\alpha_0 = \alpha(G) = w(G)$
and $\alpha(F) = w(F), \beta(F) = \alpha_0$. Since both $F$ and $G$ are normal 2-orbifolds, $\beta(F) = \beta(G) = 0$. Then, $(\alpha_0, \beta_0) = (\alpha(F), \beta(F))$.

Suppose $(\alpha(F), \beta(F)) = (\alpha_0, \beta_0)$. Since $\alpha(F) = w(F), \alpha(F') = w(F')$,
\[
w(F) = \alpha(F) = \alpha_0 \leq \alpha(F') = w(F').
\]

\[
\square
\]

Definition 2.8. We call $H \in \mathcal{F}_\Omega$ a decreased normalization of $F \in \mathcal{F}_\Omega$ if $H$ is a normal 2-orbifold and $\text{Area}(H) < \text{Area}(F)$.

Proposition 2.9. If $F \in \mathcal{F}_\Omega$ is not normal, then there is a decreased normalization $H$ of $F$.

Proof. Suppose $G \in \mathcal{F}_\Omega$ achieves the minimum of the complexity $(\alpha, \beta)$ in $\mathcal{F}_\Omega$. By Lemma 2.6, $G$ is a normal 2-orbifold and $w(G) \leq w(F)$.

If $w(G) < w(F)$, we complete the proof.

Suppose $w(G) = w(F)$. Then $\alpha(G) = \alpha(F)$. Since $F$ must not attain the minimum of $(\alpha, \beta)$ in $\mathcal{F}_\Omega$, $\beta(G) > \beta(G)$. Since $\beta(G) = 0, 1 \leq \beta(F)$. Let $e \in K^{(2)}_M$ intersect with $F$ in simple closed curves and $C$ be the innermost component of such simple closed curves. Let $E$ be the disc on $e$ bounded by $C$. We cut open $F$ along $C$ and attach the copies of $E$ to get S.G.P. 2-orbifolds $f_1 : F_1 \to M, f'_1 : F'_1 \to M$. We may assume $f_1 : F_1 \to M \in \mathcal{F}_\Omega$.

Suppose $w(F'_1) \neq 0$, then $w(F_1) < w(F)$. By Lemma 2.6, $w(G) < w(F_1)$, then $w(G) < w(F)$. It is a contradiction.

Hence $w(F'_1) = 0$. Then $|F_1| \cap K^{(2)}_M$ is derived from $|F| \cap K^{(2)}_M$ by the deletion of some simple closed curves (see Figure 2.6). (In fact, $F_1$ and $F$ are ambient isotopic.
in \( M \). Therefore \( \alpha(F_1) = \alpha(F) \) and \( \beta(F_1) < \beta(F) \). The second inequality leads to \( \ell(F_1) < \ell(F) \). If \( F_1 \) is normal, we complete the proof.

Otherwise, since \( w(G) = w(F) = w(F_1) \), we can iterate the preceding process to get \( F_2 \in \mathcal{F}_\Omega \) s.t. \( \alpha(F_2) = \alpha(F) \), \( \beta(F_2) < \beta(F_1) \), and \( \ell(F_2) < \ell(F_1) \). Since \( \beta \in \mathbb{Z}_+ \), by iterating the above process, we can get \( H \in \mathcal{F}_\Omega \) s.t. \( \alpha(H) = \alpha(F) \), \( \beta(H) < \beta(F) \), \( \ell(H) < \ell(F) \), and \( H \) is normal. Since \( (\alpha(H), \beta(H)) < (\alpha(F), \beta(F)) \), \( H \) is the desired normalization. \( \square \)

**Remark 2.10.** If \( F \in \mathcal{F}_\Omega \) is the least area 2-orbifold in \( \mathcal{F}_\Omega \), then \( F \) is normal.

### 3. Intersections of minimal 2-orbifolds

Throughout this section we assume that all orbifolds are locally orientable again.

**Definition 3.1.** A simplicial division \( K_M \) of a 3-orbifold \( M \) is **sufficiently refined** if

1. for any \( e^{(1)} \in K_M^{(1)} \), there are at least three 2-simplices \( \Delta_1^{(2)}, \Delta_2^{(2)}, \Delta_3^{(2)} \in K_M^{(2)} \) s.t. for \( \forall i = 1, 2, 3 \), \( e^{(1)} \) is a 1-face of \( \Delta_i^{(2)} \), and

2. \( K_M \) satisfies \((*)\) in Lemma 2.6.

**Remark 3.2.** By taking a barycentric subdivision, we always have a sufficiently refined division.

Let \( M \) be a 3-orbifold and \( K_M \) a simplicial division of \( M \) which is equipped with a P.H. metric and is sufficiently refined. Let \( F \) be an embedded normal 2-orbifold in \( M \). From the fact that \( F \) is embedded and transverse to \( K_M^{(1)} \), it stands that for any cells \( e_1, e_2 \) of normal 2-orbifold \( F \) which are included in the same 3-simplex \( \Delta \in K_M \), \( e_1 \cap e_2 = \phi \). Let \( F_1, F_2 \) be properly embedded normal 2-orbifolds in \( M \). Then, each component of \( \Delta \cap F_1 \cap F_2 \), \( \Delta \in K_M^{(3)} \), is an intersection of a cell \( e_1 \in K_M^{(2)} \) and a cell \( e_2 \in K_M^{(2)} \). Furthermore, suppose that \( F_1 \) and \( F_2 \) are minimal 2-orbifolds in \( N(F_1) \) and \( N(F_2) \), respectively. We can classify such intersections as follows.

**Lemma 3.3.** If \( e_1 \cap e_2 \neq \phi \), \( e_1 \in K_{F_1}^{(2)}, e_2 \in K_{F_2}^{(2)} \), then the pair \( e_1, e_2 \) is as one of (1)–(23) in Figure 3.1. (The thick lines, slant lines, and thick points indicate the intersections.)

**Proof.** Since \( F_i \)'s are minimal 2-orbifolds, each \( e_i^{(1)} \in K_{F_i} \) is uniquely embedded in a 2-simplex \( \Delta_i^{(2)} \in K_M^{(2)} \) as the geodesic line between \( \partial e_i^{(1)} \). Hence, we only have to investigate the configuration of the vertices of \( e_1 \) and \( e_2 \) in \( \Delta \). \( \square \)
For example, Figure 3.1(17), (20) and (21) are realized as in Figure 3.2. Hence, we can regard $|F_1| \cap |F_2|$ as a 2-complex in $|F_i|$, $i = 1, 2$, of which vertices are thick points, edges are straight thick lines, and faces are (5) and (23) in Figure 3.1. Put $\Gamma = |F_1| \cap |F_2|$. Note that the situation of local intersection coincides each other, i.e. for each $x \in \Gamma$, there are appropriate disc neighborhoods $U_i \subset |F_i|$ of $x$, $i = 1, 2$, s.t. $(U_1, U_1 \cap \Gamma)$ is homeomorphic to $(U_2, U_2 \cap \Gamma)$.

**Lemma 3.4.** Let $F_i$ be the minimal 2-orbifolds in $\mathfrak{M}(F_i)$, $i = 1, 2$. Put $\Gamma = |F_1| \cap |F_2|$. Then, no components of $\Gamma$ are points.

**Proof.** Suppose that a component of $\Gamma$ is a point $P$. Then, $F_1$ touches $F_2$ at $P$. By Remark 1.2, $P$ must not be an interior point of a 3-simplex of $K_M^{(3)}$. By the minimality of $\ell(F_1)$ and $\ell(F_2)$, $P$ must not be an interior point of a 2-simplex of $K_M^{(2)}$. Hence $P$ must be an interior point of a 1-simplex $\ell$ of $K_M^{(1)}$ and there are disc (or half-disc) neighborhoods $U_i$ of $P$ in $|F_i|$, $i = 1, 2$, s.t. $U_1 \cap U_2 = P$ (see Figure 3.3). Let $\Delta_1^{(2)}, \ldots, \Delta_k^{(2)}$ be all 2-cells of $\text{St}(\ell, K_M^{(2)})$. Let $I_1, \ldots, I_k$ (respectively $I_1', \ldots, I_k'$) be 1-cells of $K_{F_1}$ (respectively $K_{F_2}$) s.t. $I_j = U_1 \cap \Delta_j^{(2)}$ (respectively $I_j' = U_2 \cap \Delta_j^{(2)}$, $j = 1, \ldots, k$ (see Figure 3.4). Let $V$ be a unit vector tangent to $\ell$ at origin $P$. Let $T_j$ (respectively $T_j'$) be the unit vector tangent to $I_j$ (respectively $I_j'$) at origin $P$ oriented to the interior of $I_j$ (respectively $I_j'$). Let $t_j V$ (respectively $t_j' V$) be the orthogonal projection of $T_j$ (respectively $T_j'$). Since $U_1 \cap U_2 = P$, we may assume that $t_j > t_j'$, $i = 1, 2, \ldots, k$. On the other hand, since both $\ell(F_1)$ and $\ell(F_2)$ are minimal, by Definition 1.12, $H_{F_1}(P) = H_{F_2}(P) = 0$. 


