SMOOTH \( SU(p, q) \)-ACTIONS ON THE
\((2p + 2q - 1)\)-SPHERE
AND ON THE COMPLEX PROJECTIVE
\((p + q - 1)\)-SPACE

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Dedicated to Professor Fuich Uchida on his sixtieth birthday

1. Introduction

We consider the orthogonal \( S(U(p) \times U(q)) \)-action on the natural \((2p + 2q - 1)\)-sphere and the induced \( S(U(p) \times U(q)) \)-action on the standard complex projective \((p + q - 1)\)-space. The actions are said to be standard. If \( p, q \geq 2 \), then each action has codimension-one principal orbits and the action restricted to the principal isotropy subgroup has the fixed point set \( F \) diffeomorphic to \( S^3 \) or \( P_1(\mathbb{C}) \), respectively.

In this paper, we shall study smooth \( SU(p, q) \)-actions on the spaces for \( p, q \geq 3 \) whose restricted action to the maximal compact subgroup \( S(U(p) \times U(q)) \) is standard.

The characterization of such non-compact Lie group actions on a standard sphere was first introduced by Asoh [1]. It was improved by Uchida [6] and generalized by [3, 7].

In Section 2 we give relevant facts and basic properties for later convenience.

Actions on each space are characterized by pairs \((\phi, f)\) constructed by [7], where \( \phi \) is an action of some subgroup of \( SU(p, q) \) on \( F \) and \( f : F \rightarrow P_1(\mathbb{C}) \) is a smooth map. In the case of the projective space, the pair \((\phi, f)\) is also characterized by a triple \((S, \varphi, f_1)\) constructed by [3], where \( S \) is a one-dimensional submanifold of \( F \) diffeomorphic to \( P_1(\mathbb{R}) \), \( \varphi \) is a one-parameter transformation group on \( S \) and \( f_1 : S \rightarrow P_1(\mathbb{R}) \) is a smooth map. This implies the infiniteness of numbers of smooth \( SU(p, q) \)-actions on the projective space using the construction due to Asoh [1]. These are studied in Sections 3 and 4.

In the last section we give some relation between smooth \( SU(p, q) \)-actions on the sphere and on the projective space.
2. Preliminaries

2.1. Subgroups and subalgebras of $SU(p, q)$

Let $SU(p, q)$ denote the group of matrices in $SL(p + q, \mathbb{C})$ which leave invariant the form

$$-|z_1|^2 - |z_2|^2 - \cdots - |z_p|^2 + |z_{p+1}|^2 + \cdots + |z_{p+q}|^2$$

for any $z = (z_1, z_2, \ldots, z_{p+q}) \in \mathbb{C}^{p+q}$. Let $I_n$ be the identity matrix of order $n$ and let $I_{p,q}$ denote the matrix defined by

$$I_{p,q} = \begin{pmatrix} -I_p & O \\ O & I_q \end{pmatrix}.$$

Then

$$SU(p, q) = \{ g \in M(p + q, \mathbb{C}) \mid g^* I_{p,q} g = I_{p,q}, \det g = 1 \},$$

where $M(p + q, \mathbb{C})$ denotes the set of complex matrices of order $p + q$ and $g^*$ the Hermitian conjugate of $g$. $SU(p, q)$ is simple and contains $SU(p) \times U(q)$ as a maximal compact subgroup.

Let $\mathfrak{su}(p, q)$ and $\mathfrak{su}(u(p) \times u(q))$ denote Lie algebras of $SU(p, q)$ and $SU(p) \times U(q)$ respectively. Then

$$\mathfrak{su}(p, q) = \{ A \in M(p + q, \mathbb{C}) \mid A^* I_{p,q} + I_{p,q} A = O, \text{ trace } A = 0 \}$$

$$= \left\{ A = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix} \mid A_1 \in M(p, \mathbb{C}), A_3 \in M(q, \mathbb{C}) \right\},$$

with $A_1$ and $A_3$ skew Hermitian, trace $A = 0$.

$$\mathfrak{su}(u(p) \times u(q)) = \left\{ A = \begin{pmatrix} A_1 & O \\ O & A_3 \end{pmatrix} \in \mathfrak{su}(p, q) \right\}.$$

We consider the standard representations of $SU(p, q)$ and $\mathfrak{su}(p, q)$ on $\mathbb{C}^{p+q}$. Let $H(a : b)$ (respectively $h(a : b)$) denote the isotropy subgroup (respectively subalgebra) at $ae_1 + be_{p+1}$ for $(a, b) \neq (0, 0)$. Here $\{e_1, e_2, \ldots, e_{p+q}\}$ is the standard basis of $\mathbb{C}^{p+q}$. We see that $h(a : b)$ is the subalgebra of $\mathfrak{su}(p, q)$ consisting of matrices in the form

$$\begin{pmatrix} |b|^2 n & \bar{b}U^* & -a\bar{b}n & \bar{b}V^* \\ -bU & X & aU & Y^* \\ \bar{a}bn & \bar{a}U^* & -|a|^2 n & \bar{a}V^* \\ bV & Y & -aV & Z \end{pmatrix},$$

(2.1)
where $U \in \mathbb{C}^{p-1}$, $V \in \mathbb{C}^{q-1}$ and $n$ is a purely imaginary number. Let $H'(a : b)$ denote the subgroup of $SU(p, q)$ defined by

$$H'(a : b) = \{ g \in SU(p, q) \mid g(\alpha e_1 + b e_{p+1}) = s\alpha e_1 + sbe_{p+1} \text{ for some } s \in U(1) \}$$

and let $\mathfrak{h}'(a : b)$ be its Lie algebra. Then we have

$$\mathfrak{h}'(a : b) = \mathfrak{h}(a : b) \oplus \theta_0^1,$$

where $\theta_0^1$ is generated by the matrix $i(E_{1,1} + E_{p+1,p+1}) - 2i(E_{p+q,p+q})$. Here $i = \sqrt{-1}$ and $E_{i,j}$ is the matrix unit.

We see that $H(a : b)$ and $H'(a : b)$ are both connected for any $(a, b)$, $H(1 : 0) = SU(p - 1, q)$, $H(0 : 1) = SU(p, q - 1)$ and $\mathfrak{h}(a : b) = \mathfrak{su}(p - 1, q - 1)$.

**Lemma 2.1.** Suppose $p, q \geq 3$. Let $\mathfrak{h}$ be a proper subalgebra of $\mathfrak{su}(p, q)$ which contains $\mathfrak{su}(p - 1) \oplus \mathfrak{su}(q - 1)$. If

$$\dim \mathfrak{su}(p, q) - \dim \mathfrak{h} \leq 2p + 2q - 1,$$

then

$$\mathfrak{h} = \mathfrak{h}(a : b) \quad \text{or} \quad \mathfrak{h}'(a : b) \quad \text{for some } (a : b) \in P_1(\mathbb{C}),$$

or

$$\mathfrak{h} = \mathfrak{h}(a : b) \oplus \theta_1^1 \quad \text{or} \quad \mathfrak{h}'(a : b) \oplus \theta_1^1 \quad \text{for some } (a : b) \in P_1(\mathbb{C}) \text{ satisfying } |a| = |b|,$$

where $\theta_1^1$ is generated by the matrix $abE_{p+1,1} + abE_{1,p+1}$.

**Proof.** By the $Ad(SU(p - 1) \times SU(q - 1))$-action on $\mathfrak{su}(p, q)$, we can decompose $\mathfrak{su}(p, q)$ into $Ad(SU(p - 1) \times SU(q - 1))$-invariant subspaces:

$$\mathfrak{su}(p, q) = \mathfrak{su}(p - 1) \oplus \mathfrak{su}(q - 1) \oplus \rho_{p-1} \oplus \rho_{q-1} \oplus 2\rho_{p-1} \oplus 2\rho_{q-1} \oplus \theta^5,$$

where

$$\theta^5 = \left\{ \begin{pmatrix} si & \bar{c} \\ ui & t \\ c & vi \\ ti & \end{pmatrix} \in \mathfrak{su}(p, q) \mid s, t, u, v \in \mathbb{R}, c \in \mathbb{C} \right\}$$

and $Ad(SU(p - 1) \times SU(q - 1))$ acts trivially on $\theta^5$. Hence we see that

$$\mathfrak{h} = \mathfrak{su}(p - 1) \oplus \mathfrak{su}(q - 1) \oplus (\mathfrak{h} \cap (\rho_{p-1} \oplus \rho_{q-1})) \oplus (\mathfrak{h} \cap 2\rho_{p-1}) \oplus (\mathfrak{h} \cap 2\rho_{q-1}) \oplus (\mathfrak{h} \cap \theta^5).$$

Then the result follows by the Lie algebra structure of $\mathfrak{su}(p, q)$ and the bracket operations on these $Ad(SU(p - 1) \times SU(q - 1))$-invariant subspaces (cf. Uchida [5, Section 2]).
We denote subgroups $T^2$, $T_0$ and $T_1$ of $S(U(p) \times U(q))$ by

\[ T^2 = \{ \text{diag}(t_1, 1, 1, \ldots, 1, t_{p+1}, 1, 1, \ldots, 1, (t_1t_{p+1})^{-1}) \mid t_1, t_{p+1} \in U(1) \}, \]
\[ T_0 = \{ \text{diag}(t, 1, 1, \ldots, 1, t, 1, 1, \ldots, 1, t^{-2}) \mid t \in U(1) \}, \]
\[ T_1 = \{ \text{diag}(t, 1, 1, \ldots, 1, t^{-1}, 1, 1, \ldots, 1, 1) \mid t \in U(1) \}, \]

respectively. Here $\text{diag}(\cdot)$ is the diagonal matrix of $M(p + q, \mathbb{C})$. Note that $T^2 = T_0T_1$ and $\text{Lie}(T_0) = \theta_0^1$, where $\text{Lie}(T_0)$ is the Lie algebra of $T_0$.

