EXPLICIT CONSTRUCTION OF HARMONIC TWO-SPHERES IN $SU(3)/SO(3)$

Hui MA

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0. Introduction

The study of harmonic maps from the two-sphere S^2 into a compact symmetric space has produced many results in recent years [4, 8, 9, 10]. In this setting, harmonic maps are obtained from holomorphic curves in some auxiliary complex manifold (a twistor space) such as a flag manifold [4, 8] or a loop space [9]. In [11] and [12], Wood gave an explicit construction of harmonic maps from the two-sphere to a complex Grassmannian and the unitary group. Subsequently, Burstall and Guest [3] gave a new description of harmonic maps from the two-sphere to a compact symmetric space, using a method of Morse theory. In their paper, they described explicitly the harmonic maps in the case of an inner symmetric space in terms of meromorphic functions on S^2 (i.e. rational functions). However, there is no concrete result on non-inner symmetric spaces. In this paper, we study harmonic maps from S^2 to $SU(3)/SO(3)$, which is a non-inner compact symmetric space with dimension 5 and rank 2. Its geometric realization is a set of the $\mathbb{R}P^2$'s in $\mathbb{C}P^2$ [1]. We shall follow the work of Wood [12] and analyze the corresponding Cartan embedding to give a straightforward description of all harmonic maps from S^2 to $SU(3)/SO(3)$ (see Theorem 4.1).

The paper is arranged as follows. In Section 1, we review the harmonic sequence for harmonic maps from S^2 to the complex projective plane $\mathbb{C}P^2$. In Section 2, we recall the classification of all harmonic maps from S^2 to $U(3)$ given by Wood [12]. In Section 3, we describe the Cartan embedding of $SU(3)/SO(3)$ into $SU(3)$. In the final section, which contains the main result, we discuss the harmonic maps $S^2 \rightarrow SU(3)/SO(3)$.

Throughout this paper let $SU(3)$ and $SO(3)$ denote the Lie groups of 3×3 special unitary matrices and 3×3 special orthogonal matrices respectively. Let $su(3)$ and $so(3)$ denote their Lie algebras respectively.

1. The harmonic sequence

Let $\phi : M^2 \rightarrow \mathbb{C}P^n$ be a harmonic map from a Riemann surface and let $\underline{\mathbb{C}}^{n+1}$ be the trivial bundle $M \times \mathbb{C}^{n+1}$ over M . Assume that ϕ is not anti-holomorphic. Let $\Phi : U \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be a lift of ϕ over a coordinate neighborhood (U, z) , i.e. a representation of ϕ in homogeneous coordinates. Define $G'(\Phi) : U \rightarrow \mathbb{C}^{n+1}$ by

$$G'(\Phi) = \frac{\partial \Phi}{\partial z} - \frac{\langle \partial \Phi / \partial z, \Phi \rangle}{|\Phi|^2} \Phi,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on $\mathbb{C}^{n+1}$. Then $G'(\Phi)$ may vanish at isolated points, but because of the harmonicity of ϕ , it can be shown that it defines a holomorphic section of the subbundle of $\underline{\mathbb{C}}^{n+1}$ defined by $\phi^\perp$ equipped with its Koszul–Malgrange holomorphic structure, in particular its zeros are all of holomorphic type and so $G'(\Phi)$ gives a well-defined map $G'(\phi) : M^2 \rightarrow \mathbb{C}P^n$ called the ∂' -Gauss map of ϕ . Assuming instead that ϕ is not holomorphic and replacing z by $\bar{z}$ gives the ∂'' -Gauss map $G''(\phi) : M^2 \rightarrow \mathbb{C}P^n$ of ϕ . These two transformations are inverse to each other, viz. $G''(G'(\phi)) = \phi$ if ϕ is not anti-holomorphic and $G'(G''(\phi)) = \phi$ if ϕ is not holomorphic. Furthermore, from the harmonicity of ϕ , we can obtain the harmonicity of $G'(\phi)$ and $G''(\phi)$. Thus the sequence of harmonic maps

$$\dots \leftarrow G''(G''(\phi)) \leftarrow G''(\phi) \leftarrow \phi \rightarrow G'(\phi) \rightarrow G'(G'(\phi)) \rightarrow \dots$$