That is, \((V, \sum_{j=1}^{k} T_j) = (V, \sum_{j=1}^{k} T'_j) = 0\). Hence \(\sum_{j=1}^{k} t_j = \sum_{j=1}^{k} t'_j = 0\). It is a contradiction. \(\square\)

**Definition 3.5.** Let \(F\) be a 2-manifold and \(\Gamma\) a 2-complex in \(F\). A vertex \(P \in \Gamma^{(0)}\) is called a **free vertex of \(\Gamma^{(1)}\)** if there is a 1-simplex \(I \in \Gamma^{(1)}\) s.t. \(St(P, \Gamma) = I\) (Figure 3.5(1)). Such \(I\) is called a **terminal edge of \(\Gamma\)**. A vertex \(P \in \Gamma^{(0)}\) is called a **free vertex of \(\Gamma^{(2)}\)** if there are 2-simplices \(E_1, \ldots, E_r \in \Gamma^{(2)}\) s.t. \(St(P, \Gamma) = \bigcup_{i=1}^{r} E_i\) and \(\bigcup_{i=1}^{r} E_i\) is a 2-manifold (Figure 3.8). Let \(E_1, \ldots, E_r\) be 2-simplices of \(\Gamma^{(2)}\). \(\bigcup_{i=1}^{r} E_i\) is called a **terminal domain of \(\Gamma\)** if \(\bigcup_{i=1}^{r} E_i\) is a 2-manifold and \((\bigcup_{i=1}^{r} E_i) \cap \text{cl}(\Gamma - \bigcup_{i=1}^{r} E_i)\) consists of only one vertex.

**Lemma 3.6.** Let \(F_i\) be the minimal 2-orbifolds in \(\mathcal{M}(F_i), i = 1, 2\). Put \(\Gamma = |F_1| \cup |F_2|\). Let \(I\) be a terminal edge of \(\Gamma\) and \(P\) a free vertex of \(I\). Then \(P \in \partial F_i\) and \(\text{Int } I \cap \partial F_i = \phi, i = 1, 2\) (see Figure 3.5(2)).

**Proof.** Suppose \(P \in \text{Int } F_1\). Since \(F_i\) are properly embedded, \(P \in \text{Int } F_2\). Hence, there are disc neighborhoods \(U_i\) of \(P\) in \(F_i, i = 1, 2\), s.t. \(U_1\) touches \(U_2\) with \(I, I \cap \partial U_i = \partial I - \{P\}, P \in \text{Int } U_i, i = 1, 2\). By the same reason as in the proof of Lemma 3.4, \(P\) must be an interior point of a 1-simplex \(\ell\) of \(K_M\). See Figure 3.6. Since \(P\) is on the 1-simplex \(\ell\) of \(K_M\), \(P\) must be a vertex of a cell of \(K_{F_i}\). Furthermore, \(I\) must be an edge of a cell of \(K_{F_i}\), since the edge of \(\Gamma\) properly embedded in a cell of \(K_{F_i}\) is realized as the transversal intersection of \(F_1\) and \(F_2\). Hence \(I\) must be a thick
Figure 3.4.

Figure 3.5.
edge of a cell of $K_{F_1}$ indicated in Figures 3.1(3), (7), (15), (17) and (18), and $P$ is the boundary vertex of it. (In Figures 3.1(17) and (18), $P$ must still be the boundary point of thick lines.) Let $\Delta_1^{(2)}, \ldots, \Delta_k^{(2)}$ be 2-cells of $\text{St}(\ell, K_M^{(2)})$. Let $I_1, \ldots, I_k$ (respectively $I'_1, \ldots, I'_k$) be 1-cells of $K_{F_1}$ (respectively $K_{F_2}$) s.t. $I_j = U_1 \cap \Delta_j^{(2)}$ (respectively $I'_j = U_2 \cap \Delta_j^{(2)}$), $j = 1, \ldots, k$. Let $V$ be the unit vector tangent to $\ell$ at origin $P$. Let $T_j$ (respectively $T'_j$) be the unit vector tangent to $I_j$ (respectively $I'_j$) at origin $P$ oriented to the interior of $I_j$ (respectively $I'_j$). Let $t_jV$ (respectively $t'_jV$) be the orthogonal projection of $T_j$ (respectively $T'_j$). Since $U_1$ touches $U_2$ in $I$, we may assume that $I_1 = I'_1 = I$, hence $t_1 = t'_1$, and $t_\nu > t'_\nu$, $\nu = 2, \ldots, k$. By the same reason as in the proof of Lemma 3.4, this is a contradiction. Therefore $P \in \partial F_i$, $i = 1, 2$.

Suppose $I \subset \partial F_1$. Then, we can take half-disc neighborhoods $U_1, U_2$ of $P$ in $|F_1|, |F_2|$, respectively, s.t. $U_1$ touches $U_2$ with $I$ and $I \subset \partial U_i$, $i = 1, 2$ (see Figure 3.7). By the same argument as above, we can derive a contradiction.

The case that $\text{Int} I \subset \text{Int} F_i$ and $\partial I \subset \partial F_i$ must not occur since $K_M$ is sufficiently refined.

\begin{lemma}
Let $F_i$ be the minimal 2-orbifolds in $\mathfrak{M}(F_i)$, $i = 1, 2$. Put $\Gamma = |F_1| \cap |F_2|$. Let $P$ be a free vertex of $\Gamma^{(2)}$ and $E_1, \ldots, E_r$ be 2-simplices of $\Gamma$ s.t. $\text{St}(P, \Gamma) = \bigcup_{j=1}^r E_j$ (see Figure 3.8). Then either (i) or (ii) holds:
\end{lemma}
(i) $P \in \text{Int} F_1$ and $\text{St}(P, \Gamma) = \text{St}(P, K_{F_1}) - (\text{one rectangle of } \text{St}(P, K_{F_1}))$, $i = 1, 2$ (see Figure 3.8(2));

(ii) $P \in \partial F_i$, and $(\bigcup_{j=1}^{i} E_j) \cap \partial F_i = (\text{either } P \text{ or two edges of } \partial(\bigcup_{j=1}^{i} E_j) \text{ including } P)$, $i = 1, 2$ (see Figure 3.8(3)).

Proof. Suppose $P \in \text{Int} F_1$. Since $K_M$ is sufficiently refined, we can use the same argument as in the proof of Lemma 3.6, and derive conclusion (i).

Suppose $P \in \partial F_1$. Since $K_M$ is sufficiently refined, and (i) holds, $(\bigcup_{j=1}^{i} E_j) \cap \partial F_1 = (\text{either } P, \text{ one edge of } \partial(\bigcup_{j=1}^{i} E_j) \text{ including } P, \text{ or two edges of } \partial(\bigcup_{j=1}^{i} E_j) \text{ including } P)$. By the same argument as in the proof of Lemma 3.6, we have conclusion (ii). \qed
Definition 3.8. Let $F$ be a 2-manifold and $\Gamma$ be a 2-complex in $F$. $\Gamma$ is tamely embedded in $F$ if (i)–(iv) hold.

(i) There is no component of $\Gamma$ which is a point.
(ii) If $P$ is a free vertex of a terminal edge $I$ of $\Gamma$, then $P \subseteq \partial F$ and $\text{Int} I \cap \partial F = \emptyset$.
(iii) If $P \in \Gamma^{(2)}$ is a free vertex of $\Gamma^{(2)}$ and $P \in \text{Int} F$, then no vertices of $\text{Lk}(P, \partial \Gamma)$ are free vertices of $\Gamma^{(2)}$ included in $\text{Int} F$ and at least one vertex of $\text{Lk}(P, \partial \Gamma)$ is included in $\text{Int} F$.
(iv) If for a simplex $I \in \Gamma^{(1)}$ which is not a terminal edge, $\partial I \subset \partial F$ then $I \subset \partial F$.

Remark 3.9.

(1) Suppose a 2-complex $\Gamma$ is tamely embedded in a 2-manifold $F$. If $\Gamma$ is contractible, then, by Definition 3.8(i), $\Gamma$ is a one- or two-dimensional tree. By Definition 3.8(ii), each free vertex of a terminal edge of $\Gamma$ meets $\partial F$ transversely. Let $E$ be a terminal domain of $\Gamma$. By Definition 3.8(iii) and the fact that $\partial E \cap \Gamma^{(0)}$ consists of more than three vertices, there is at least one free vertex as Figure 3.8(3) in each terminal domain.

(2) If $F_i$’s are the minimal 2-orbifolds in $N(F_i)$, then $|F_1| \cap |F_2|$ is tamely embedded in $|F_i|$, $i = 1, 2$. (By Lemma 3.4 (respectively Lemma 3.6), Definition 3.8(i) (respectively Definition 3.8(ii)) is derived. By Lemmas 3.3 and 3.7 and the fact that $M$ is sufficiently refined, Definition 3.8(iii) is derived. Definition 3.8(iv) is derived from the fact that $M$ is sufficiently refined.)