Let $N'(p, q)$ denote the subgroup of $SU(p, q)$ consisting of matrices in the form

\[
\begin{pmatrix}
  g_1 & g_2 \\
  I_{p-1} & g_1 \\
  g_2 & I_{q-1}
\end{pmatrix}
\]

and put $N(p, q) = T_0N'(p, q)$. Then $N'(p, q)$ and $N(p, q)$ are in the normalizer of $S(U(p - 1) \times U(q - 1))$ and canonically isomorphic to $SU(1, 1)$ and $U(1, 1)$, respectively.

We denote $m(\theta) \in N'(p, q)$ by

\[
m(\theta) = \begin{pmatrix}
  \cosh \theta & \sinh \theta \\
  \sinh \theta & \cosh \theta \\
  I_{p-1} & I_{q-1}
\end{pmatrix}
\]

and put $M(p, q) = \{ m(\theta) \mid \theta \in \mathbb{R} \}$.

Considering the orbit of $ae_1 + be_{p+1}$, we obtain the following (cf. [3, Proof of Lemma 1.10]):

\[
SU(p, q) = S(U(p) \times U(q))N(p, q)H(a : b) \tag{2.2}
\]

for each $(a : b) \in P_1(\mathbb{C})$. If $(a : b) \in P_1(\mathbb{R}) = \{(a : b) \in P_1(\mathbb{C}) \mid a, b \in \mathbb{R} \}$, then we also obtain the next decomposition:

\[
SU(p, q) = S(U(p) \times U(q))M(p, q)H(a : b). \tag{2.3}
\]

Since $N(p, q) = N'(p, q)T_0$ and $H'(a : b) = T_0H(a : b)$, we also have following decompositions:

\[
SU(p, q) = S(U(p) \times U(q))N'(p, q)H'(a : b),
SU(p, q) = S(U(p) \times U(q))M(p, q)H'(a : b), \tag{2.4}
\]

for each $(a : b) \in P_1(\mathbb{C})$ or for each $(a : b) \in P_1(\mathbb{R})$, respectively.
2.2. Standard actions

Let $\Phi_0 : SU(p, q) \times S^{2p+2q-1} \to S^{2p+2q-1}$ and $\Phi_1 : SU(p, q) \times P_{p+q-1}(\mathbb{C}) \to P_{p+q-1}(\mathbb{C})$ denote the standard actions defined by

$$\Phi_0(g, z) = \|gz\|^{-1}gz, \quad z \in S^{2p+2q-1},$$

and

$$\Phi_1(g, [z]) = [gz], \quad [z] \in P_{p+q-1}(\mathbb{C}),$$

respectively. The restricted $S(U(p) \times U(q))$-action on $S^{2p+2q-1}$ is orthogonal and has codimension-one principal orbits with $S(U(p-1) \times U(q-1))$ as the principal isotropy subgroup. Hereafter we put

$$G = SU(p, q), \quad K = S(U(p) \times U(q)),
H = S(U(p-1) \times U(q-1)), \quad \psi_0 = \Phi_0|_{K \times S^{2p+2q-1}} \tag{2.5}$$

and put

$$H' = \left\{ \begin{pmatrix} t & g_1 \\ g_2 & t \end{pmatrix} \in K \mid t \in U(1), g_1 \in U(p-1), g_2 \in U(q-1) \right\},$$

$$\psi_1 = \Phi_1|_{K \times P_{p+q-1}(\mathbb{C})}. \tag{2.6}$$

Then the $K$-action $\psi_1$ has codimension-one principal orbits with $H'$ as the principal isotropy subgroup.

Let $F(H)$ and $F(H')$ denote the fixed points set of the $H$-action $\psi_0$ on $S^{2p+2q-1}$ and that of the $H'$-action $\psi_1$ on $P_{p+q-1}(\mathbb{C})$, respectively. Then

$$F(H) = \{ (u, v) = ue_1 + ve_{p+1} \mid |u|^2 + |v|^2 = 1 \},$$

$$F(H') = \{ (u : v) = [ue_1 + ve_{p+1}] \mid |u|^2 + |v|^2 = 1 \}.$$

These are naturally diffeomorphic to $S^3$ and $P_1(\mathbb{C})$, respectively.

Each orbit space of $K$-actions $\psi_0, \psi_1$ is identified with an interval $I$ and there exist natural diffeomorphisms

$$S^{2p+2q-1}/K \cong F(H)/T \cong I,$$

$$P_{p+q-1}(\mathbb{C})/K \cong F(H')/T_1 \cong I,$$

respectively (see [2, p. 191]).
Let \( \pi : F(H) \to F(H') \) be the restriction of the natural projection \( \Pi : S^{2p+2q-1} \to P_{p+q-1}(\mathbb{C}) \). Then we may regard \( \pi : F(H) \to F(H') \) as a principal bundle with structure group \( T_0 \).

We identify \( F(H') = P_1(\mathbb{C}) \) with \( S^2 \) by the map \( \xi : F(H') \to S^2 \) defined by

\[
\xi((u : v)) = (-2uv, 1 - 2v\overline{v}) = (r_1, r_2, r_3),
\]

where

\[
r_1 = -2(u_1v_1 + u_2v_2), \quad r_2 = -2(u_2v_1 - u_1v_2),
\]

\[
r_3 = 1 - 2(u_1^2 + u_2^2) = -1 + 2(u_1^2 + u_2^2),
\]

for \( u = u_1 + u_2i, v = v_1 + v_2i \).

The standard representation of \( U(1, 1) \) on \( \mathbb{C}^2 \) induces a smooth action of \( U(1, 1) \) on the complex projective space \( P_1(\mathbb{C}) \). It also induces a smooth action of \( SU(1, 1) \) on the space. Via the canonical isomorphisms of \( N(p, q) \) onto \( U(1, 1) \) and \( N'(p, q) \) onto \( SU(1, 1) \), we may regard \( P_1(\mathbb{C}) \) as a \( N(p, q) \)- and \( N'(p, q) \)-manifold.

3. Smooth \( SU(p, q) \)-actions on the \((2p + 2q - 1)\)-sphere

3.1. Construction of pairs

Let \( \Phi : G \times S^{2p+2q-1} \to S^{2p+2q-1} \) be a smooth \( G \)-action on \( S^{2p+2q-1} \) such that its restricted \( K \)-action coincides with the action \( \psi_0 \), i.e., \( \Phi|_{(K \times S^{2p+2q-1})} = \psi_0 \).

We shall construct a map \( f : F(H) \to P_1(\mathbb{C}) \) as follows. Let \( z \in F(H) \) and \( g_z \) be the isotropy subalgebra at \( z \) with respect to the given \( G \)-action \( \Phi \). Since \( g_z \) is a proper subalgebra of \( \text{su}(p, q) \) containing \( \text{su}(p - 1) \oplus \text{su}(q - 1) \), there exists a unique \( (a : b) \in P_1(\mathbb{C}) \) such that \( h((a : b)) \subset g_z \) by Lemma 2.1. Therefore we define the map \( f \) by \( f(z) = (a : b) \).

**Lemma 3.1.** The map \( f : F(H) \to P_1(\mathbb{C}) \) is smooth.

**Proof.** Let \( H_i, U_{i,j}, V_{i,j}, U_{p+i,q+j}, V_{p+i,q+j}, X_{i,p+j} \) and \( Y_{i,p+j} \) denote the (real) basis of \( \text{su}(p, q) \) defined by

\[
H_i = i(E_{i,i} - E_{p+q,p+q}), \quad 1 \leq i \leq p + q - 1,
\]

\[
U_{i,j} = E_{i,j} - E_{j,i}, \quad V_{i,j} = i(E_{i,j} + E_{j,i}), \quad 1 \leq i < j \leq p,
\]

\[
U_{p+i,p+j} = E_{p+i,p+j} - E_{p+j,p+i}, \quad V_{p+i,p+j} = i(E_{p+i,p+j} + E_{p+j,p+i}), \quad 1 \leq i < j \leq q,
\]

\[
X_{i,p+j} = E_{i,p+j} + E_{p+j,i}, \quad Y_{i,p+j} = i(E_{i,p+j} - E_{p+j,i}), \quad 1 \leq i \leq p, 1 \leq j \leq q.
\]
Let $f(z) = (a : b)$ and $\xi \circ f(z) = (r_1, r_2, r_3)$, where $\xi$ is the diffeomorphism defined by (2.7). Since $\mathfrak{h}(a : b) \subset \mathfrak{g}_z$, we have

$$b_1 U_{1,i} - b_2 V_{1,i} + a_1 X_{i,p+1} + a_2 Y_{i,p+1} \in \mathfrak{g}_z, \quad 2 \leq i \leq p,$$

$$b_2 U_{1,i} + b_1 V_{1,i} + a_2 X_{i,p+1} - a_1 Y_{i,p+1} \in \mathfrak{g}_z, \quad 2 \leq i \leq p,$$

$$a_1 U_{p+1,p+i} - a_2 V_{p+1,p+i} + b_1 X_{1,p+i} - b_2 Y_{1,p+i} \in \mathfrak{g}_z, \quad 2 \leq i \leq q,$$

$$a_2 U_{p+1,p+i} + a_1 V_{p+1,p+i} + b_2 X_{1,p+i} + b_1 Y_{1,p+i} \in \mathfrak{g}_z, \quad 2 \leq i \leq q,$$

by (2.1), where $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$. By these relations and the fact that each element of $\mathfrak{su}(p, q)$ can be considered naturally as a smooth vector field on $S^{2p+2q-1}$ (see [4, Chapter II]), we obtain the following equations:

$$r_1 (\|U\|_z^2 + \|X\|_z^2) + r_2 (\langle X, Y \rangle_z + \varepsilon \langle U, V \rangle_z) - 2 \langle U, X \rangle_z = 0,$$

$$r_1 (\langle X, Y \rangle_z - \varepsilon \langle U, V \rangle_z) + r_2 (\|U\|_z^2 + \|Y\|_z^2) - 2 \langle U, Y \rangle_z = 0,$$

$$r_3 (\|U\|_z^2 + \|X\|_z^2) - r_2 (\langle V, X \rangle_z - \varepsilon \langle U, Y \rangle_z) - \varepsilon (\|U\|_z^2 - \|X\|_z^2) = 0,$$

for $\varepsilon = \pm 1$. Here $U = U_{1,i}$, $V = V_{1,i}$, $X = X_{i,p+1}$, $Y = Y_{i,p+1}$ in the case $\varepsilon = 1$, $U = U_{p+1,p+i}$, $V = V_{p+1,p+i}$, $X = X_{1,p+i}$ and $Y = Y_{1,p+i}$ in the case $\varepsilon = -1$ and $\langle \cdot, \cdot \rangle$ denotes the standard Riemannian metric on $S^{2p+2q-1}$. Hence we see that $\xi \circ f$ is smooth. So is $f$. \hfill $\Box$