(together with natural maps between them) is called the harmonic sequence of ϕ . On the other hand, any harmonic maps from the two-sphere to $\mathbb{C}P^n$ can be obtained from a holomorphic map by repeated application of the ∂' -Gauss map. In particular, if $f : S^2 \rightarrow \mathbb{C}P^2$ is a full holomorphic map, then we have the harmonic sequence of f as follows:

$$0 \leftarrow f \rightarrow G'(f) \rightarrow G^{(2)}(f) \rightarrow 0,$$

where $G^{(2)}(f) = G'(G'(f))$.

2. Harmonic maps from S^2 to $SU(3)$

2.1. Harmonic maps from S^2 to $U(3)$

For $i = 1, 2$ let $G_i(\mathbb{C}^3)$ denote the Grassmannian with its standard Kähler structure. We regard an element $a \in G_i(\mathbb{C}^3)$ as an i -dimensional complex subspace of $\mathbb{C}^3$ and π_a as the (Hermitian) orthogonal projection onto the subspace a . Then π_a satisfies

$\pi_a^2 = \pi_a^* = \pi_a$, where $*$ denotes adjoint (conjugate transpose). Note that there is an anti-holomorphic isometry $\perp : G_i(\mathbb{C}^3) \rightarrow G_{3-i}(\mathbb{C}^3)$ given by sending a subspace a to its Hermitian orthogonal complement $a^\perp$.

Let $\underline{\mathbb{C}}^3$ denote the trivial holomorphic vector bundle $S^2 \times \mathbb{C}^3$ over S^2 . As in [5], we identify a (smooth) map $\phi : S^2 \rightarrow G_i(\mathbb{C}^3)$ with the (smooth complex) subbundle $\underline{\phi}$ of rank i of the trivial bundle $\underline{\mathbb{C}}^3$ with fiber at any point $x \in S^2$ given by $\underline{\phi}_x = \phi(x)$. Let ∂ denote the flat connection on $\underline{\mathbb{C}}^3$ and let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian metric. The Cartan embedding $c : G_i(\mathbb{C}^3) \rightarrow U(3)$ is defined by $a \mapsto \pi_a - \pi_a^\perp = 2\pi_a - I$; note that $c(a^\perp) = -c(a)$. Then the Cartan embedding is totally geodesic and identifies $G_i(\mathbb{C}^3)$ with the subset $\{A \in U(3) : A^2 = I\} = \{A \in U(3) : A = A^*\}$.

Using the above notation, Wood gave explicitly all the harmonic maps from S^2 to $U(3)$. The following theorem is Theorem 4.3 in [12].

THEOREM 2.1. *Let $\phi : S^2 \rightarrow U(3)$ be a harmonic map. Then one of the following holds.*

- (i) *The map ϕ is of the form $\phi = \phi_0 c(f)$ for some harmonic map $f : S^2 \rightarrow G_i(\mathbb{C}^3)$ ($i = 1$ or 2), where c denotes the Cartan embedding of $G_i(\mathbb{C}^3)$ into $U(3)$ and $\phi_0 : S^2 \rightarrow U(3)$ is a constant map. Note that since $c(f^\perp) = -c(f)$, we may assume that $i = 1$, that is, f is a harmonic map from S^2 to $\mathbb{C}P^2$. This form of map is called a Grassmannian (solution).*
- (ii) *The map ϕ is of the form $\phi = \phi_0(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$, where $\phi_0 : S^2 \rightarrow U(3)$ is a constant map, $\beta_1 : S^2 \rightarrow G_2(\mathbb{C}^3)$ is a holomorphic map and β_2 is a basic flag factor for $\phi_1 = \phi_0(\pi_{\beta_1} - \pi_{\beta_1}^\perp)$. Explicitly, $\underline{\beta}_1$ and $\underline{\beta}_2$ are one of the following:*