Lemma 3.10. Suppose a 2-complex $\Gamma$ is tamely embedded in a 2-manifold $F$. If a component of $\Gamma$ is a topological 2-manifold, then $\Gamma = F$.

Proof. Since all vertices of $\partial \Gamma$ are free vertices, from Definition 3.8(iii), all vertices are included in $\partial F$. Hence, by Definition 3.8(iv), $\partial \Gamma \subset \partial F$. □

Definition 3.11. Let $F$ be a 2-manifold and $\Gamma$ a subset in $F$. A simple curve (a closed curve or a properly embedded arc) $C$ in $F$ is called an innermost curve of $(F, \Gamma)$ if $C \subseteq \Gamma$, $C$ separates $F$, and for a subset $F'$ of $F$ bounded by $C$, $F' \cap \Gamma = C$.

Proposition 3.12. Let $F$ be a 2-disc or 2-sphere. Let $\Gamma$ be a 2-complex tamely embedded in $F$. Suppose $\Gamma^{(1)}$ includes a simple closed curve $C$. Let $E$ be any 2-disc in $F$ bounded by $C$. If $E \not\subseteq \Gamma$, then there is an innermost simple closed curve of $(F, \Gamma)$ in $E$.

Proof. By Definition 3.8(i), dim $\Gamma \geq 1$. In case dim $\Gamma = 1$, the proposition is clear by Definition 3.8(ii). Suppose dim $\Gamma = 2$. Let $E$ be the set of all simple closed curves in
\[ \Gamma^{(1)} \cap E. \text{ Put} \]

\[ \mathcal{C}_0 = \{ C \in \mathcal{C} \mid C \text{ does not bound any subcomplex in } \Gamma^{(2)} \cap E \}. \]

Since \( E \not\subset \Gamma \), \( \mathcal{C}_0 \neq \emptyset \). For \( C_1, C_2 \in \mathcal{C}_0 \), define \( C_1 \preceq C_2 \) if \( C_1 \) is included in the disc in \( F \) bounded by \( C_2 \). \((\mathcal{C}_0, \preceq)\) is a partially ordered set.

**Claim.** The minimum element of \((\mathcal{C}_0, \preceq)\) is the desired simple closed curve.

Indeed, let \( C \) be the minimum element. Let \( D \) be the disc in \( F \) bounded by \( C \). If \( C \) is not innermost, then there is a subcomplex \( \Gamma' \) of \( \Gamma \) s.t. \( \Gamma' \) is a connected component of \( \text{cl}((\text{Int} \ D) \cap \Gamma) \). Suppose \( \Gamma' \cap C = \emptyset \). By Remark 3.9(1), \( \Gamma' \) is not contractible. Hence there is an element of \( \mathcal{C}_0 \) in \( \Gamma' \). This contradicts the fact that \( C \) is the minimum element. Thus, \( \Gamma' \cap C \neq \emptyset \).

Suppose that \( \Gamma' \cap C \) includes more than two vertices. Let \( P \) and \( Q \) be vertices of \( \Gamma' \cap C \) and \( L \) a simple arc in \( \Gamma' \) joining \( P \) and \( Q \). We may assume that \((\text{Int} \ L) \cap C = \emptyset \). Let \( D_1 \) and \( D_2 \) be discs in \( D \) s.t. \( D_1 \cup D_2 = D \) and \( D_1 \cap D_2 = L \). Since \( D \not\subset \Gamma' \), at least one of \( D_1 \) and \( D_2 \) is not included in \( \Gamma' \) (see Figure 3.9). We may assume \( D_1 \not\subset \Gamma' \). Then there is an element of \( \mathcal{C}_0 \) in \( D_1 \). It is a contradiction.

Suppose that \( \Gamma' \cap C \) includes only one vertex. Since \( \Gamma' \) does not include an element of \( \mathcal{C}_0 \), \( \Gamma' \) must be contractible. By Remark 3.9(1), \( \Gamma' \) has only one free vertex of \( \Gamma^{(1)} \) or \( \Gamma^{(2)} \). This is impossible. \( \square \)
PROPOSITION 3.13. Let $F$ be a 2-disc and $\Gamma$ be a 2-complex tamely embedded in $F$. Suppose $\Gamma^{(1)}$ includes a simple arc $A$ properly embedded in $F$. Let $E$ be any 2-disc in $F$ bounded by $A$. If $E \not\subset \Gamma$ and $H_1(E \cap \Gamma) = 0$, then there is an innermost simple arc of $(F, \Gamma)$ in $E$.

Proof. Let $\mathfrak{A}$ be the set of all simple arcs in $\Gamma^{(1)} \cap E$. Put $\mathfrak{A}_0 = \{ A \in \mathfrak{A} \mid [A] \neq 0 \text{ in } H_1(\Gamma \cap (\partial E - \text{Int } A), \partial E - \text{Int } A)\}$. Since $E \not\subset \Gamma$, $\mathfrak{A}_0 \neq \emptyset$. For $A_1, A_2 \in \mathfrak{A}_0$, define $A_1 \leq A_2$ if $A_1$ is included in the disc in $E$ bounded by $A_2$. $(\mathfrak{A}_0, \leq)$ is a partially ordered set. By the argument similar to the proof of Proposition 3.12 and using the fact that $H_1(E \cap \Gamma) = 0$, we can show that the minimal element of $(\mathfrak{A}_0, \leq)$ is the desired arc. 

Definition 3.14. Let $F$ be a 2-manifold and $\Gamma$ be a 2-complex in $F$. $\Gamma$ evenly includes innermost curves of $(F, \Gamma)$ if for any simple arc $A \subset \Gamma^{(1)}$ properly embedded in $F$ (respectively a simple closed curve $C \subset \Gamma^{(1)}$) and for the closure $F'$ of any component of $F - A$ (respectively $F - C$) s.t. $F' \not\subset \Gamma$, there is an innermost curve of $(F, \Gamma)$ in $F'$.

COROLLARY 3.15. Let $D_i$ be an embedded normal 2-orbifold whose underlying space is a 2-disc and which is minimal in $N(D_i)$, $i = 1, 2$. Let $\Gamma$ be the 2-complex $|D_1| \cap |D_2|$. Then $\Gamma$ evenly includes innermost curves of $(|D_i|, \Gamma)$, $i = 1, 2$.

Proof. By Remark 3.9(2), $\Gamma$ is tamely embedded in $|D_i|$. Let $C$ be a simple closed curve in $\Gamma^{(1)}$ and $D'_i$ be the 2-disc in $|D_i|$ bounded by $C$, $i = 1, 2$. If $D'_i \not\subset \Gamma$ then, by Proposition 3.12, there is an innermost simple closed curve of $(|D_i|, \Gamma)$ in $D'_i$, $i = 1, 2$.

Let $A$ be a simple arc in $\Gamma^{(1)}$ properly embedded in $|D_i|$ and $D'_i$ be a 2-disc separated by $A$, $i = 1, 2$. Suppose $D'_i \not\subset \Gamma$. If $H_1(D'_i \cap \Gamma) \neq 0$, $D'_i \cap \Gamma$ includes a simple closed curve $C$ in $\Gamma^{(1)}$ s.t. $D''_i \not\subset \Gamma$, where $D''_i$ is the disc in $D'_i$ bounded by $C$. Hence, by Proposition 3.12, there is an innermost simple closed curve of $(|D_i|, \Gamma)$ in $D''_i$ (hence, in $D'_i$), $i = 1, 2$. If $H_1(D'_i \cap \Gamma) = 0$, then, by Proposition 3.13, there is an innermost simple arc of $(|D_i|, \Gamma)$ in $D'_i$, $i = 1, 2$.

COROLLARY 3.16. Let $S_i$ be an embedded normal 2-orbifold whose underlying space is a 2-sphere and which is minimal in $N(S_i)$, $i = 1, 2$. Let $\Gamma$ be a 2-complex $|S_1| \cap |S_2|$. Then $\Gamma$ evenly includes innermost curves of $(|S_i|, \Gamma)$, $i = 1, 2$.

Proof. The proof is similar to the former part of Corollary 3.15. 

\[\Box\]
4. Least area 2-orbifolds

Throughout this section, we assume that all orbifolds are locally orientable again.

Definition 4.1. A class Ω of 2-suborbifolds of a 3-orbifold $M$ is surgery-invariant if for any $F, G \in \Omega$ and any innermost curve $C$ of $(|G|, |F| \cap |G|)$,

(a) $C$ separates $F$ into two subspaces $F_1$ and $F_2$,

(b) if $s(E) \leq s(F_i), i = 1, 2$, at least one of $(F_1 \cup E)'$ and $(F_2 \cup E)'$ belongs to $\Omega$, where $E$ is the closure of the subspace of a component of $G - C$ s.t. $E \cap F = C$, and $(X)'$ is the suborbifold of $M$ derived from perturbing $X$ to delete points tangent to $\Sigma M$.