The subgroup $N(p, q)$ acts on $F(H)$ via $\Phi|_{(N(p,q) \times F(H))}$, since $N(p, q)$ is contained in $N(H, G)$, which is the normalizer of $H$ in $G$. We denote the restricted $N(p, q)$-action of $\Phi$ on $F(H)$ by $\phi$. The pair $(\phi, f)$ satisfies the following.

**Lemma 3.2.**

1. The map $f$ is $N(p, q)$-equivariant, that is,

$$f(\phi(n, z)) = nf(z) \quad \text{for each } n \in N(p, q), z \in F(H).$$

2. For each $z \in F(H)$, we have

$$N(p, q)z \supset N(p, q) \cap H(a : b), \quad (3.1)$$

where $f(z) = (a : b)$.

**Proof:** (1) Let $f(z) = (a : b)$. Then $\mathfrak{h}(a : b) \subset \mathfrak{g}_z$. Suppose $f(\phi(n, z)) = (a' : b')$ for $n \in N(p, q)$. Then $\mathfrak{h}(a' : b') \subset \mathfrak{g}_{\phi(n, z)} = n\mathfrak{g}_zn^{-1}$. Hence we have

$$\mathfrak{h}(a' : b') = n\mathfrak{h}(a : b)n^{-1}.$$ 

Therefore $n(a : b) = (a' : b')$.

(2) If $n \in N(p, q) \cap H(a : b)$, then $n \in N(p, q) \cap G_z$. \hfill $\Box$
We denote two circles in \( F(H) \) by \( S_1 = \{(u, 0)\} \) and \( S_2 = \{(0, v)\} \). Considering the isotropy subgroup of the restricted \( K \cap N(p, q) \)-action, we have
\[
f(z) = (1 : 0) \Leftrightarrow z \in S_1, \\
f(z) = (0 : 1) \Leftrightarrow z \in S_2.
\]  

(3.2)

3.2. Properties of \((\phi, f)\)

Let \((\phi, f)\) be a pair of a smooth \( N(p, q) \)-action \( \phi \) on \( F(H) \) whose restriction on \( K \cap N(p, q) \) coincides with the restriction of the standard \( K \)-action \( \psi_0 \) and a \( N(p, q) \)-equivariant smooth map \( f : F(H) \to P_1(\mathbb{C}) \) satisfying the condition (3.1).

Let \( j_i \in N(H, K) \) \((i = 1, 2)\) denote matrices defined by
\[
j_1 = \begin{pmatrix} -1 & I_{p-1} \\ I_{q-1} & -1 \end{pmatrix}, \quad j_2 = \begin{pmatrix} I_p & -1 \\ -I_{q-2} & 0 \end{pmatrix}
\]
and put \( J_i = \psi_0(j_i, -) \). Then \( J_1 \) are involutions on \( F(H) \) satisfying \( J_1(u, v) = (-u, v), J_2(u, v) = (u, -v) \).

Let \( Q(z) \) denote a matrix defined by
\[
Q(z) = (|a|^2 + |b|^2)^{-1}(ae_1 + be_{p+1})(ae_1 + be_{p+1})^* 
\]
for \( f(z) = (a : b) \). We denote the identity component of \( \{g \in G \mid gQ(z)g^* = Q(z)\} \) by \( U(z) \). Then we see that \( U(z) \supset H(a : b) \) for \( f(z) = (a : b) \) and there exists a positive real number \( \lambda(n, z) \) such that
\[
nQ(z)n^* = \lambda(n, z)Q(\phi(n, z))
\]
for each \( z \in F(H) \) and \( n \in N(p, q) \). Notice that \( U(z) \supset H(c : d) \) if and only if \( f(z) = (c : d) \) by Lemma 2.1.

Since \( T_0 \) is a central subgroup of \( N(p, q) \) and \( F(H)/T_0 = F(H') \), there exist the \( N(p, q) \)-equivariant smooth map \( f' : F(H') \to P_1(\mathbb{C}) \) and the smooth \( N(p, q) \)-action \( \phi' : N(p, q) \times F(H') \to F(H') \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
N(p, q) \times F(H) & \xrightarrow{\phi} & F(H) & \xrightarrow{f} & P_1(\mathbb{C}) \\
\downarrow \text{id} \times \pi & & \downarrow \pi & & \\
N(p, q) \times F(H') & \xrightarrow{\phi'} & F(H') & \xrightarrow{f'} & P_1(\mathbb{C}).
\end{array}
\]
Since $T_0$ acts trivially on $F(H')$ and on $P_1(\mathbb{C})$, we may regard $\phi'$ and $f'$ as a $N'(p, q)$-action on $F(H')$ and a $N'(p, q)$-equivariant smooth map, respectively.

By (3.2), the map $f'$ satisfies the following conditions:

$$f'(z) = (1 : 0) \Leftrightarrow z = [e_1],$$
$$f'(z) = (0 : 1) \Leftrightarrow z = [e_{p+1}].$$

Let $P = f^{-1}(P_1(\mathbb{R}))$ and $S = f'^{-1}(P_1(\mathbb{R}))$. Then $P = \pi^{-1}(S)$. Since $f'$ is $t$-regular to the $T_1$-orbits on $P_1(\mathbb{R}) - \{(1 : 0), (0 : 1)\}$ and $S$ is covered by four open subsets $f'^{-1}(P_1(\mathbb{R}) - \{(1 : 0), (0 : 1)\}), \{\phi'(m(\theta), [e_1]) | \theta \in \mathbb{R}\}$ and $\{\phi'(m(\theta), [e_{p+1}]) | \theta \in \mathbb{R}\}$, $S$ is diffeomorphic to a one-dimensional real projective space in $F(H')$. Therefore $P$ is a two-dimensional closed submanifold of $F(H)$ diffeomorphic to $T_0 \times S$.

3.3. **Construction of $SU(p, q)$-actions on $S^{2p+2q-1}$**

Let $(\phi, f)$ be a pair defined in Section 3.2. We construct a smooth $G$-action on $S^{2p+2q-1}$ from the pair $(\phi, f)$.

Take $(g, p) \in G \times S^{2p+2q-1}$. Let us choose

$$k \in K, z \in F(H); \quad \psi_0(k, z) = p,$$
$$k' \in K, n \in N(p, q), u \in H(a : b); \quad gk = k'n'u,$$

where $f(z) = (a : b)$. Put

$$\Phi(g, p) = \psi_0(k', \phi(n, z)). \quad (3.4)$$

We shall show that $\Phi$ is well defined and is a smooth $G$-action.

**Lemma 3.3.** Let $z \in F(H)$ and $f(z) = (a : b)$. Suppose $(a : b) \in P_1(\mathbb{R})$ and $km(\theta)u = k'm(\theta')u'$ for $k, k' \in K, \theta, \theta' \in \mathbb{R}$ and $u, u' \in H(a : b)$. Then

$$\psi_0(k, \phi(m(\theta), z)) = \psi_0(k', \phi(m(\theta'), z)).$$

The proof is same as that of [3, Lemma 5.5]. Hence we omit it.

**Lemma 3.4.** Let $z \in F(H)$ and $f(z) = (a : b)$. Suppose

$$knu = k'n'u' \quad (*)$$

for $k, k' \in K, n, n' \in N(p, q)$ and $u, u' \in H(a : b)$. Then

$$\psi_0(k, \phi(n, z)) = \psi_0(k', \phi(n', z)).$$
Proof. First assume that \((a : b) \in P_1(\mathbb{R})\). Then we obtain the decomposition
\[
N(p, q) = T^2 M(p, q)(N(p, q) \cap H(a : b))
\]
by (2.3). We put
\[
n = t_1 m(\vartheta) u_1, \quad n' = t'_1 m(\vartheta') u'_1,
\]
where \(t_1, t'_1 \in T^2, \vartheta, \vartheta' \in \mathbb{R}\) and \(u_1, u'_1 \in N(p, q) \cap H(a : b)\). Then we have
\[
(k t_1) m(\vartheta)(u_1 u) = (k' t'_1) m(\vartheta')(u'_1 u') \in K M(p, q) H(a : b)
\]
by the condition (*). Hence
\[
\psi_0(k t_1, \phi(m(\vartheta), z)) = \psi_0(k' t'_1, \phi(m(\vartheta'), z))
\]
by Lemma 3.3. Thus \(\psi_0(k, \phi(n, z)) = \psi_0(k', \phi(n', z))\).