- (a) *$\underline{\beta}_1 = \underline{c} \oplus \underline{f}$, where $\underline{c}$ is a rank-1 constant subbundle of $\underline{\mathbb{C}}^3$ and $\underline{f}$ is a non-constant holomorphic subbundle of $\underline{c}^\perp \cap \underline{\mathbb{C}}^3 \cong \underline{\mathbb{C}}^2$ (so $f : S^2 \rightarrow \mathbb{C}P^1$ is a non-constant holomorphic map).*

Explicitly, let C be a constant section of $\underline{c}$ and let F be a meromorphic section of $\underline{f}$ (thus C and F represent $\underline{c}$ and $\underline{f}$ in homogeneous coordinates). Then

$$\underline{\beta}_1 = \text{span}\{C, F\}, \quad \underline{\beta}_2 = \text{span}\{\lambda_1 C + \lambda_2 G'(F)\},$$

where λ_1 and λ_2 are arbitrary non-identically-zero meromorphic functions.

- (b) *$\underline{\beta}_1 = \underline{f} \oplus G'(f)$, where $f : S^2 \rightarrow \mathbb{C}P^2$ is a full holomorphic map. Let F be a meromorphic section of $\underline{f}$. Then*

$$\underline{\beta}_1 = \text{span}\{F, G'(F)\}, \quad \underline{\beta}_2 = \text{span}\{\lambda_1 F + \lambda_2 G^{(2)}(F)\},$$

where $\lambda_1, \lambda_2 : S^2 \rightarrow \mathbb{C} \cup \{\infty\}$ are arbitrary non-identically-zero meromorphic functions. That is, β_1 is a Frenet pair associated to a full holomorphic map $f : S^2 \rightarrow \mathbb{CP}^2$ and β_2 is a rank-1 holomorphic subbundle of the mixed pair $\underline{f} \oplus G^{(2)}(f) = G'(f)^\perp (\beta_2 \neq f)$.

- (iii) The map ϕ is the composition of conjugation $z \mapsto \bar{z}$ of $S^2 = \mathbb{C} \cup \{\infty\}$ and a map of type (ii).

2.2. Harmonic maps from S^2 to $SU(3)$

Since $U(3) = SU(3) \times S^1$, the inclusion $i : SU(3) \rightarrow U(3)$ is a totally geodesic embedding. Thus by the composition law of harmonic maps, we find that an arbitrary map $\phi : S^2 \rightarrow SU(3)$ is harmonic if and only if the composition $i \circ \phi : S^2 \rightarrow U(3)$ is harmonic. Therefore an arbitrary harmonic map $S^2 \rightarrow SU(3)$ is a harmonic map $S^2 \rightarrow U(3)$ valued in $SU(3)$. That is, we have the following.

COROLLARY 2.2. *Let $\phi : S^2 \rightarrow SU(3)$ be a harmonic map. Then one of the following holds (note that we still use the notation of Theorem 2.1 except that ϕ_0 is a constant map valued in $SU(3)$):*

- (i) the map ϕ is of the form $\phi = \varepsilon_i \phi_0 c(f)$ ($i = 1$ or 2), where $\varepsilon_1 = 1, \varepsilon_2 = -1$;
- (ii) the map ϕ is of the form $\phi = -\phi_0(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$;
- (iii) the map ϕ is the composition of conjugation $z \mapsto \bar{z}$ of $S^2 = \mathbb{C} \cup \{\infty\}$ and a map of type (ii).