Example 4.2. The examples in Example 2.5 are also surgery-invariant classes.

Proof. Definition 4.1(a) and (b). The fact that $\Omega$ is surgery invariant is equivalent to the fact that $\Omega$ is $D$-invariant. Hence the conclusion is derived from (a) and (b) of Example 2.5.

(c). Let $S_1, S_2 \in \Omega$. Let $C$ be the innermost curve of $(|S_2|, |S_1| \cap |S_2|)$. Let $D$ be the subspace of $S_2$ bounded by $C$ s.t. $|D| \cap |S_1| = C$. Suppose $s(D) \leq s(D_1)$, where $D_1$ s are the subspaces of $S_2$ separated by $C$. Noting that $s(D) \leq s(D_1)$, we illustrate the types of $\Sigma S_1$ relative to $\partial D$ in Figure 4.1 (respectively Figure 4.2) when $\Sigma M \cap \partial D = \phi$ (respectively $\Sigma M \cap \partial D \neq \phi$). Figures 4.1 and 4.2(1)–(4) show that at least one of $D \cup D_1$ and $D \cup D_2$ is a bad subsphere. Figure 4.2(5) shows that $(D \cup D_1)'$ and $(D \cup D_2)'$ are bad spheres.

(d). Let $S_1', C, D$, and $D_1$ be as same as in (c). Noting that $s(D) \leq s(D_1)$ and there are no bad sub spheres in $M$, we illustrate the types of $\Sigma S_1$ relative to $\partial D$ in Figure 4.3 (respectively Figure 4.4) when $\Sigma M \cap \partial D = \phi$ (respectively $\Sigma M \cap \partial D \neq \phi$). Figure 4.3 shows that both of $D \cup D_1$ and $D \cup D_2$ are spherical sub orbifolds. Since $S_1$ does not bound any ballic sub orbifold, at least one of $D \cup D_1$ and $D \cup D_2$ does not bound any ballic sub orbifolds. Figure 4.4 shows that one of $D \cup D_1$ and $D \cup D_2$ is a spherical sub orbifold and another is a subspace tangent to $\Sigma M$. If the spherical
suborbifold part does not bound a ballic suborbifold, we are done. If the spherical suborbifold part bounds a ballic suborbifold, then, by perturbing the other subspace part and using the fact that \( S_1 \) does not bound any ballic suborbifold in \( M \), we can find an incompressible spherical suborbifold \( S' \) in \( M \).

\[ \square \]

**Theorem 4.3.** Let \( M \) be a 3-orbifold and \( \Omega \) be a surgery-invariant class of 2-suborbifolds in \( M \). Let \( K_M \) be a simplicial division of \( M \) which is equipped with a P.H. metric and is sufficiently refined. Let \( \mathcal{F}_\Omega = \{ F \in \Omega \mid F \text{ is an S.G.P. 2-orbifold w.r.t. } K_M \} \). Suppose \( \mathcal{F}_\Omega \) satisfies the following:
(1) for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
(2) for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $f_0 : F_0 \to M \in N(F)$ is an orb-embedding;
(3) for any normal 2-orbifolds $F, G \in \mathcal{F}_\Omega$, the 2-complex which is the intersection of the underlying spaces of the minimal 2-orbifolds $F_0, G_0 \in N(F), N(G)$ evenly includes innermost curves of $(|F_0|, |F_0| \cap |G_0|)$ and $(|G_0|, |F_0| \cap |G_0|)$.

Suppose $F_1, F_2 \in \mathcal{F}_\Omega$ are the least area 2-orbifolds of $\mathcal{F}_\Omega$. Then either $F_1 \cap F_2 = \emptyset$ or $F_1 = F_2$.

**Proof.** Note that, by (1), $F_i$ is normal 2-orbifold and that, by (2), $F_i$ is minimal in $N(F_i), i = 1, 2$. Let $K_{F_i}$ be the normal division of normal 2-orbifold $F_i, i = 1, 2$. Let $\Gamma$ be the 2-complex s.t. $\Gamma = |F_i| \cap |F_2|$. Put

$$\mathcal{E} = \{ E \mid E \text{ is a subspace of } F_i, |E| \text{ is a 2-manifold in } |F_i| \text{ bounded by an innermost curve of } (|F_i|, \Gamma), i = 1 \text{ or } 2 \}.$$

Suppose $F_1 \cap F_2 \neq \emptyset$ and $F_1 \neq F_2$. Then, by Lemma 3.10, $\Gamma \neq |F_1|, \Gamma \neq |F_2|$, and $\Gamma \neq \emptyset$. Hence, by (3), $\mathcal{E} \neq \emptyset$. Since $\mathcal{E}$ is a finite set, there is an element $D \in \mathcal{E}$ s.t. $\text{Area}(D) = \min\{\text{Area}(E) \mid E \in \mathcal{E}\}$. We may assume $D \subset F_2$. Let $D'$ be the closure of any component of $F_1 - \partial D$. Note that $\partial|D|$ is not necessarily an innermost curve of $(|F_1|, \Gamma)$. Since $\partial|D|$ is an innermost curve of $(|F_2|, \Gamma)$, we may assume that $|D'| \subset \Gamma$. By (3), we can take an innermost curve of $(|F_1|, \Gamma)$ in $|D'|$. Hence it holds that $\text{Area}(D) \leq \text{Area}(D')$. Otherwise the area of the subspace of $D'$ bounded by the innermost curve of $(|F_1|, \Gamma)$ is smaller than $\text{Area}(D)$. Put $D_1 = (F_1 - \text{Int } D') \cup D$. Since $D_1$ is an S.G.P. 2-orbifold and $\Omega$ is a surgery-invariant class, we may assume that $(D_1)' \in \mathcal{F}_\Omega$. If $(D_1)' \neq D_1$, then $s((D_1)') < s(D_1) \leq s(F_1)$. It is a contradiction. Hence, it holds that $(D_1)' = D_1$. Thus $D_1 \in \mathcal{F}_\Omega$.

**Claim 1.** $D_1$ is normal.

Otherwise, by (1), there is a decreased normalization $D_1^* \in \mathcal{F}_\Omega$ of $D_1$. Since $\text{Area}(D) \leq \text{Area}(D')$,

$$\text{Area}(F_1) = (s(F_1) - s(D') + s(D'), w(F_1) - w(D') + w(D'),$$
$$\ell(F_1) - \ell(D') + \ell(D'),$$
$$\geq (s(F_1) - s(D') + s(D), w(F_1) - w(D') + w(D),$$
$$\ell(F_1) - \ell(D') + \ell(D))$$
$$= \text{Area}(D_1).$$
Therefore if $\overline{D}_1$ is not normal, $\text{Area}(F_1) \geq \text{Area}(\overline{D}_1) > \text{Area}(D_1^*)$. It is a contradiction. Hence $\overline{D}_1$ is an embedded normal 2-orbifold.

**Case 1.** $\Gamma^{(0)} \cap \partial D' \not\subset K_{F_1}^{(0)}$ (see Figure 4.5).

There is a point $P \in \partial D'$ included in the interior of a 1-simplex of $K_{F_1}$. By observing Figure 3.1, $\partial D'$ meets $K_{F_1}^{(1)}$ transversely at $P$. $P$ is included in the interior of a 2-face of a 3-simplex $\Delta \in K_M$. There are cells $e_1 \in K_{F_1}^{(2)}$ and $e_2 \in K_{F_2}^{(2)}$ included in $\Delta$ s.t. $e_1$ meets $e_2$ transversely and $P \in e_1 \cap e_2$ (see Figure 4.6). Then we can modify $\overline{D}_1$ by a normal orbi-homotopy as indicated in Figure 4.7 to have a smaller length $\ell$ than that of $\overline{D}_1$. It is a contradiction.

**Case 2.** $\Gamma^{(0)} \cap \partial D' \subset K_{F_1}^{(0)}$.

**Claim 2.** There is a vertex $v \in \partial D' \cap K_{F_1}^{(0)}$ and an edge $I$ of $K_{F_1}^{(1)}$ s.t. $v \in \partial I$ and $\text{Int} I \subset \text{Int} D'$. 
Indeed, suppose $\partial D' \subset K_{F_i}^{(1)}$ (see Figure 4.8). If there is no such a point, then $D'$ is a cell of $K_{F_i}$ and $\partial D'$ is its boundary. By observing Figure 3.1, it holds that $D' \subset \Gamma$. By the preceding remark of Lemma 3.4 and the fact that $D$ is bounded by an innermost curve of $(|F_2|, \Gamma)$, for a sufficiently small neighborhood $A$ of $\partial D'$, $A \cap \Gamma \subset D'$. Hence, by Lemma 3.10, $\Gamma = |F_1|$. It is a contradiction.