In general, for any \((a : b) \in P_1(\mathbb{C})\), there exist \(t \in T^2\) and \((a' : b') \in P_1(\mathbb{R})\) satisfying \(t(a : b) = (a' : b')\). Since \(f(\phi(t, z)) = tf(z) = (a' : b')\) and the equality
\[
k (nt^{-1})(tu t^{-1}) = k'(n't^{-1})(tu' t^{-1}) \in K N(p, q) H(a' : b')
\]
holds by (*), we have
\[
\psi_0(k, \phi(nt^{-1}, \phi(t, z))) = \psi_0(k', \phi(n't^{-1}, \phi(t, z))).
\]
Hence \(\psi_0(k, \phi(n, z)) = \psi_0(k', \phi(n', z))\).

Proposition 3.5. \(\Phi\) of (3.4) is well defined and is an abstract \(G\)-action on \(S^{2p+2q-1}\) such that \(\Phi|_{K \times S^{2p+2q-1}} = \psi_0\).

Proof. The proof that \(\Phi\) of (3.4) defines a \(G\)-action on \(S^{2p+2q-1}\) such that \(\Phi|_{K \times S^{2p+2q-1}} = \psi_0\) is similar to [3]. We shall show only the well definedness of \(\Phi\). For \((g, p) \in G \times S^{2p+2q-1}\) let us choose
\[
k_i \in K, z_i \in F(H); \quad p = \psi_0(k_1, z_1) = \psi_0(k_2, z_2),
\]
\[
g k_i = k'_i n_1 u_i \in K N(p, q) H(a_i : b_i)
\]
for \(f(z_i) = (a_i : b_i)\) \((i = 1, 2)\). By the first equation, there exists \(t \in T^2\) satisfying \(\psi_0(t, z_2) = z_1\). Hence we have \(tk_2^{-1} k_1 \in K z_1 \subset H(a_1 : b_1)\). We put \(tk_2^{-1} k_1 = u'_1\). Then
\[
k'_i n_1 u_1 = g k_1 = g k_2 t^{-1} u'_1 = k'_2 n_2 u_2 t^{-1} u'_1
\]
\[
= (k'_2 t^{-1})(tn_2 t^{-1})(tu_2 t^{-1} u'_1) \in K N(p, q) H(a_1 : b_1).
\]
Hence by Lemma 3.4, we have
\[
\psi_0(k'_2, \phi(n_1, z_1)) = \psi_0(k'_2 t^{-1}, \phi(t n_2 t^{-1}, z_1))
\]
\[
= \psi_0(k'_2, \phi(n_2 t^{-1}, z_1)) = \psi_0(k'_2, \phi(n_2, z_2)).
\]
Hence \(\Phi\) is well defined. \(\square\)
We next define

\[ S_1(\Phi) = \{ \Phi(g, e_1) \mid g \in G \}, \quad S_2(\Phi) = \{ \Phi(g, e_{p+1}) \mid g \in G \}, \]
\[ S_1(\Phi_0) = \{ \Phi_0(g, e_1) \mid g \in G \}, \quad S_2(\Phi_0) = \{ \Phi_0(g, e_{p+1}) \mid g \in G \}, \]

for the \( G \)-action \( \Phi \) of (3.4) and the standard \( G \)-action \( \Phi_0 \), respectively. Then clearly

\[ S_1(\Phi_0) = \{ u \oplus v \in S(\mathbb{C}^p \oplus \mathbb{C}^q) \mid \|u\| > \|v\| \}, \]
\[ S_2(\Phi_0) = \{ u \oplus v \in S(\mathbb{C}^p \oplus \mathbb{C}^q) \mid \|u\| < \|v\| \}, \]

where \( S(\mathbb{C}^p \oplus \mathbb{C}^q) \) is the \((2p + 2q - 1)\)-sphere. By the definition of \( \Phi \) and the conditions of \( \phi \) and \( f \), we see that there exist positive numbers \( r_1, r_2 < 1 \) such that

\[ S_1(\Phi) = \{ u \oplus v \in S(\mathbb{C}^p \oplus \mathbb{C}^q) \mid \|v\| < r_1 \}, \]
\[ S_2(\Phi) = \{ u \oplus v \in S(\mathbb{C}^p \oplus \mathbb{C}^q) \mid \|u\| < r_2 \}. \]

**Lemma 3.6.** \( \Phi \) is smooth on \( G \times S_i(\Phi) \) (\( i = 1, 2 \)).

**Proof.** Let \( U_1 = \{ (u, v) \in F(H) \mid |v| < r_1 \}. \) Then \( U_1 = \{ \psi_0(t, \phi(m(\theta), e_1)) \mid t \in T^2, \theta \in \mathbb{R} \}. \) Set \( \phi(m(\theta), e_1) = u(\theta)e_1 + v(\theta)e_{p+1} = (u(\theta), v(\theta)), \) where \( u(\theta) \) and \( v(\theta) \) are smooth complex functions. Considering the involution \( J_2, \) we see that \( u(\theta) \) is an even function and \( v(\theta) \) is an odd function. Hence there exists the smooth even function \( w(\theta) \) such that \( v(\theta) = \theta w(\theta), \) where \( w(\theta) \neq 0 \) for each \( \theta \) by the conditions of \( f. \) We define \( \sigma_1 : \mathbb{R} \to U(1) \) and \( \tau : \mathbb{R} \to (-r_1, r_1) \) by

\[ \sigma_1(\theta) = \frac{w(\theta)}{|w(\theta)|}, \quad \tau(\theta) = \theta |w(\theta)|. \quad (3.5) \]

Then \( \sigma_1(\theta)v(\theta) = \tau(\theta). \) Since the curve \( \phi(m(\theta), e_1) \) is transverse to each \( T^2 \)-orbit on \( U_1, \) we see that \( \tau \) is a diffeomorphism. In addition there exists a smooth map \( \sigma_2 : \mathbb{R} \to U(1) \) determined by \( \sigma_2(\theta)u(\theta) = (1 - \tau(\theta)^2)^{1/2}. \) Now we define \( K \)-equivariant diffeomorphisms \( h_1, h_2 : S_1(\Phi) \to S_1(\Phi) \) by

\[ h_1(u \oplus v) = u \oplus \sigma_1^{-1}(\|v\|)v, \]
\[ h_2(u \oplus v) = \sigma_2^{-1}(\|v\|)u \oplus v, \]

respectively. We also define a \( K \)-equivariant diffeomorphism \( h_3 : S_1(\Phi) \to S_1(\Phi_0) \) by

\[ h_3(u \oplus v) = (1 + \|v\|^2\mu(\|v\|)^2)^{-1/2}(u/\|u\| \oplus \mu(\|v\|)v), \]

where \( \mu : (-r_1, r_1) \to \mathbb{R} \) is the smooth even function defined by \( \mu(x) = (\tanh \tau^{-1}(x))/x. \) Put \( F_1 = h_3 \circ h_2 \circ h_1. \) Since \( F_1(\phi(m(\theta), e_1)) = \Phi_0(m(\theta), e_1), \)
\( F_1 : S_1(\Phi) \to S_1(\Phi_0) \) is a \( G \)-equivariant diffeomorphism. Hence \( \Phi \) is smooth on \( G \times S_1(\Phi). \) Similarly we can show that \( \Phi \) is smooth on \( G \times S_2(\Phi) \).
Put $P = f^{-1}(P_1(\mathbb{R}))$. Then $P$ is the two-dimensional closed submanifold of $F(H)$ (see Section 3.2). Let $z = (u, v) \in P$ and $f(z) = (a : b) \in P_1(\mathbb{R})$. Then there exists a smooth function $\beta$ on $[z = (u, v) \in P | v \neq 0]$ defined by $f(z) = (\beta(z) : 1)$.
Define

$$P_+ \quad \text{(respectively } P_-) = \{z = (u, v) \in P | v \neq 0, \beta(z) > 0 \quad \text{(respectively } \beta(z) < 0)\}. $$
Then we see that $\phi(m(\theta), z) \in P_+$ (respectively $P_-$) if and only if

$$2\beta(z) \cosh 2\theta + (1 + \beta(z)^2) \sinh 2\theta > 0 \quad \text{(respectively } < 0), \quad (3.6)$$
by Lemma 3.2. We define

$$D_+ = \{(\theta, z) \in \mathbb{R} \times P_+ | \phi(m(\theta), z) \in P_+\},
$$

$$W_+ = \{(g, z) \in G \times P_+ | \text{trace}(gQ(z)g^*) \neq |(1 - \beta(z)^2)(1 + \beta(z)^2)^{-1}|\}. $$
Then $D_+$ is an open set of $\mathbb{R} \times P_+$ and $W_+$ is an open set of $G \times P_+$.

Using (3.3) and (3.6), we can show the following two lemmas (see [3, Lemmas 5.10, 5.12]).

**Lemma 3.7.** Let $(g, z) \in G \times P_+$ and $f(z) = (a : b)$. Then we have $(g, z) \in W_+$ if and only if there is a decomposition $g = km(\theta)u$, where $k \in K$, $\theta \in \mathbb{R}$ and $u \in H(a : b)$ such that $(\theta, z) \in D_+$.

**Lemma 3.8.** Let $(g, z) \in W_+$ and $f(z) = (a : b)$. Then the map $\Delta : W_+ \to (K/H) \times D_+$ defined by $\Delta(g, z) = (kH, \theta, z)$ is smooth, where $g = km(\theta)u$; $k \in K$, $\theta \in \mathbb{R}$ and $u \in H(a : b)$.