3. Cartan embedding of $SU(3)/SO(3)$ into $SU(3)$

THEOREM 3.1. [7, p. 424] *A compact symmetric space of semi-simple type G/K is inner if and only if K contains a maximal torus of G , that is $\text{rank } G = \text{rank } K$.*

We know that $\text{rank } SU(3) = 2$ and $\text{rank } SO(3) = 1$, so $SU(3)/SO(3)$ is a non-inner symmetric space. Thus the study of harmonic maps from S^2 to $SU(3)/SO(3)$ is out of the scope of known results.

Now we give some details of the Cartan embedding in the case of $SU(3)/SO(3)$. Define an involutive automorphism $\sigma : SU(3) \rightarrow SU(3)$ as

$$\sigma(y) = \bar{y}.$$

Then $SO(3)$ is the fixed point set of σ . The decomposition of $\mathfrak{su}(3)$ into ± 1 -eigenspace of the derivative of σ gives the symmetric decomposition $\mathfrak{su}(3) = \mathcal{K} \oplus \mathcal{P}$.

It is easy to know that for $X \in su(3)$, we have $d\sigma(X) = \bar{X} = -X^t$. Then

$$\mathcal{K} = \{X \in su(3) \mid -X^t = X\} = so(3), \quad \mathcal{P} = \{X \in su(3) \mid X^t = X\}.$$

Following [2] and [6], we have the following.

PROPOSITION 3.2. *The map $i : [gSO(3)] \mapsto \bar{g}\bar{g}^t$ defines an embedding of $SU(3)/SO(3)$ onto $\exp \mathcal{P}$ which is a totally geodesic submanifold of $SU(3)$ (where the map i is called the Cartan embedding).*

Set $\hat{N} = \exp \mathcal{P} = \{g \in SU(3) \mid g^t = g\}$. By Proposition 3.2, we have $i(SU(3)/SO(3)) = \hat{N}$. From the fact that $\hat{N}$ is a totally geodesic submanifold of $SU(3)$, one can derive the following basic fact concerning harmonic maps.

PROPOSITION 3.3. *$\psi : S^2 \rightarrow SU(3)/SO(3)$ is harmonic if and only if $i \circ \psi : S^2 \rightarrow SU(3)$ is harmonic.*

The above proposition shows that the theory of harmonic maps $S^2 \rightarrow SU(3)/SO(3)$ is a special case of the theory of harmonic maps $S^2 \rightarrow SU(3)$.

To find harmonic maps into the symmetric space $SU(3)/SO(3)$ we must obtain conditions that ensure that harmonic maps into $SU(3)$ factor through the Cartan embedding and so give rise to harmonic maps into $SU(3)/SO(3)$. How to do this has a pleasantly simple answer: it is necessary and sufficient that $\phi : S^2 \rightarrow SU(3)$ take values in $\hat{N} = \{g \in SU(3) \mid g = g^t\}$.

Subsequently, to construct the harmonic maps into $SU(3)/SO(3)$, it remains to present the inverse map $i^{-1} : \hat{N} \rightarrow SU(3)/SO(3)$. As a consequence, our desired maps are just the composition of the harmonic maps into $SU(3)$ which factor through the Cartan embedding and the inverse map i^{-1} . We will give the map $i^{-1} : \hat{N} \rightarrow SU(3)/SO(3)$ as follows.

Given an element A of $\hat{N} = \{A \in SU(3) \mid A = A^t\}$, we shall find $[g] \in SU(3)/SO(3)$ such that $A = \bar{g}\bar{g}^t$, then $i^{-1}(A) = [g]$. Since A is a special unitary symmetric matrix there is a real orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal. If we put $Q^{-1}AQ = DD$ for a special unitary diagonal matrix D and then let $P = QDQ^{-1}$ we see that $P \in SU(3)$ and $A = PP^t$. Set $g = \bar{P}$ and $A = i([g])$. By the injectivity of i we can define $i^{-1}(A) = [g]$.

4. Harmonic maps from S^2 to $SU(3)/SO(3)$

4.1. Let $\phi : S^2 \rightarrow SU(3)$ be of type (i) given by Corollary 2.2. Note that since ϕ_0 is a constant map, we may assume ϕ_0 is equal to I , the identity element of $SU(3)$.