Suppose $\partial D' \not\subset K_{F_i}^{(1)}$ (see Figure 4.8). By Figure 3.1, there is at least one cell $e \in K_{F_i}$ s.t. $(e, e \cap \partial D')$ is as Figure 3.1(22). If there are no points in Claim 2, $\partial D'$ must be included in a cell of $K_{F_i}^{(1)}$ as described in Figure 4.9. That is impossible.

Let $U_j$ be a sufficiently small discal neighborhood of $v$ in $F_i$, $i = 1, 2$. Let $\ell$ be the 1-simplex of $K_M$ which includes $v$ and $\triangle_1^{(2)}, \ldots, \triangle_k^{(2)}$ be 2-cells of $St(\ell, K_M^{(2)})$. Let $I_1, \ldots, I_k$ (respectively $I_1', \ldots, I_k'$) be 1-cells of $K_{F_1}$ (respectively $K_{F_2}$) s.t. $I_j = U_1 \cap \triangle_j^{(2)}$ (respectively $I_j' = U_2 \cap \triangle_j^{(2)}$), $j = 1, \ldots, k$. By Claim 2, we may assume that $Int I_1, \ldots, Int I_q \subset Int D'$ and $Int I_1', \ldots, Int I_q' \subset Int D$, $q \geq 1$. Let $V$ be a unit vector tangent to $\ell$ at the origin $v$. Let $T_j$ (respectively $T_j'$) be the unit vector tangent to $I_j$ (respectively $I_j'$) at the origin $v$ oriented to interior of $I_j$ (respectively $I_j'$).
Let \( t_j V \) (respectively \( t'_j V \)) be the normal projection of \( T_j \) (respectively \( T'_j \)) to \( V \). Since \( (U_1 \cap D') \cap (U_2 \cap D) \subset \partial D' \), we may assume that \( t_j > t'_j, 1 \leq j \leq q \). From the minimality of \( \ell(F_1) \), it stands that \( \sum_{j=1}^{k} t_j = 0 \). Hence

\[
H_{\overline{D}_1}(v) = \left( V, \sum_{j=1}^{k} T_j - \sum_{j=1}^{q} T_j + \sum_{j=1}^{q} T'_j \right)
\]

\[
= \sum_{j=1}^{k} t_j - \sum_{j=1}^{q} t_j + \sum_{j=1}^{q} t'_j
\]

\[
< 0.
\]

This means that \( \overline{D}_1 \) is not a minimal 2-orbifold. Hence by Proposition 1.14, we can discover an element of \( \mathcal{S}_\Omega \) whose length \( \ell \) is smaller than \( \ell(\overline{D}_1) \) and \( (s, w) \) is equal to \( (s(\overline{D}_1), w(\overline{D}_1)) \). This contradicts the minimality of \( \text{Area}(\overline{D}_1) = \text{Area}(F_1) \).

Thus, \( F_1 \cap F_2 = \emptyset \) or \( F_1 = F_2 \).

\[\square\]

5. The existence of the least area 2-orbifold

Let \( M \) be a (not necessarily locally orientable) 3-orbifold and \( p : \tilde{M} \to M \) any regular covering s.t. \( \tilde{M} \) is locally orientable. Let \( K_M \) be a simplicial division of \( M \) and \( \tilde{K_M} \) the simplicial division of \( \tilde{M} \) induced by the map \( p \). If \( K_M \) is equipped with a P.H. metric, then \( \tilde{K_M} \) has the P.H. metric naturally induced by the map \( p \). If \( K_M \) is sufficiently refined, then \( \tilde{K_M} \) is sufficiently refined. In Sections 5 and 6, we assume that \( \tilde{M} \) is equipped with such triangulation and P.H. metric.

Fix a 2-orbifold \( F \). Let

\[
\mathcal{F}(m, n) = \{ f : F \to \tilde{M} \text{ is a normal 2-orbifold w.r.t. } K_{\tilde{M}}, s(f) = m, w(f) = n \}.
\]
Define for $f, f' \in \mathcal{F}(m, n)$, $f \sim f'$ if there exists an element $\sigma \in \text{Aut}(M, p)$ s.t. $\sigma \circ f'$ and $f$ are normally orbi-homotopic.

**Lemma 5.1.** If for $f, f' \in \mathcal{F}(m, n)$, $p \circ f$ and $p \circ f'$ are normally orbi-homotopic w.r.t. $K_M$, then $f \sim f'$.

**Proof.** We can lift the normal orbi-homotopy between $p \circ f$ and $p \circ f'$ to $K_M$ along $f$.

**Lemma 5.2.** If $M$ is compact, then $\mathcal{F}(m, n)/\sim$ is a finite set.

**Proof.** Note that $s(p \circ f) = s(f)$ and $w(p \circ f) = w(f)$. By Lemma 5.1, we only have to show that there exist only finitely many normal orbi-homotopy types of normal 2-orbifolds whose singular weight and weight are $m$ and $n$, respectively. This is clear from the compactness of $M$.

Note that for a compact 3-orbifold $M$, we can always define a P.H. metric for its simplicial division $K_M$ by mapping $K_M^{(0)}$ into the boundary of the Poincaré disc mutually disjoint and extending them skeleton-wise.

**Theorem 5.3.** (The existence of the least area 2-orbifold) Let $M$ be a compact (not necessarily locally orientable) 3-orbifold and $p : \tilde{M} \to M$ be a regular covering s.t. $\tilde{M}$ is locally orientable. Let $\Omega$ be a class of 2-suborbifolds of $M$. Let $\mathcal{F}_\Omega = \{ F \in \Omega \mid F$ is an S.G.P. 2-orbifold w.r.t. $K_M^{(0)} \}$. Suppose $\mathcal{F}_\Omega$ satisfies the following:

1. for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
2. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, $F$ is not a linking 2-orbifold;
3. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $F_0 \in N(F)$ is an orbi-embedding.

If $\Omega \neq \emptyset$, then there is a least area 2-orbifold of $\mathcal{F}_\Omega$.

**Proof.** Let $F$ be an element of $\Omega$. By perturbing $F$, we may assume $F$ is normal w.r.t. $K_M$. Hence, there is an S.G.P. 2-orbifold $F_0 \in \Omega$. Let $m, n$ be the singular weight, weight of $F_0$, respectively. Put

$$\mathcal{F}(m, n) = \{ F \in \mathcal{F}_\Omega \mid (s(F), w(F)) \leq (m, n) \}.$$ 

Since $F_0 \in \mathcal{F}(m, n)$, $\mathcal{F}(m, n) \neq \emptyset$. Since $s(\mathcal{F}(m, n))$ and $w(\mathcal{F}(m, n))$ are non-negative integers, there is a minimal value $(m_0, n_0)$. Put

$$\mathcal{F}(m_0, n_0) = \{ F \in \mathcal{F}(m, n) \mid F$ is a normal 2-orbifold, $(s(F), w(F)) = (m_0, n_0) \}.$$

Note that, by (1), $\mathcal{F}(m_0, n_0) \neq \emptyset$. 

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Define for $F, F' \in \mathcal{F}(m_0, n_0)$, $F \sim F'$ if there is an element $\sigma \in \text{Aut}(\tilde{M}, p)$ s.t. $\sigma(F')$ and $F$ are normally orbi-homotopic. By Lemma 5.2, $\mathcal{F}(m_0, n_0)/\sim$ is a finite set. By (2), $F \in \mathcal{F}(m_0, n_0)$ is not a linking 2-orbifold. Hence by Theorem 1.13, for each $F \in \mathcal{F}(m_0, n_0)$, there is one and only one minimal immersed 2-orbifold in $N(F)$. By (3), such an immersion is an orbi-embedding, hence it belongs to $\mathcal{F}(m_0, n_0)$. Hence we can define a set

$$\mathcal{F}_\ell(m_0, n_0) = \{F \in \mathcal{F}(m_0, n_0) \mid F \text{ is the minimal 2-orbifold in } N(F)\}.$$ 

Claim. The cardinality of $\text{Area}(\mathcal{F}_\ell(m_0, n_0)) = (m_0, n_0, \ell(\mathcal{F}_\ell(m_0, n_0)))$ is finite.

Since $\mathcal{F}(m_0, n_0)/\sim$ is a finite set, $\mathcal{F}_\ell(m_0, n_0)/\text{Aut}(\tilde{M}, p)$ is also a finite set. Hence, we only have to show that, for any $\text{Aut}(\tilde{M}, p)$-equivariant 2-orbifolds $F_1, F_2 \in \mathcal{F}_\ell(m_0, n_0)$, $\ell(F_1) = \ell(F_2)$. Suppose $F_2 = g(F_1)$, $g \in \text{Aut}(\tilde{M}, p)$. Since the P.H. metric on $K^{(2)}_M$ is $\text{Aut}(\tilde{M}, p)$-equivariant, $\ell(F_1) = \ell(g(F_1)) = \ell(F_2)$.