We put

$$W(\Phi) = \{(g, \psi(0, k, z)) \in G \times S^{2p+2q-1} | k \in K, (\psi, z) \in W_+\}. $$

Then we see that $W(\Phi)$ is an open set of $G \times S^{2p+2q-1}$, since $W_+$ is an open set of $G \times S_+$. Moreover we see that $\Phi|_{W(\Phi)}$ is smooth, since $\Delta$ is smooth by Lemma 3.8.

Since $G \times S^{2p+2q-1}$ is covered by three open sets $G \times S_1(\Phi)$, $G \times S_2(\Phi)$ and $W(\Phi)$ and $\Phi$ is smooth on each open set, we get the following.

**Proposition 3.9.** $\Phi$ of (3.4) is a smooth $G$-action.

Thus we have proved the following.

**Theorem 3.10.** Suppose $p, q \geq 3$. Then there is a one-to-one correspondence between the set of smooth $SU(p, q)$-actions on $S^{2p+2q-1}$ whose restricted $SU(p)$-
Smooth $SU(p, q)$-actions

$U(q)$-action is standard and the set of pairs $(\phi, f)$, where $\phi$ is a smooth $N(p, q)$-action on $F(H) = S^3$ whose restriction on $S(U(p) \times U(q)) \cap N(p, q)$ is standard and $f : S^3 \to P_1(\mathbb{C})$ is a smooth $N(p, q)$-equivariant map satisfying the condition (3.1).

4. Smooth $SU(p, q)$-actions on the complex projective $(2p + 2q - 1)$-space

4.1. Construction of pairs

Let $\Phi' : G \times P_{p+q-1}(\mathbb{C}) \to P_{p+q-1}(\mathbb{C})$ be a smooth $G$-action on $P_{p+q-1}(\mathbb{C})$ such that its restricted $K$-action coincides with the action $\psi_1$. Let $H'$ be the subgroup defined in (2.6). Then $F(H')$ is diffeomorphic to $P_1(\mathbb{C})$ (see Section 2.2). In the same way as in Section 3.1, we can construct a smooth map $f' : F(H') \to P_1(\mathbb{C})$ using Lemma 2.1. Here $f'(z) = (a : b)$ if and only if $\mathfrak{h}'(a : b) \subset \mathfrak{g}_z$, where $\mathfrak{g}_z$ is the isotropy subalgebra at $z$ with respect to the given $G$-action $\Phi'$. The proof of the smoothness of $f'$ is same as that of $f$ in Lemma 3.1. The subgroup $N'(p, q)$ acts on $F(H')$ via $\Phi'_{|N'(p, q) \times F(H')}$, since $N'(p, q)$ is contained in $N(H', G)$. We denote the restricted $N'(p, q)$-action of $\Phi'$ on $F(H')$ by $\phi'$. Then we see that the pair $(\phi', f')$ satisfies the following three conditions (see Sections 3.1 and 3.2):

$$f'(\phi'(n, z)) = n f'(z), \quad \text{for each } n \in N'(p, q), z \in F(H'),$$

that is the map $f'$ is $N'(p, q)$-equivariant,

$$N'(p, q)_z \supset N'(p, q) \cap H'(a : b) \quad (4.1)$$

for each $z \in F(H')$, where $f'(z) = (a : b)$, and

$$f'(z) = (1 : 0) \iff z = [e_1],$$

$$f'(z) = (0 : 1) \iff z = [e_{p+1}]. \quad (4.2)$$

Let $j \in N(H', K)$ denote the matrix defined by

$$j = \begin{pmatrix} i & \  & \ \  & I_{p-1} & \  \ \  & \ & -i \ \  & \ & \ & I_{q-1} \end{pmatrix}$$

and put $J = \psi_1(j, -)$. Then $J$ is the involution on $F(H')$ satisfying $J(u : v) = (-u : v)$. 

Let $Q'(z)$ denote a matrix defined by

$$Q'(z) = (|a|^2 + |b|^2)^{-1} (ae_1 + be_{p+1})(ae_1 + be_{p+1})^*$$

for $f'(z) = (a : b)$. We denote the identity component of \{ $g \in G$ | $gQ'(z)g^* = Q'(z)$ \} by $U'(z)$. Then we see that $U'(z) = H'(a : b)$ and there exists a positive real number $\lambda'(n, z)$ such that

$$nQ'(z)n^* = \lambda'(n, z)Q'(\phi'(n, z)),$$

for each $z \in F(H')$ and $n \in N'(p, q)$.

4.2. *Construction of SU(p, q)-actions on $P_{p+q-1}(\mathbb{C})$*

Let $(\phi', f')$ be a pair of a smooth $N'(p, q)$-action $\phi'$ on $F(H')$ whose restriction on $K \cap N'(p, q)$ coincides with the restriction of the standard $K$-action $\psi_1$ and a $N'(p, q)$-equivariant smooth map $f' : F(H') \to P_1(\mathbb{C})$ satisfying conditions (4.1) and (4.2).

We construct a smooth $G$-action on $P_{p+q-1}(\mathbb{C})$ from the pair $(\phi', f')$.

By (2.4) and the fact that $U'(z) = H'(a : b)$ for $f'(z) = (a : b)$, we have decompositions

$$G = KN'(p, q)U'(z), \quad \text{for each } z \in F(H'),$$

$$G = KM(p, q)U'(z), \quad \text{for each } z \in S,$$

where $S = f'^{-1}(P_1(\mathbb{R}))$, which is a one-dimensional closed submanifold containing $[e_1]$ and $[e_{p+1}]$ and diffeomorphic to a real projective space in $F(H')$ (see Section 3.2).

Take $(g, p) \in G \times P_{p+q-1}(\mathbb{C})$. Let us choose

$$k \in K, z \in F(H') : \psi_1(k, z) = p,$$

$$k' \in K, n \in N'(p, q), u \in U'(z) : gk = k'nu.$$

Put

$$\Phi'(g, p) = \psi_1(k', \phi'(n, z)). \quad (4.3)$$

The proof that $\Phi'$ is well defined and is an abstract $G$-action on $P_{p+q-1}(\mathbb{C})$ is quite similar to that of Proposition 3.5 in Section 3. Hence we show only the smoothness of $\Phi'$. 
We define
\[ S_1(\Phi') = \{ \Phi'(g, [e_1]) \mid g \in G \}, \quad S_2(\Phi') = \{ \Phi'(g, [e_{p+1}]) \mid g \in G \}, \]
\[ S_1(\Phi_1) = \{ \Phi_1(g, [e_1]) \mid g \in G \}, \quad S_2(\Phi_1) = \{ \Phi_1(g, [e_{p+1}]) \mid g \in G \}, \]
for the $G$-action $\Phi'$ of (4.3) and the standard $G$-action $\Phi_1$, respectively. Then clearly
\[ S_1(\Phi_1) = \{ [u + v] \in C P(C^p \oplus C^q) \mid \|u\| > \|v\| \}, \]
\[ S_2(\Phi_1) = \{ [u + v] \in C P(C^p \oplus C^q) \mid \|u\| < \|v\| \}, \]
where $C P(C^p \oplus C^q)$ is the standard complex projective $(p + q - 1)$-space. We see that there exist positive numbers $r_1, r_2 < 1$ such that
\[ S_1(\Phi') = \{ [u + v] \in C P(C^p \oplus C^q) \mid \|v\| < r_1 \}, \]
\[ S_2(\Phi') = \{ [u + v] \in C P(C^p \oplus C^q) \mid \|u\| < r_2 \}. \]

**Lemma 4.1.** $\Phi'$ is smooth on $G \times S_1(\Phi')$ ($i = 1, 2$).

**Proof.** Put $S_1 = \{ (u : v) \in S \mid \|v\| < r_1 \}$. Then $S_1 = \{ \phi'(m(\theta), [e_1]) \mid \theta \in \mathbb{R} \}$. We set $\phi'(m(\theta), [e_1]) = (r(\theta), v(\theta))$, where $r(\theta) > 0$ and $|v(\theta)| < r_1$. Considering the involution $J$, we see that $r(\theta)$ is an even function and $v(\theta)$ is an odd function. Hence there exists the smooth even function $w(\theta)$ such that $v(\theta) = \theta w(\theta)$, where $w(\theta) \neq 0$ for each $\theta$ by the conditions of $f'$ and $\phi'$. Using this function $w(\theta)$, we define smooth maps $\sigma_1 : \mathbb{R} \rightarrow U(1)$ and $\tau : \mathbb{R} \rightarrow (-r_1, r_1)$ by (3.5).

We now define $K$-equivariant diffeomorphisms $h_1 : S_1(\Phi') \rightarrow S_1(\Phi')$ and $h_2 : S_1(\Phi') \rightarrow S_1(\Phi_1)$ by
\[ h_1([u + v]) = [u + \sigma_1 \tau^{-1}(\|v\|)v], \]
\[ h_2([u + v]) = [u/\|u\| \oplus \mu(\|v\|)v], \]
where $\mu : (-r_1, r_1) \rightarrow \mathbb{R}$ is the smooth function defined by $\mu(x) = (\tanh \tau^{-1}(x))/x$. Put $F_1 = h_2 \circ h_1 : S_1(\Phi') \rightarrow S_1(\Phi_1)$. Then we see that $F_1$ is a $G$-equivariant diffeomorphism and that $\Phi'$ is smooth on $G \times S_1(\Phi')$. \qed

Put $z = (u : v) \in S$ and $f'(z) = (a : b)$. Then there is a smooth function $\beta$ on $\{ (u : v) \in S \mid v \neq 0 \}$ such that $f'(z) = (\beta(z) : 1)$.