Therefore $\phi = c \circ f$, where $f : S^2 \rightarrow \mathbb{C}P^2$ is a harmonic map and $c(a) = 2\pi_a - I$, $\forall a \in \mathbb{C}P^2$, is the Cartan embedding of $\mathbb{C}P^2$ in $U(3)$. Assume $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$. Let $F(z) = (b_1(z), b_2(z), b_3(z)) : U \rightarrow \mathbb{C}^3 \setminus \{0\}$ be a lift of $f : S^2 \rightarrow \mathbb{C}P^2$ over a coordinate neighborhood (U, z) . The canonical meromorphic basis of $\mathbb{C}^3$ is the standard basis e_i ($i = 1, 2, 3$) defined by the constant sections $e_i = (\delta_{1i}, \delta_{2i}, \delta_{3i})$, where $\delta_{ij} = 1$ if $i = j$ otherwise $\delta_{ij} = 0$. Thus the projective matrix π_f can be obtained from the following formula:

$$\pi_f(e_i) = \frac{\langle e_i, F \rangle}{|F|^2} F = \frac{\sum_{k=1}^3 \bar{b}_i b_k}{|F|^2} e_k, \quad (4.1)$$

$$\pi_f = \frac{1}{|F|^2} \begin{pmatrix} |b_1|^2 & \bar{b}_2 b_1 & \bar{b}_3 b_1 \\ \bar{b}_1 b_2 & |b_2|^2 & \bar{b}_3 b_2 \\ \bar{b}_1 b_3 & \bar{b}_2 b_3 & |b_3|^2 \end{pmatrix}. \quad (4.2)$$

Since ϕ is given by $\phi = 2\pi_f - I$, the harmonic maps of type (i) which factor through the Cartan embedding must satisfy the condition $\pi_f = \pi_f^t$, that is, $\bar{b}_1 b_2, \bar{b}_1 b_3, \bar{b}_2 b_3$ are real. Therefore $f = [F] = [b_1, b_2, b_3] \in \mathbb{R}P^2$. Accordingly, $\phi = 2\pi_f - I : S^2 \rightarrow SU(3)$ factors through $SU(3)/SO(3)$ for a full harmonic map $f : S^2 \rightarrow \mathbb{C}P^2$ if and only if f is valued in $\mathbb{R}P^2$.

4.2. Let $\phi : S^2 \rightarrow SU(3)$ be of case (a) of type (ii) given by Corollary 2.2, i.e. $\phi = -(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$, where $\beta_1 : S^2 \rightarrow G_2(\mathbb{C}^3)$ is a holomorphic map such that $\underline{\beta}_1 = \text{span}\{C, F\}$ where $F = (b_1(z), b_2(z), b_3(z)) : U \rightarrow \mathbb{C}^3 \setminus \{0\}$ is a local lift of the holomorphic map $f : S^2 \rightarrow \mathbb{C}P^1 \subset \mathbb{C}P^2$ and $C = (c_1, c_2, c_3)$ is a constant section of $\underline{c}$ and $\underline{\beta}_2 = \text{span}\{\lambda_1 C + \lambda_2 G'(F)\}$, where λ_1 and λ_2 are arbitrary non-identically-zero meromorphic functions. Using the formula

$$\pi_{\beta_1}(e_i) = \frac{\langle e_i, C \rangle}{|C|^2} C + \frac{\langle e_i, F \rangle}{|F|^2} F \quad (4.3)$$

we get the projective matrix

$$\pi_{\beta_1} = \frac{1}{|C|^2} \begin{pmatrix} |c_1|^2 & \bar{c}_2 c_1 & \bar{c}_3 c_1 \\ \bar{c}_1 c_2 & |c_2|^2 & \bar{c}_3 c_2 \\ \bar{c}_1 c_3 & \bar{c}_2 c_3 & |c_3|^2 \end{pmatrix} + \frac{1}{|F|^2} \begin{pmatrix} |b_1|^2 & \bar{b}_2 b_1 & \bar{b}_3 b_1 \\ \bar{b}_1 b_2 & |b_2|^2 & \bar{b}_3 b_2 \\ \bar{b}_1 b_3 & \bar{b}_2 b_3 & |b_3|^2 \end{pmatrix}. \quad (4.4)$$