Hence, there is an element $F^*$ of $\mathcal{F}_\ell(m_0, n_0)$ whose length is minimal in $\mathcal{F}_\ell(m_0, n_0)$. Clearly $F^*$ is the desired least area 2-orbifold of $\mathcal{F}_\Omega$. □

6. Some applications to 3-orbifolds

**Theorem 6.1.** Let $M$ be a compact (not necessarily locally orientable) 3-orbifold and $p : \tilde{M} \to M$ be a regular covering s.t. $\tilde{M}$ is locally orientable. Let $\Omega$ be a surgery-invariant class of 2-suborbifolds of $\tilde{M}$. Put $\mathcal{F}_\Omega = \{F \in \Omega \mid F \text{ is an S.G.P. 2-orbifold w.r.t. } K_{\tilde{M}}\}$. Suppose $\mathcal{F}_\Omega$ satisfies the following:

1. for any not normal $F \in \mathcal{F}_\Omega$, there is a decreased normalization $F_0 \in \mathcal{F}_\Omega$ of $F$;
2. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, $F$ is not a linking 2-orbifold;
3. for any normal 2-orbifold $F \in \mathcal{F}_\Omega$, the minimal 2-orbifold $f_0 : F_0 \to M \in N(F)$ is an orbi-embedding;
4. for any normal 2-orbifolds $F, G \in \mathcal{F}_\Omega$, the 2-complex which is the intersection of the underlying spaces of the minimal 2-orbifolds $F_0, G_0 \in N(F), N(G)$ evenly includes innermost curves of $(|F_0|, |F_0| \cap |G_0|)$ and $(|G_0|, |F_0| \cap |G_0|)$, respectively.

If $\Omega \neq \emptyset$, then there is an element $F_0 \in \Omega$ which is equivariant under $\text{Aut}(\tilde{M}, p)$.

**Proof.** By Theorem 5.3, there is a least area element $F_0 \in \mathcal{F}_\Omega$. Since Area is $\text{Aut}(\tilde{M}, p)$-equivariant, for any $g \in \text{Aut}(\tilde{M}, p)$, $g(F_0)$ is also a least area element of $\mathcal{F}_\Omega$. Then, by Theorem 4.3, $F_0$ is the desired equivariant element of $\Omega$. □

**Example 6.2.** Put $\Omega_A \sim \Omega_D$ as follows. Then $\Omega_A \sim \Omega_D$ are surgery invariant and $\mathcal{F}_\Omega_A \sim \mathcal{F}_\Omega_D$ satisfy Theorem 6.1(1)-(4).
(A) Let $M$ be a 3-manifold and $F$ be a 2-submanifold in $\partial M$. Put $\Omega_A = \{ D \mid D$ is a properly embedded 2-disc in $M$ s.t. $\partial D \subset F$ and $[\partial D] \neq 1$ in $\pi_1(F)\}.$

(B) Let $M$ be a simply connected 3-manifold. Put $\Omega_B = \{ S \mid S$ is a 2-sphere in $M$ s.t. $[S] \neq 0$ in $\pi_2(M)\}.$

(C) Let $M$ be a 3-orbifold with $\pi_1(|M|) = 1$. Put $\Omega_C = \{ S \mid S$ is adab subsphere in $M\}.$

(D) Let $M$ be an adab 3-orbifold with $\pi_1(|M|) = 1$. Put $\Omega_D = \{ S \mid S$ is an incompressible spherical 2-suborbifold in $M\}.$

Where, in each example, we suppose to take an appropriate simplicial division $K_M$ which equipped with a P.H. metric and sufficiently refined.

Proof. Note that $M$ in (C) and (D) is locally orientable since $\pi_1(|M|) = 1$. By Example 4.2, we see that $\Omega_A \sim \Omega_D$ are surgery invariant. By Proposition 2.9 (respectively Remark 1.15, respectively Corollaries 3.15 and 3.16), we see that $\mathfrak{F}_{\Omega_A} \sim \mathfrak{F}_{\Omega_D}$ satisfy (1) (respectively (3), respectively (4)). We may assume that for any $v \in K_M^{(0)}, \text{St}(v, K_M) \subset F$ is either empty or contractible, by taking a barycentric subdivision of $K_M$ if necessary. Hence, a normal disc whose boundary is a non-trivial curve in $F$ is not a linking 2-orbifold. Thus $\mathfrak{F}_{\Omega_A}$ satisfies (2). Since a linking 2-orbifold is a cone on a vertex, $\mathfrak{F}_{\Omega_B} \sim \mathfrak{F}_{\Omega_D}$ satisfy (2). □

Let $N$ be a suborbifold of an orbifold $M$. Let $p : \tilde{M} \to M$ be the universal covering. Let $\tilde{N} \to \tilde{N}$ be the universal covering. Fix a base point $x_0$ in $|N| - \Sigma N$. Furthermore, fix a base point $\tilde{x}_0$ in $p^{-1}(x_0)$ and $\tilde{x}_0$ in $(p \circ q)^{-1}(x_0)$. For an element $\sigma \in \text{Aut}((\tilde{N}, p \circ q) (= \pi_1(N)))$, we correspond the element $\tau \in \text{Aut}(\tilde{M}, p) (= \pi_1(M))$ which translates $\tilde{x}_0$ to $q(\sigma(\tilde{x}_0)).$ We denote the above correspondence $\eta_* : \pi_1(N) \to \pi_1(M)$. It is easy to see that $\eta_*$ is a homomorphism. We call such $\eta_*$ the homomorphism induced by an inclusion orb-map. We also write it simply $\pi_1(N) \to \pi_1(M)$.

LEMMA 6.3. Let $M$ be a good 3-orbifold and $q : M' \to M$ be a 2-sheeted orbicovering whose underlying space is orientable. Let $F$ be a 2-suborbifold of $\partial M$ and $F'$ be a component of $q^{-1}(F)$. If there is a discal 2-suborbifold $D'$ properly embedded in $M'$ s.t. $\partial D' \subset F'$ and $\partial D'$ does not bound any discal suborbifold in $F'$, then there is a discal 2-suborbifold $D$ properly embedded in $M$ s.t. $\partial D \subset F$ and $\partial D$ does not bound any discal suborbifold in $F$.

Proof. Let $\sigma$ be the generator of $\text{Aut}(M', q)$. Put $\mathcal{D} = \{ D_1 \subset M' \mid D_1$ is a discal 2-suborbifold properly embedded in $M'$ s.t. $\partial D_1 \subset F'$ and $\partial D_1$ does not bound any
discal 2-suborbifold in \( F' \). By the hypothesis, \( \mathcal{D} \neq \emptyset \). We may assume that for 
\( \forall D_1 \in \mathcal{D}, \sigma(D_1) \cap D_1 \) consists of a finite number of simple closed curves.

Define, for \( D_1 \in \mathcal{D}, n(D_1) = (\text{the number of the components of } \sigma(D_1) \cap D_1). \) 
Let \( D_0 \) be the element of \( \mathcal{D} \) which attains the minimal value of \( n \in \mathcal{D} \). We see that 
\( D_0 \) is Aut(\( M', q \))-equivariant as follows.

Put \( \mathcal{E} = \{ E \mid E \text{ is a suborbifold of } D_0 \text{ or } \sigma(D_0) \text{ bounded by a component of } 
D_0 \cap \sigma(D_0) \text{ s.t. } (\text{Int } E) \cap (D_0 \cap \sigma(D_0)) = \emptyset \}. \) Define, for \( E \in \mathcal{E}, m(E) = \#(\Sigma^{(0)} E) \). 
Let \( E_0 \) be the element of \( \mathcal{E} \) which attains the minimal value of \( m \) in \( \mathcal{E} \). We may 
assume that \( E_0 \subset D_0 \). Noting that \( M' \) is abad and \( m(E_0) \) is minimal in \( \mathcal{E} \), we describe 
all types of the relations of \( E_0 \) and \( \sigma(D_0) \) in Figure 6.1. We cut \( \sigma(D_0) \) along \( E_0 \) 
into 2-orbifolds \( D_{0,1} \) and \( D_{0,2} \). By Figure 6.1(1), (2), (4), (5) and (6), we see that 
\( D_{0,1} \in \mathcal{D} \). By Figure 6.1(3), (7) and (8), we see that both of \( D_{0,1} \) and \( D_{0,2} \) are 
discal 2-suborbifolds properly embedded in \( M' \). Furthermore, since \( \partial(\sigma(D_0)) \) does 
not bound any discal 2-suborbifold in \( F' \) and \( M' \) is abad, at least one of \( \partial D_{0,1} \) and 
\( \partial D_{0,2} \) does not bound any discal 2-suborbifold in \( F' \). Hence, in any case, at least one 
of \( D_{0,1} \) and \( D_{0,2} \) belongs to \( \mathcal{D} \). On the other hand, it is clear that \( n(D_{0,1}), n(D_{0,2}) \) 
\( \leq n(\sigma(D_0)) \) (\( = n(D_0) \)). It is a contradiction.