We define the subset $S_+$ (respectively $S_-$) of $S$ by
\[ S_+ (\text{respectively } S_-) = \{ z = (u : v) \in S \mid v \neq 0, \beta(z) > 0 (\text{respectively } \beta(z) < 0) \}. \]
Then we see that $\phi(m(\theta), z) \in S_+$ (respectively $S_-$) if and only if
\[ 2\beta(z) \cosh 2\theta + (1 + \beta(z)^2) \sinh 2\theta > 0 (\text{respectively } < 0). \]
We define the open set $D_+$ of $\mathbb{R} \times S_+$ and the open set $W_+$ of $G \times S_+$ by

$$D_+ = \{ (\theta, z) \in \mathbb{R} \times S_+ | \phi'(m(\theta), z) \in S_+ \},$$

$$W_+ = \{ (g, z) \in G \times S_+ | \text{trace}(gQ'(z)g^*) \neq |(1 - \beta(z)^2)(1 + \beta(z)^2)^{-1}| \},$$

respectively and put

$$W(\Phi') = \{ (g, \psi_1(k, z)) \in G \times P_{p+q-1}(\mathbb{C}) | k \in K, (gk, z) \in W_+ \}.$$

Then we see that $\Phi'$ is smooth on $W(\Phi')$. The proof of the smoothness is similar to that of $\Phi$ in Section 3.3 (see also [3]). Thus we obtain the following result.

**Proposition 4.2.** $\Phi'$ of (4.3) is a smooth $G$-action.

Therefore we have proved the following.

**Theorem 4.3.** Suppose $p, q \geq 3$. Then there is a one-to-one correspondence between the set of smooth $SU(p, q)$-actions on $P_{p+q-1}(\mathbb{C})$ whose restricted $SU(p) \times U(q)$-action is standard and the set of pairs $(\phi', f')$, where $\phi'$ is a smooth $N'(p, q)$-action on $F(H') = P_1(\mathbb{C})$ whose restricted $SU(p) \times U(q) \cap N'(p, q)$-action is standard and $f' : P_1(\mathbb{C}) \to P_1(\mathbb{C})$ is a smooth $N'(p, q)$-equivariant map satisfying the conditions (4.1) and (4.2).

### 4.3. Construction of triples

Let $(\phi', f')$ be a pair of a smooth $N'(p, q)$-action $\phi'$ on $F(H')$ whose restriction on $K \cap N'(p, q)$ coincides with the restriction of the standard $K$-action $\psi_1$ and a $N'(p, q)$-equivariant smooth map $f' : F(H') \to P_1(\mathbb{C})$ satisfying conditions (4.1) and (4.2).

We construct a triple $(S, \varphi', f'_1)$ as follows. Set $S = f'^{-1}(P_1(\mathbb{R}))$ and $f'_1 = f'|_S$. Then $S$ contains $[e_1]$ and $[e_{p+1}]$. By the restricted $T_1(= K \cap N'(p, q))$-action of $\phi'$, we see that $S - \{ [e_1], [e_{p+1}] \}$ intersects each $T_1$-orbit on $P_1(\mathbb{R}) - \{ (1 : 0), (0 : 1) \}$ transversely. By the definition of $f'_1$, we have

$$f'_1(z) = (1 : 0) \Leftrightarrow z = [e_1],$$

$$f'_1(z) = (0 : 1) \Leftrightarrow z = [e_{p+1}],$$

$$f'_1(J(z)) = (-a : b) \quad \text{for} \quad f'_1(z) = (a : b).$$

We next denote the smooth $\mathbb{R}$-action $\varphi'$ on $S$ by $\varphi'(\theta, z) = \phi'(m(\theta), z)$. Then we have
the following:
\[ f'_{1}(\varphi'(\theta, z)) = m(\theta) f'_{1}(z), \quad \text{for each } \theta \in \mathbb{R}, z \in S, \]
\[ f'_{1}(\psi_{1}(t, z)) = t f'_{1}(z), \quad \text{for each } t \in T_{1}, z \in S, \]
\[ J(\varphi'(\theta, z)) = \varphi'(-\theta, J(z)). \]

Hence we see that \( S \) is \( J \)-invariant and diffeomorphic to \( P_{1}(\mathbb{R}) \).

### 4.4. Relation between pairs and triples

Let \( (S, \varphi', f'_{1}) \) be a triple of a one-dimensional closed submanifold \( S \) of \( F(H') = P_{1}(\mathbb{C}) \), a smooth \( \mathbb{R} \)-action \( \varphi' : \mathbb{R} \times S \to S \) and a smooth map \( f'_{1} : S \to P_{1}(\mathbb{R}) \) satisfying the following conditions.

(i) \( S \) is \( J \)-invariant and diffeomorphic to \( P_{1}(\mathbb{R}) = \{(u : v) \in F(H') \mid u, v \in \mathbb{R}\} \) containing \([e_{1}],[e_{p+1}]\), where \( J \) is the involution on \( F(H') \) defined in Section 4.1. \( S - \{(e_{1}),(e_{p+1})\} \) intersects each \( T_{1} \)-orbit on \( P_{1}(\mathbb{R}) - \{(1 : 0), (0 : 1)\} \) transversely.

(ii) \( J(\varphi'(\theta, z)) = \varphi'(-\theta, J(z)). \)

(iii) \( f'_{1}(J(z)) = (-a : b) \) for \( f'_{1}(z) = (a : b) \).

(iv) \( f'_{1}(\varphi'(\theta, z)) = m(\theta) f'_{1}(z). \)

(v) \( f'_{1}(z) = (1 : 0) \Leftrightarrow z = [e_{1}], f'_{1}(z) = (0 : 1) \Leftrightarrow z = [e_{p+1}]. \)

We shall construct a pair \( (\phi', f') \) from the triple \( (S, \varphi', f'_{1}) \), where \( \phi' \) is a smooth \( N'(p, q) \)-action on \( F(H') \) and \( f' : F(H') \to P_{1}(\mathbb{C}) \) is a \( N'(p, q) \)-equivariant smooth map whose restriction on \( K \cap N'(p, q) \) coincides with the restriction of the standard \( K \)-action \( \psi_{1} \) satisfying conditions (4.1) and (4.2).

Suppose \( \psi_{1}(t_{1}, x_{1}) = \psi_{1}(t_{2}, x_{2}) \) for \( t_{i} \in T_{1} \) and \( x_{i} \in S \). Then one of the following occurs:

(a) \( x_{1} = x_{2} = [e_{1}] \) or \([e_{p+1}]\),

(b) \( x_{2} = J^{a}(x_{1}) \) and \( t_{1}^{-1} t_{2} = j^{a} (a = 0, 1) \).

Hence we can define a map \( f' : F(H') \to P_{1}(\mathbb{C}) \) by
\[ f'(\psi_{1}(t, x)) = tf'_{1}(x) \quad \text{for any } t \in T_{1}, x \in S. \]

By the conditions of \( f'_{1} \), we see that \( f' \) is smooth and is the unique extension of \( f'_{1} \) as a \( T_{1} \)-equivariant map.

Next we construct a \( N'(p, q) \)-action \( \phi' : N'(p, q) \times F(H') \to F(H') \). Let \( x \in S \). Then we have the decomposition
\[ N'(p, q) = T_{1} M(p, q) (N'(p, q) \cap U'(x)), \]
by (2.4).

Take \((n, z) \in \mathcal{N}'(p, q) \times F(H')\). Let us choose

\[
t \in T_1, x \in S : \quad \psi_1(t, x) = z,
\]

\[
t' \in T_1, \theta \in \mathbb{R}, u \in (\mathcal{N}'(p, q) \cap U')(x) : \quad nt = t'm(\theta)u.
\]

Put

\[
\phi'(n, z) = \psi_1(t', \varphi'(\theta, x)).
\] (4.4)

Then we have the following.

**Proposition 4.4.** \(\phi' : \mathcal{N}'(p, q) \times F(H') \to F(H')\) of (4.4) is a smooth \(\mathcal{N}'(p, q)\)-action on \(F(H')\) such that \(\phi'|_{\mathcal{N}'(p, q) \cap K} : F(H') = \psi_1\) and it satisfies conditions (4.1) and (4.2).

**Proof.** The proof that \(\phi'\) of (4.4) is well defined and defines a smooth \(\mathcal{N}'(p, q)\)-action on \(F(H')\) such that \(\phi'|_{\mathcal{N}'(p, q) \cap K} = \psi_1\) is similar to that of \(\Phi\) in Section 3.

Suppose \(n \in \mathcal{N}'(p, q) \cap U'(z)\). Then there exist \(t_1 \in T_1\) and \(x \in S\) satisfying \(\psi_1(t_1, x) = z\). Since

\[
nt_1 = t_1m(0)(t_1^{-1}nt_1) \in T_1M(p, q)(\mathcal{N}'(p, q) \cap U'(x)),
\]

we have \(\phi'(n, z) = z\). Thus the condition (4.1) holds.

By the definitions of \(\phi'\) and \(f'\), we have

\[
f'(\phi'(n, z)) = f'(\psi_1(t', \varphi'(\theta, x))) = t'f_1'(\varphi'(\theta, x)) = t'm(\theta)f_1'(x)
\]

\[
= ntf_1'(x) = nf_1'(\psi_1(t, x)) = nf'(z).
\]

Hence \(f'\) is \(\mathcal{N}'(p, q)\)-equivariant.

Thus we have proved the following.

**Theorem 4.5.** Suppose \(p, q \geq 3\). Then there is a one-to-one correspondence between the set of pairs \((\phi', f')\) given in Theorem 4.3 and the set of triples \((S, \varphi', f_1')\), where \(S\) is a one-dimensional closed manifold of \(F(H')\), \(\varphi'\) is a smooth one-parameter group on \(S\) and \(f_1' : S \to P_1(\mathbb{R})\) is a smooth map satisfying conditions (i)–(v) in Section 4.4.