By definition,

$$G'(F) = \sum_{i=1}^3 \left(\frac{\partial b_i}{\partial z} - \alpha b_i \right) e_i := \sum_{i=1}^3 g_i e_i, \quad (4.5)$$

where

$$\alpha = \frac{\langle \partial F / \partial z, F \rangle}{|F|^2} = \frac{\sum_{j=1}^3 (\partial b_j / \partial z) \bar{b}_j}{|F|^2}.$$

Set

$$\lambda_1 C + \lambda_2 G'(F) = \sum_{i=1}^3 d_i e_i, \quad \text{where } d_i = \lambda_1 c_i + \lambda_2 g_i, \quad i = 1, 2, 3. \quad (4.6)$$

Similarly, we have the projective matrix

$$\pi_{\beta_2} = \frac{1}{|\beta_2|^2} \begin{pmatrix} |d_1|^2 & \bar{d}_2 d_1 & \bar{d}_3 d_1 \\ \bar{d}_1 d_2 & |d_2|^2 & \bar{d}_3 d_2 \\ \bar{d}_1 d_3 & \bar{d}_2 d_3 & |d_3|^2 \end{pmatrix}. \quad (4.7)$$

By direct computation, $\phi : S^2 \rightarrow SU(3)$ satisfies the condition $\phi = \phi^t$, that is, $(2\pi_{\beta_1} - I)(2\pi_{\beta_2} - I) = (2\pi_{\beta_2}^t - I)(2\pi_{\beta_1}^t - I)$, if and only if the following equations are satisfied:

$$\frac{2\lambda_1}{|\beta_2|^2} (c_1 \cdot \bar{d}_2) - \frac{d_1 \bar{d}_2}{|\beta_2|^2} - \frac{c_1 \bar{c}_2}{|C|^2} - \frac{b_1 \bar{b}_2}{|F|^2} = \frac{2\lambda_1}{|\beta_2|^2} (c_2 \cdot \bar{d}_1) - \frac{\bar{d}_1 d_2}{|\beta_2|^2} - \frac{\bar{c}_1 c_2}{|C|^2} - \frac{\bar{b}_1 b_2}{|F|^2}, \quad (4.8)$$

$$\frac{2\lambda_1}{|\beta_2|^2} (c_1 \cdot \bar{d}_3) - \frac{d_1 \bar{d}_3}{|\beta_2|^2} - \frac{c_1 \bar{c}_3}{|C|^2} - \frac{b_1 \bar{b}_3}{|F|^2} = \frac{2\lambda_1}{|\beta_2|^2} (c_3 \cdot \bar{d}_1) - \frac{\bar{d}_1 d_3}{|\beta_2|^2} - \frac{\bar{c}_1 c_3}{|C|^2} - \frac{\bar{b}_1 b_3}{|F|^2}, \quad (4.9)$$

$$\frac{2\lambda_1}{|\beta_2|^2} (c_2 \cdot \bar{d}_3) - \frac{d_2 \bar{d}_3}{|\beta_2|^2} - \frac{c_2 \bar{c}_3}{|C|^2} - \frac{b_2 \bar{b}_3}{|F|^2} = \frac{2\lambda_1}{|\beta_2|^2} (c_3 \cdot \bar{d}_2) - \frac{\bar{d}_2 d_3}{|\beta_2|^2} - \frac{\bar{c}_2 c_3}{|C|^2} - \frac{\bar{b}_2 b_3}{|F|^2}. \quad (4.10)$$

From (4.8) we know that $\lambda_1 (c_1 \cdot \bar{d}_2) - \lambda_1 (c_2 \cdot \bar{d}_1)$ is purely imaginary. By (4.6),

$$\lambda_1 \bar{\lambda}_2 (c_1 \bar{g}_2 - c_2 \bar{g}_1) \quad (4.11)$$

is also purely imaginary.