Thus, \( q(D_0) \) is the desired discal 2-suborbifold.
Corollary 6.4. (Loop theorem for orbifolds) Let $M$ be a good 3-orbifold with boundaries. Let $F$ be a connected 2-suborbifold in $\partial M$. If $\text{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$, then there is a discal 2-suborbifold $D$ properly embedded in $M$ s.t. $\partial D \subset F$ and $\partial D$ does not bound any discal 2-suborbifold in $F$.

Proof. At first, we show the theorem when $M$ is orientable. In this case, the elements of $\pi_1(M)$ (respectively $\pi_1(F)$) are represented by loops in $[M] - \Sigma M$ (respectively $[F] - \Sigma F$) (see [Ta]). Let $p : \tilde{M} \to M$ be the universal covering and $\tilde{F}$ a component of $p^{-1}(F)$. Since $M$ is good, $\tilde{M}$ and $\tilde{F}$ are manifolds.

Since $\text{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$, there is an element $\alpha \in \pi_1(\tilde{F})$ s.t. $\alpha \neq 1$ in $\pi_1(\tilde{F})$. Since $\tilde{M}$ is simply connected, $\text{Ker}(\pi_1(\tilde{F}) \to \pi_1(\tilde{M})) \neq 1$. By the Loop theorem for manifolds, there is a 2-disc $E_0$ properly embedded in $\tilde{M}$ s.t. $\partial E_0 \subset \tilde{F}$ and $[\partial E_0] \neq 1$ in $\pi_1(\tilde{F})$.

Suppose $M$ and $F$ are compact. We can take a simplicial division $K_M$ of $(M, F)$ which is equipped with a P.H. metric and is sufficiently refined. Let $K_{\tilde{M}}$ be the simplicial division of $\tilde{M}$ induced by the map $p$. Note that $K_{\tilde{M}}$ is also sufficiently refined and that $K_{\tilde{M}}$ has the P.H. metric naturally induced by the map $p$.

Put $\Omega = \{ E \subset \tilde{M} \mid E \text{ is a 2-disc properly embedded in } \tilde{M} \text{ s.t. } \partial E \subset \tilde{F} \text{ and } [\partial E] \neq 1 \text{ in } \pi_1(\tilde{F}) \}$. By the above paragraph, we see that $\Omega \neq \phi$. Hence, by Theorem 6.1 and Example 6.2, there is an element $D_0 \in \Omega$ which is equivariant under $\text{Aut}(\tilde{M}, p)$. Therefore, $p(D_0)$ is the desired suborbifold.

In the general case. Put $\partial E_0 = \tilde{C}$ and $p(\tilde{C}) = C$. Let $A$ be the regular neighborhood of $C$ in $F$. Let $C_1, \ldots, C_k$ be the components of $\partial A$ s.t. there are discal suborbifolds $B_1, \ldots, B_k$ in $F - \text{Int} A$ with $\partial B_i = C_i$. Let $G = A \cup B_1 \cup \cdots \cup B_k$. Let $N$ be the regular neighborhood of $(p(E_0)) \cup G$ in $M$. Note that $N$ and $G$ are compact and $\partial N \supset G$. Note also that $[\partial p(E_0)] \neq 1$ in $\pi_1(G)$ and $[\partial p(E_0)] = 1$ in $\pi_1(N)$. Hence, by the result of the compact case, there is a discal suborbifold $D$ properly embedded in $N$ s.t. $\partial D$ does not bound any discal suborbifold in $G$. By the construction, $\partial D$ does not bound any discal suborbifold in $F$.

Next, we deal with the case that $|M|$ is orientable (i.e. $M$ possibly has some 'mirror boundaries'). Let $q : M' \to M$ be the double covering w.r.t. the mirror boundaries. Let $F'$ be a component of $q^{-1}(F)$. By the hypothesis and the definition of the homomorphism induced by the inclusion orbi-map, $\text{Ker}(\pi_1(F') \to \pi_1(M')) \neq 1$. Hence, by the orientable case and Lemma 6.3, we can get a desired discal 2-suborbifold in $M$.

Finally, we consider the case that $|M|$ is possibly non-orientable. Let $q : M' \to M$ be the double covering induced by the orientable double covering of
$|M|$. Let $F'$ be a component of $q^{-1}(F)$. As similar to the above case, it holds that $\text{Ker}(\pi_1(F') \to \pi_1(M')) \neq 1$. Hence, we reduce to the above case. 

\textbf{Corollary 6.5.} (Dehn's lemma for orbifolds) Let $M$ be a good 3-orbifold with boundaries. Let $\gamma$ be a simple closed curve in $\partial M - \Sigma M$ s.t. the order of $[\gamma]$ is $n$ in $\pi_1(M)$. Then there exists a discal suborbifold $D^2(n)$ properly embedded in $M$ with $\partial D^2(n) = \gamma$.

\textit{Proof.} Let $F$ be a regular neighborhood of $\gamma$ in $\partial M$. Since $F$ is an annulus, $\pi_1(F) \cong \mathbb{Z}$. By the hypothesis, $\text{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$. Then by Lemma 6.3, there is a discal suborbifold $D^2(m)$ properly embedded in $M$ s.t. $\partial D^2(m) \subset F$ and $\partial D^2(m)$ does not bound any discal suborbifold in $F$. Since $\partial D^2(m)$ is a simple closed curve, $\gamma$ and $\partial D^2(m)$ are ambient isotopic in $F$. Hence, we may assume that $\gamma$ bounds $D^2(m)$. Hence $[\gamma]$ and the normal loop around $\Sigma D^2(m)$ are conjugate in $\pi_1(M)$. Since $[\gamma]$ is order $n$ in $\pi_1(M)$ and $M$ is good, $m$ must equal to $n$. □

\textbf{Lemma 6.6.} Let $M$ be a good 3-orbifold and $q : M' \to M$ be a 2-sheeted orbicovering whose underlying space is orientable. If there is a spherical 2-suborbifold $S'$ of $M'$ s.t. $[\tilde{S}'] \neq 0$ in $\pi_2(\tilde{M})$, then there is a spherical 2-suborbifold $S$ of $M$ s.t. $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{M})$, where $p : M \to M'$ is the universal covering and $\tilde{S}'$ (respectively $\tilde{S}$) is a component of $p^{-1}(S')$ (respectively $(q \circ p)^{-1}(S))$.

\textit{Proof.} The proof is similar to Lemma 6.3. □

\textbf{Corollary 6.7.} (Sphere theorem for orbifolds) Let $M$ be a good 3-orbifold. Let $p : \tilde{M} \to M$ be the universal covering of $M$. If $\pi_2(\tilde{M}) \neq 0$, then there is a spherical suborbifold $S$ in $M$ s.t. $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{M})$, where $\tilde{S}$ is any component of $\pi_2(\tilde{M})$.

\textit{Proof.} At first, we show the theorem when $M$ is orientable. By the Sphere theorem for manifolds, there is a 2-sphere $\tilde{S}_1$ in $\tilde{M}$ s.t. $[\tilde{S}_1] \neq 0$ in $\pi_2(\tilde{M})$.

Suppose $\tilde{M}$ is compact. Take a simplicial division $K_M$ of $M$ which is equipped with a P.H. metric and is sufficiently refined. Let $K_{\tilde{M}}$ be the simplicial division of $\tilde{M}$ induced by the map $p$. $K_{\tilde{M}}$ is sufficiently refined and has the P.H. metric induced by $p$. Put

$$\Omega = \{ S \mid S \text{ is an embedded subsphere in } \tilde{M} \text{ s.t. } [S] \neq 0 \text{ in } \pi_2(\tilde{M}) \}.$$ 

By the above paragraph we see that $\Omega \neq \emptyset$. Hence, by Theorem 6.1 and Example 6.2, there is an element $S_0 \in \Omega$ which is equivariant under $\text{Aut}(\tilde{M}, p)$. Therefore $p(S_0)$ is the desired spherical suborbifold.
In the general case. Let $A$ be the regular neighborhood of $p(\tilde{S}_1)$ in $M$. Let $C_1, \ldots, C_k$ be all components of $\partial A$ s.t. there are compact suborbifolds $B_1, \ldots, B_k$ in $M - \text{Int} A$ with $\partial B_i = C_i$. Put $N = A \cup B_1 \cup \cdots \cup B_k$. Let $\tilde{N}$ be a component of $p^{-1}(N)$. Note that $[\tilde{S}_1] \neq 0$ in $\pi_2(\tilde{N})$. Hence, by the result of the compact case, there is a spherical 2-suborbifold $\tilde{S}$ in $N$ s.t. for any component $\tilde{S}$ of $p^{-1}(S)$, $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{N})$. This $S$ is the desired spherical 2-suborbifold, i.e. $[\tilde{S}] \neq 0$ in $\pi_2(M)$. Otherwise, there is a homotopy 3-cell $H$ in $\tilde{M}$ s.t. $\partial H = \tilde{S}$. Since $H$ is equivariant, $p(H)$ is a suborbifold of $M$. Since $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{N})$, $H$ is not included in $\tilde{N}$. Hence, $p(H)$ is not included in $N$. Hence, a component of $\partial N$ is included in Int($p(H)$). Since $M$ is orientable, the component of $\partial N$ must bound a compact suborbifold in $p(H)$. This contradicts the construction.