Let \(m\) be any positive integer. Then we see that the triple \((S^1, \varphi, f)\) in [3, Example 2] induces the triple \((P_1(\mathbb{R}), \varphi', f_1')\) such that the corresponding \(SU(p, q)\)-action constructed by (4.3) has \((4m - 1)\) orbits. Hence we have the following.

**Corollary 4.6.** There exist infinitely many smooth \(SU(p, q)\)-actions on \(P_{p+q-1}(\mathbb{C})\) which are topologically mutually distinct.
5. Relation between $SU(p, q)$-actions on two spaces

We shall relate smooth $G$-actions on $S^{2p+2q-1}$ to those on $P_{p+q-1}(\mathbb{C})$. Let $(\phi, f)$ be a pair of a smooth $N(p, q)$-action $\phi$ on $F(H)$ and a $N(p, q)$-equivariant smooth map $f : F(H) \to P_1(\mathbb{C})$ such that the restriction of $\phi$ on $K \cap N(p, q)$ coincides with the restriction of the standard $K$-action $\psi_0$ satisfying the condition \eqref{eq:3.1}. We denote the set of the pairs by $\mathcal{A}$. Let $(\phi', f')$ be a pair of a smooth $N'(p, q)$-action $\phi'$ on $F(H')$ and a $N'(p, q)$-equivariant smooth map $f' : F(H') \to P_1(\mathbb{C})$ such that the restriction of $\phi'$ on $K \cap N'(p, q)$ coincides with the restriction of the standard $K$-action $\psi_1$ satisfying the conditions \eqref{eq:4.1} and \eqref{eq:4.2}. We denote the set of the pairs by $\mathcal{B}$. By Theorem 3.10 and Theorem 4.3, we identify $\mathcal{A}$ (respectively $\mathcal{B}$) with the set of smooth $G$-actions on $S^{2p+2q-1}$ (respectively $P_{p+q-1}(\mathbb{C})$) whose restricted $K$-action is standard.

We shall construct a mapping $\rho : \mathcal{A} \to \mathcal{B}$ as follows. Let $(\phi, f) \in \mathcal{A}$. Then by the conditions of the pair, we have the smooth map $f' : F(H') \to P_1(\mathbb{C})$ and the smooth $N(p, q)$-action $\phi' : N(p, q) \times F(H') \to F(H')$ such that the following diagram is commutative (see Section 3.2):

$$
\begin{array}{ccc}
N(p, q) \times F(H) & \xrightarrow{\phi} & F(H) \\
\downarrow{id \times \pi} & & \downarrow{f} \\
N(p, q) \times F(H') & \xrightarrow{\phi'} & F(H') \xrightarrow{f'} P_1(\mathbb{C}).
\end{array}
$$

(5.1)

Since $T_0$ acts trivially on $F(H')$ and $P_1(\mathbb{C})$, we may consider that $\phi'$ is the $N'(p, q)$-action on $F(H')$ and $f'$ is the $N'(p, q)$-equivariant smooth map.

It is clear that $(\phi', f')$ satisfies the condition \eqref{eq:4.2}. Now let $z' = \pi(z) \in F(H')$ and $f'(z') = (a : b)$. Suppose $n \in N'(p, q) \cap H(a : b)$. Since $n(a, b) = t_0(a, b)$ for some $t_0 \in T_0$, we have

$$
t_0^{-1}n \in N(p, q) \cap H(a : b) = N(p, q) \cap H(a : b).
$$

Hence $t_0^{-1}n \in N(p, q)_{z'}$. Since $\phi(t_0^{-1}n, z) = z$, we have $\phi'(n, z') = z'$. Hence $n \in N'(p, q)_{z'}$ and the condition \eqref{eq:4.1} holds. Thus $(\phi', f') \in \mathcal{B}$. Therefore we define the mapping $\rho$ by

$$
\rho((\phi, f)) = (\phi', f').
$$

**Proposition 5.1.** The mapping $\rho$ is surjective.

**Proof.** Let $(\phi', f') \in \mathcal{B}$ be given. We may consider that $\phi'$ is a smooth $N(p, q)$-action and $f'$ is a smooth $N(p, q)$-equivariant map. Now we shall construct a pair $(\phi, f) \in \mathcal{A}$ such that the diagram \eqref{eq:5.1} is commutative.
First we define the map \( f \) by \( f = f' \circ \pi \). Then \( f \) is a smooth map.

Next we shall define a smooth \( N(p, q) \)-action \( \phi \) on \( F(H) \). Put \( S = f'^{-1}(P_1(\mathbb{R})) \), \( \varphi' = \phi'|_{M(p,q) \times S} \) and \( f'_1 = f'|_S \). Then the triple \((S, \varphi', f'_1)\) satisfies conditions (i)--(v) in Section 4.4 (see Section 4.3). Let \( P_1(\mathbb{R}) = \{(x : y) \in F(H') \mid x, y \in \mathbb{R}\} \) and \( S^1 = \{(x, y) \in F(H) \mid x, y \in \mathbb{R}\} \). Now we construct a two-fold covering space \( \tilde{S} \) of \( S \) and a \( \mathbb{R} \)-action \( \tilde{\varphi} \) on \( \tilde{S} \) as follows. Using the restricted \( T_1 \)-action on \( F(H') \) and condition in Section 4.4(i), we can define the smooth map \( \sigma : P_1(\mathbb{R}) \to T_1 \) satisfying the following conditions:

\[
\psi_1(\sigma(x : y), (x : y)) = (u : v) \in S, \\
\psi_1(\sigma(x : y), (-x : y)) = (-u : v) \in S, \\
\sigma([e_1]) = \sigma([e_{p+1}]) = 1 \in T_1.
\]

Then we see that the map \( \sigma \) induces the well-defined smooth map \( \sigma' : P_1(\mathbb{R}) \to T_1/\mathbb{Z}_2 \). Designating one component of \( P_1(\mathbb{R}) - \{[e_1], [e_{p+1}]\} \) and that of \( S - \{[e_1], [e_{p+1}]\} \), we can lift the map \( \sigma' \) to the smooth map \( \tilde{\sigma} : S^1 \to T_1 \). Put

\[
\tilde{S} = \{\psi_0(\tilde{\sigma}(x, y), (x, y)) \mid (x, y) \in S^1\} \subset F(H).
\]

Then \( \tilde{S} \) is a two-fold covering space of \( S \) diffeomorphic to \( S^1 \) and \( J_i \)-invariant \((i = 1, 2)\). Since \( \{\varphi'(\theta, (u : v)) \mid \theta \in \mathbb{R}\} \) is an open interval or a point, we can lift the smooth \( \mathbb{R} \)-action \( \varphi' \) on \( S \) to that on \( \tilde{S} \). We denote the smooth \( \mathbb{R} \)-action on \( \tilde{S} \) by \( \tilde{\varphi} \).

By definition we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R} \times \tilde{S} & \xrightarrow{\tilde{\varphi}} & \tilde{S} \\
\downarrow{id \times \pi} & & \downarrow{\pi} \\
\mathbb{R} \times S & \xrightarrow{\psi'} & S \\
\downarrow{f_1} & & \downarrow{f'_1} \rightarrow P_1(\mathbb{R}),
\end{array}
\]

where \( f_1 \) is the map defined by \( f_1 = f|_{\tilde{S}} \). We notice that \( F(H) = \psi_0(T^2, \tilde{S}) \).

Take \((n, z) \in N(p, q) \times F(H)\). Let us choose

\[
t \in T^2, s \in \tilde{S} : \quad \psi_0(t, s) = z,
\]

\[
t' \in T^2, \theta \in \mathbb{R}, u \in N(p, q) \cap H(a : b) : \quad nt = t'm(\theta)u,
\]

where \( f_1(s) = (a : b) \) and set

\[
\phi(n, z) = \psi_0(t', \tilde{\varphi}(\theta, s)). \quad (5.2)
\]
Smooth $SU(p, q)$-actions

Then $\phi$ is well defined and defines a smooth $N(p, q)$-action on $F(H)$ whose restriction on $K \cap N(p, q)$ coincides with the restriction of $\psi_0$. The proof is similar to that of $\Phi$ in Section 3 (see also Section 4.4).

By definition, we have

$$\pi \phi(n, z) = \pi \psi_0(t', \phi(\theta, s)) = \psi_1(t', \pi \overline{\phi}(\theta, s)) = \psi_1(t', \phi'(\theta, \pi(s))).$$

Here we decompose $t, t'$ and $n$ as

$$t = t_1t_0 \in T_1T_0, \quad t' = t_1't_0' \in T_1T_0, \quad n = t_0''n' \in T_0N'(p, q),$$

respectively. Then it is clear that $f'(\pi(s)) = (a : b)$, $\psi_1(t, \pi(s)) = \pi(z)$ and $u \in N'(p, q)$. Hence $\psi_1(t_1, \pi(s)) = \pi(z)$ and $n't_1 = t_1'm(\theta)(t_0''t_0'\mu_0^{-1}) \in T_1M(p, q)(N'(p, q) \cap H'(a : b))$. Therefore we have

$$\phi'(n, \pi(z)) = \phi'(n', \pi(z)) = \psi_1(t_1', \phi'(\theta, \pi(s))) = \psi_1(t', \phi'(\theta, \pi(s))) = \pi \phi(n, z).$$

Hence the diagram (5.1) is commutative.

Finally, we shall show that the pair satisfies the condition (3.1). Let $z \in F(H)$ and $f(z) = (a : b)$. Suppose that $n \in N(p, q) \cap H(a : b)$. If $z \notin \tilde{S}$, then $n \in (p, q)_z$ is clear. If $z \notin \tilde{S}$, then there exist $t \in T^2$ and $a_0, b_0 \in \mathbb{R}$ such that $t(a_0, b_0) = (a, b)$. Since $f(\psi_0(t^{-1}, z)) = (a_0, b_0)$ and $\psi_0(t^{-1}, z) \in \tilde{S}$, we have $t^{-1}nt \in N(p, q) \cap H(a_0 : b_0) \subset N(p, q)_{\psi_0(t^{-1}, z)}$. Hence $\phi(t^{-1}nt, \psi_0(t^{-1}, z)) = \psi_0(t^{-1}, z)$. Thus $\phi(n, z) = z$ and $n \in N(p, q)_z$.