Since $[c_1, c_2, c_3]$ is a homogenous coordinate of C , we can take $[\sqrt{-1}c_1, \sqrt{-1}c_2, \sqrt{-1}c_3]$ instead of $[c_1, c_2, c_3]$ in (4.11). Then (4.11) is still purely imaginary. Because λ_1, λ_2 are non-identically-zero meromorphic functions on S^2 ,

$$c_1 \bar{g}_2 - c_2 \bar{g}_1 \equiv 0.$$

Similarly, from (4.9) and (4.10) we can get

$$c_1 \bar{g}_3 - c_3 \bar{g}_1 \equiv 0, \quad c_2 \bar{g}_3 - c_3 \bar{g}_2 \equiv 0.$$

Thus $G'(f) = [(g_1, g_2, g_3)] = [(\bar{c}_1, \bar{c}_2, \bar{c}_3)]$ is a constant map. Hence $f = G''G'(f) = 0$. This is a contradiction, therefore this case cannot induce a harmonic map $\psi : S^2 \rightarrow SU(3)/SO(3)$ via factorization through the Cartan embedding of $SU(3)/SO(3)$ into $SU(3)$.

4.3. Let $\phi : S^2 \rightarrow SU(3)$ be of case (b) of type (ii) given by Corollary 2.2, i.e. $\phi = -(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$, where $\beta_1 : S^2 \rightarrow G_2(\mathbb{C}^3)$ is a holomorphic map such that $\underline{\beta}_1 = \text{span}\{F, G'(F)\}$ where $F = (b_1(z), b_2(z), b_3(z))$ is a local lift of the full holomorphic map $f : S^2 \rightarrow \mathbb{C}P^2$ and $\underline{\beta}_2 = \text{span}\{\lambda_1 F + \lambda_2 G^{(2)}(F)\}$, where λ_1 and λ_2 are arbitrary non-identically-zero meromorphic functions. Using the same definition of $G'(F)$ as (4.5) and the formula

$$\pi_{\beta_1}(e_i) = \frac{\langle e_i, F \rangle}{|F|^2} F + \frac{\langle e_i, G'(F) \rangle}{|G'(F)|^2} G'(F), \quad i = 1, 2, 3, \quad (4.12)$$

we get the projective matrix

$$\pi_{\beta_1} = \frac{1}{|F|^2} \begin{pmatrix} |b_1|^2 & \bar{b}_2 b_1 & \bar{b}_3 b_1 \\ \bar{b}_1 b_2 & |b_2|^2 & \bar{b}_3 b_2 \\ \bar{b}_1 b_3 & \bar{b}_2 b_3 & |b_3|^2 \end{pmatrix} + \frac{1}{|G'(F)|^2} \begin{pmatrix} |g_1|^2 & \bar{g}_2 g_1 & \bar{g}_3 g_1 \\ \bar{g}_1 g_2 & |g_2|^2 & \bar{g}_3 g_2 \\ \bar{g}_1 g_3 & \bar{g}_2 g_3 & |g_3|^2 \end{pmatrix}. \quad (4.13)$$

By definition,

$$G^{(2)}(F) = G'(G'(F)) = \sum_{i=1}^3 h_i e_i, \quad (4.14)$$

where

$$h_i = \frac{\partial^2 b_i}{\partial z^2} - \frac{\langle \partial^2 F / \partial z^2, F \rangle}{|F|^2} b_i - \frac{\langle \partial^2 F / \partial z^2, G'(F) \rangle}{|G'(F)|^2} g_i, \quad i = 1, 2, 3. \quad (4.15)$$