As similar to Corollary 6.4, we can reduce the non-orientable case to the above case by using Lemma 6.6.

**Lemma 6.8.** Let $S$ be a bad sphere and $G$ be a finite group whose elements are orbilisomorphisms on $S$. Then $S/G$ is a bad sphere.

**Proof.** Suppose that $S$ is not orientable, $G$ must be equivalent to the identity. Hence, we may assume that $S$ is orientable.

Suppose $S = S^2(m)$. Let $\Sigma S = \{x\}$. Since $g(x) = x, g \in G$ and $G$ acts on $|S|$ as diffeomorphisms, $G$ consists of either a rotation or a reflection whose fixed point set includes $x$. Hence $S/G = S^2(mr, r)$ or $D^2(; m)$, where $r = \#G$.

Suppose $S = S^2(m, n)$. Let $\Sigma S = \{x, y\}$. Since $g(x) = x$ and $g(y) = y, g \in G$ and $G$ acts on $|S|$ as diffeomorphisms, $G$ consists of either a rotation or a reflection whose fixed point set is $x, y$. Hence $S/G = S^2(mr, nr)$ or $D^2(; m, n)$, where $r = \#G$.

**Corollary 6.9.** (The lifting of abadness) Let $M$ be a 3-orbifold. Let $p : \tilde{M} \to M$ be any covering. $\tilde{M}$ is abad, if and only if $M$ is abad.

**Proof.** Suppose $\tilde{M}$ is abad. Since any covering orbifold of a bad sphere is also a bad sphere, $M$ must not include any bad spheres as suborbifolds.

Suppose $M$ is abad. Let $\hat{p} : \hat{M} \to \tilde{M}$ be the universal covering. We only have to show that $\hat{M}$ is abad.

When $M$ is compact. Take a simplicial division $K_M$ of $M$ which is equipped with a P.H. metric and is sufficiently refined. Let $K_{\hat{M}}$ be the simplicial division of $\hat{M}$ induced by $p \circ \hat{p}$.

Put $\Omega = \{S \subset \hat{M} \mid S$ is a bad subsphere$\}$. Suppose there is a bad subsphere $S \subset \hat{M}$. Then, by Theorem 6.1 and Example 6.2, there is an element $S_0 \in \Omega$ which
is equivariant under \( \text{Aut}(\hat{M}, p \circ \hat{p}) \). Therefore, by Lemma 6.8, \((p \circ \hat{p})(S_0)\) is a bad subsphere in \( M \). It is a contradiction.

In the general case. Suppose there is a bad subsphere \( S \) in \( \hat{M} \). Let \( A \) be the regular neighborhood of \((p \circ \hat{p})(S)\) in \( M \). By the hypothesis, \( A \) is abad. By the result of the compact case, each component of \((p \circ \hat{p})^{-1}(A)\) is abad. On the other hand, \( S \) is included in a component of \((p \circ \hat{p})^{-1}(A)\). It is a contradiction.

**Lemma 6.10.** Let \( S \) be a spherical 2-orbifold and \( G \) be a finite group whose elements are orbi-isomorphisms on \( S \). Then \( S/G \) is a spherical 2-orbifold.

**Proof.** Since the universal cover of \( S/G \) is a 2-sphere, \( S/G \) is a spherical 2-orbifold.

**Lemma 6.11.** Let \( B \) be a ballic 3-orbifold and \( G \) be a finite group whose elements are orbi-isomorphisms on \( B \). Then \( B/G \) is a ballic 3-orbifold.

**Proof.** Since the universal cover of \( B/G \) is a 3-ball, there is a finite group \( H \subset \text{diff}(B^3) \) s.t. \( B/G \) is orbi-isomorphic to \( B^3/H \). By [KS, 5.6], \( B^3/H \) is a ballic 3-orbifold.

**Lemma 6.12.** Let \( M \) be an abad 3-orbifold and \( q : M' \to M \) be a 2-sheeted orbicovering whose underlying space is orientable. If there is an incompressible spherical 2-suborbifold \( S' \) in \( M' \), then there is an incompressible spherical 2-suborbifold \( S \) in \( M \).

**Proof.** The proof is similar to Lemma 6.3.

Let \( S \) be the double of the 3-balls \( B_i \), \( i = 1, 2 \), and let \( G \) be a finite group which acts effectively on \( S \). Suppose that \( g(\partial B_i) = \partial B_i \) for any \( g \in G \) and that there is at least one element \( g \in G \) such that \( g(B_1) = B_2 \). Note that \( \partial B_1/G \) is a spherical 2-orbifold embedded in \( S/G \) but does not bound the cone. Indeed, \( \partial B_1/G \) is one-sided in \( S/G \). Thus, \( S/G \) is not irreducible. We call \( S/G \) a non-irreducible spherical 3-orbifold.

**Corollary 6.13.** (The lifting of irreducibility) Let \( M \) be an abad 3-orbifold. Let \( p : \hat{M} \to M \) be any covering. \( \hat{M} \) is irreducible, if \( M \) is irreducible. Furthermore, if \( M \) is not a non-irreducible spherical 3-orbifold, vice versa.

**Proof.** Suppose \( \hat{M} \) is irreducible. Let \( \hat{p} : \hat{M} \to \hat{M} \) be the universal covering. Let \( S \) be any spherical suborbifold in \( M \). Let \( \tilde{S} \) (respectively \( \hat{S} \)) be a component of \( p^{-1}(S) \) (respectively \( \hat{p}^{-1}(\tilde{S}) \)). By the irreducibility of \( \hat{M} \), \( \tilde{S} \) bounds a ballic orbifold in \( \hat{M} \), hence \( \hat{S} \) also bounds a ballic orbifold \( \hat{B} \) in \( \hat{M} \). Since \( M \) is not a non-irreducible
spherical 3-orbifold, \( g(\hat{B}) = \hat{B} \) for any \( g \in \text{Aut}(\hat{M}, \rho \circ \hat{\rho}) \). Then, by Lemma 6.11, \( S \) bounds a ballic suborbifold in \( \hat{M} \).

Suppose \( M \) is irreducible. We only have to show that the universal covering orbifold \( \hat{M} \) is irreducible. Suppose that there is a spherical orbifold \( S \) in \( \hat{M} \) s.t. \( S \) does not bound a ballic suborbifold in \( \hat{M} \). At first, we deal with the case that \( M \) is orientable.

When \( M \) is compact. Take a simplicial division \( K_M \) of \( M \) which is equipped with a P.H. metric and sufficiently refined. Let \( K_{\hat{M}} \) be the simplicial division of \( \hat{M} \) induced by \( \rho \circ \hat{\rho} \). We may assume that \( S \) is normal w.r.t. \( K_{\hat{M}} \). Put \( \Omega = \{ S \subset \hat{M} \mid S \text{ is an incompressible spherical suborbifold} \} \). Suppose \( \Omega \neq \emptyset \). Then, by Theorem 6.1 and Example 6.2, there is an element \( S_0 \in \Omega \) which is equivariant under \( \text{Aut}(\hat{M}, \rho) \). Therefore, by Lemma 6.10, \( (\rho \circ \hat{\rho})(S_0) \) is an incompressible spherical suborbifold in \( M \). It is a contradiction.

In the general case. Let \( A \) be the regular neighborhood of \( (\rho \circ \hat{\rho})(S) \) in \( M \). Let \( C_1, \ldots, C_k \) be the components of \( \partial A \) s.t. there are compact suborbifolds \( B_1, \ldots, B_k \) in \( M - \text{Int} A \) with \( \partial B_i = C_i \). Put \( N = A \cup B_1 \cup \cdots \cup B_k \). By the orientability and irreducibility of \( M \), and the construction of \( N \), it holds that \( N \) is irreducible. By the result of the compact case, each component of \( (\rho \circ \hat{\rho})^{-1}(N) \) is irreducible. Let \( \hat{N} \) be the component of \( (\rho \circ \hat{\rho})^{-1}(N) \) which includes \( S \). \( S \) must be a boundary of a ballic suborbifold in \( \hat{N} \) (hence, in \( \hat{M} \)). This contradicts the hypothesis that \( S \) is incompressible in \( \hat{M} \).

As similar to Corollary 6.4, we can show that \( M \) has an incompressible spherical 2-suborbifold in the non-orientable case by using Lemma 6.12 and the above case. \( \Box \)

**References**


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