By this proposition and Corollary 4.6, we have the following.

**Corollary 5.2.** There exist infinitely many smooth $SU(p, q)$-actions on $S^{2p+2q-1}$ which are topologically mutually distinct.

**Proposition 5.3.** Let $(\phi, f) \in A$ and $(\phi', f') \in B$ be given. Let $\Phi$ and $\Phi'$ be the smooth actions constructed from $(\phi, f)$ by (3.4) and from $(\phi', f')$ by (4.3), respectively.

If $\rho((\phi, f)) = (\phi', f')$, then the following diagram is commutative:

$$\begin{array}{ccc}
SU(p, q) \times S^{2p+2q-1} & \xrightarrow{\Phi} & S^{2p+2q-1} \\
\text{id} \times \Pi & \downarrow & \Pi \\
SU(p, q) \times P_{p+q-1}(\mathbb{C}) & \xrightarrow{\Phi'} & P_{p+q-1}(\mathbb{C})
\end{array}$$

(5.3)

**Proof.** Let $\Phi(g, p) = \psi_0(k', \phi(n, z))$, where $p = \psi_0(k, z)$ for some $k \in K$, $z \in F(H)$ and $gk = k'nu \in KN(p, q)H(a : b)$ for $f(z) = (a : b)$. We decompose $n$ as
\[ n = n't, \text{ where } n' \in N'(p, q), t \in T_0. \text{ Then } gk = k'n'(tu) \in KN'(p, q)H'(a : b) \text{ and } \psi_1(k, \pi(z)) = \Pi(p). \text{ Hence we have} \]

\[ \Pi(\Phi(g, p)) = \Pi(\psi_0(k', \phi(n, z))) = \psi_1(k', \phi'(n, \pi(z))) = \psi_1(k', \phi'(n', \pi(z))) = \Phi'(g, \Pi(p)). \]

**Corollary 5.4.** Let \( \Phi' : SU(p, q) \times P_{p+q-1}(\mathbb{C}) \to P_{p+q-1}(\mathbb{C}) \) be a smooth action whose restricted \( SU(p) \times U(q) \)-action is standard. Then the action \( \Phi' \) is lifted on \( S^{2p+2q-1} \). Namely, there exists a smooth \( SU(p, q) \)-action \( \Phi \) on \( S^{2p+2q-1} \) such that the restricted \( SU(p) \times U(q) \)-action is standard and the diagram (5.3) is commutative.

Next we consider the injectivity of the mapping \( \rho \).

**Lemma 5.5.** Let \( (\phi_i, f_i) \in A \) and \( P_i = f_i^{-1}(P_1(\mathbb{R})) \) \((i = 1, 2)\). If \( \rho((\phi_1, f_1)) = \rho((\phi_2, f_2)) \), then \( P_1 = P_2 \) and \( f_1 = f_2 \).

**Proof.** Let \( \rho((\phi_i, f_i)) = (\phi', f') \). Then \( P_i = \pi^{-1}(f' \pi^{-1}(P_1(\mathbb{R}))) \) and \( f_i = f' \circ \pi \) \((i = 1, 2)\). Hence \( P_1 = P_2 \) and \( f_1 = f_2 \). \( \square \)

**Lemma 5.6.** Let \( (\phi_i, f_i) \in A \) \((i = 1, 2)\) and \( \rho((\phi_1, f_1)) = \rho((\phi_2, f_2)) \). Then \( (\phi_1, f_1) = (\phi_2, f_2) \) if and only if \( \phi_1\mid_{M(p,q) \times P} = \phi_2\mid_{M(p,q) \times P} \), where \( P = f_i^{-1}(P_1(\mathbb{R})) \).

**Proof.** Suppose \( \phi_1\mid_{M(p,q) \times P} = \phi_2\mid_{M(p,q) \times P} \). Let \( (n, p) \in N(p, q) \times F(H) \). Then there exist \( t \in T^2 \) and \( z \in P \) such that \( p = \psi_0(t, z) \) and \( f(z) = (a : b) \in P_1(\mathbb{R}) \), where \( f = f_i \). We have a decomposition \( nt = t_1m(\theta)u \in T^2M(p, q)(N(p, q) \cap H(a : b)) \) by (2.4). Hence

\[
\phi_1(n, p) = \phi_1(n, \psi_0(t, z)) = \phi_1(nt, z) \\
= \phi_1(t_1m(\theta)u, z) = \psi_0(t_1, \phi_1(m(\theta), z)) = \psi_0(t_1, \phi_2(m(\theta), z)) \\
= \phi_2(nt, z) = \phi_2(n, p).
\]

The converse is trivial. \( \square \)

Let \( (\phi, f) \in A \) and \( \rho(\phi, f) = (\phi', f') \). We define

\[ F(\phi, f) = \{ z \in P : \phi(m(\theta), z) = z \text{ for any } \theta \in \mathbb{R} \}, \]
\[ F(\phi', f') = \{ x \in S : \phi'(m(\theta), x) = x \text{ for any } \theta \in \mathbb{R} \}, \]

where \( P = f_1^{-1}(P_1(\mathbb{R})) \) and \( S = f'^{-1}(P_1(\mathbb{R})) \). Then \( F(\phi, f) \) is \( T_0 \)-invariant. If \( z \in F(\phi, f) \), then \( \pi(z) \in F(\phi', f') \). Hence we have the following.
Lemma 5.7. If $\rho((\phi', f')) = (\phi', f')$. Then $#(F(\phi', f')/T_0) \leq #F(\phi', f')$, where $#F$ is the number of connected components of $F$.

Example 5.8. (1) Let $(\phi', f') \in B$ be given and let $(\phi, f) \in A$ be the pair constructed in the proof of Proposition 5.1 satisfying $\rho((\phi, f)) = (\phi', f')$. We use the notation in the proof of Proposition 5.1. If $S - F(\phi', f')$ has more than two components, we can construct a new pair $(\tilde{\phi}, \tilde{f}) \in A$ satisfying $(\tilde{\phi}, \tilde{f}) \neq (\phi, f)$ and $\rho((\tilde{\phi}, \tilde{f})) = (\phi', f')$ as follows. Since $\pi : P = f^{-1}(P_1(\mathbb{R})) \to S$ is the principal $T_0$-bundle, $\pi : \tilde{S} \to S$ is the restricted $Z_2$-bundle. Let $S_1$ be a component of $S - F(\phi', f')$ which does not contain $[e_1]$ or $[e_{p+1}]$ and let $\tilde{S}_1$ be a component of $\pi^{-1}(S_1) \subset \tilde{S}$. Deforming a neighborhood of a point of $\tilde{S}_1$ slightly along the fiber of the principal $T_0$-bundle $\pi$, we first construct a two-fold covering covering $\tilde{S}$ of $S$ and a one-parameter group $\tilde{\phi}$ on $\tilde{S}$ satisfying the following:

(a) $\tilde{S}_1^{-1}(S_1) = \tilde{S}_1 \cup J_1 J_2(\tilde{S}_1)$, $\tilde{S}_1^{-1}(J\tilde{S}_1) = J_1(\tilde{S}_1) \cup J_2(\tilde{S}_1)$, where $\tilde{S}_1$ is the slight deformation of $\tilde{S}_1$ and $\tilde{S} = \tilde{S}$ on $\tilde{S} - \pi^{-1}(S_1) \cup J(S_1)$;

(b) $\{\bar{\phi}(\theta, J_1^a J_2^b(z)) : \theta \in \mathbb{R}\} = J_1^a J_2^b(\tilde{S}_1)(a, b = 0, 1)$ for each $z \in \tilde{S}_1$ and $\bar{\phi} = \tilde{\phi}$ on the other place;

(c) $d\pi(\tilde{X}) = X$, where $X$ and $\tilde{X}$ are vector fields on $S$ and $\tilde{S}$ induced from the one-parameter groups $\phi'$ and $\phi$, respectively.

In addition, we put $\tilde{f} = f$. Using $\tilde{S}, \tilde{\phi}$ and $\tilde{f}$, we can construct a new pair $(\tilde{\phi}, \tilde{f}) \in A$ in the same way as in the proof of Proposition 5.1.

By the definition of the pairs, we see that $\#(F(\tilde{\phi}, \tilde{f})/T_0) = #F(\phi, f)/T_0) = #F(\phi', f')$.

(2) If $S - F(\phi', f')$ has more than three components, then we put $S_1 = S_1' \cup \{z\} \cup S_1''$, where $S_1'$ and $S_1''$ are components of $S - F(\phi', f')$ and $z \in F(\phi', f')$. Assume that $S_1$ is connected and the orientation of the vector field on $S_1'$ induced from the one-parameter group $\phi'$ is same as that on $S_1''$. Note that we can construct a pair $(\phi', f')$ satisfying the assumption, using the functions constructed by Asok [1, Section 10] (cf. [3, Section 7, Example 2]). Then we can construct a pair $(\tilde{\phi}', \tilde{f}') \in A$ satisfying $\rho((\tilde{\phi}', \tilde{f}')) = (\phi', f')$, using the same deformation around the point $z$ as in (1). Moreover we see that $\#((\tilde{\phi}', \tilde{f}')/T_0) < #F(\phi', f')$ by definition.

Thus we have the following.

Proposition 5.9. The mapping $\rho$ is not injective.

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