From the formula

$$\pi_{\beta_2}(e_i) = \frac{\langle e_i, \lambda_1 F + \lambda_2 G^{(2)}(F) \rangle}{|\beta_2|^2} (\lambda_1 F + \lambda_2 G^{(2)}(F)), \quad i = 1, 2, 3, \quad (4.16)$$

where $|\beta_2|^2 = |\lambda_1 F + \lambda_2 G^{(2)}(F)|^2$, we obtain another projective matrix

$$\pi_{\beta_2} = \frac{1}{|\beta_2|^2} \begin{pmatrix} |l_1|^2 & \bar{l}_2 l_1 & \bar{l}_3 l_1 \\ \bar{l}_1 l_2 & |l_2|^2 & \bar{l}_3 l_2 \\ \bar{l}_1 l_3 & \bar{l}_2 l_3 & |l_3|^2 \end{pmatrix}, \quad (4.17)$$

where

$$l_i = \lambda_1 b_i + \lambda_2 h_i, \quad i = 1, 2, 3. \quad (4.18)$$

By direct computation, $\phi : S^2 \rightarrow SU(3)$ satisfies the condition $\phi = \phi^\dagger$ if and only if the following equations are satisfied:

$$\begin{aligned} & \frac{2\lambda_1}{|\beta_2|^2}(b_1 \cdot \bar{l}_2) - \frac{l_1 \bar{l}_2}{|\beta_2|^2} - \frac{b_1 \bar{b}_2}{|F|^2} - \frac{g_1 \bar{g}_2}{|G'(F)|^2} \\ &= \frac{2\lambda_1}{|\beta_2|^2}(b_2 \cdot \bar{l}_1) - \frac{\bar{l}_1 l_2}{|\beta_2|^2} - \frac{\bar{b}_1 b_2}{|F|^2} - \frac{\bar{g}_1 g_2}{|G'(F)|^2}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} & \frac{2\lambda_1}{|\beta_2|^2}(b_1 \cdot \bar{l}_3) - \frac{l_1 \bar{l}_3}{|\beta_2|^2} - \frac{b_1 \bar{b}_3}{|F|^2} - \frac{g_1 \bar{g}_3}{|G'(F)|^2} \\ &= \frac{2\lambda_1}{|\beta_2|^2}(b_3 \cdot \bar{l}_1) - \frac{\bar{l}_1 l_3}{|\beta_2|^2} - \frac{\bar{b}_1 b_3}{|F|^2} - \frac{\bar{g}_1 g_3}{|G'(F)|^2}, \end{aligned} \quad (4.20)$$

$$\begin{aligned} & \frac{2\lambda_1}{|\beta_2|^2}(b_2 \cdot \bar{l}_3) - \frac{l_2 \bar{l}_3}{|\beta_2|^2} - \frac{b_2 \bar{b}_3}{|F|^2} - \frac{g_2 \bar{g}_3}{|G'(F)|^2} \\ &= \frac{2\lambda_1}{|\beta_2|^2}(b_3 \cdot \bar{l}_2) - \frac{\bar{l}_2 l_3}{|\beta_2|^2} - \frac{\bar{b}_2 b_3}{|F|^2} - \frac{\bar{g}_2 g_3}{|G'(F)|^2}. \end{aligned} \quad (4.21)$$

From (4.19) we get

$$\frac{2\lambda_1}{|\beta_2|^2}(b_1 \bar{l}_2 - b_2 \bar{l}_1) = \frac{\bar{l}_2 l_1 - l_2 \bar{l}_1}{|\beta_2|^2} + \frac{\bar{b}_2 b_1 - b_2 \bar{b}_1}{|F|^2} + \frac{\bar{g}_2 g_1 - g_2 \bar{g}_1}{|G'(F)|^2}. \quad (4.22)$$

Substituting (4.18) into (4.22), we obtain

$$\begin{aligned} & \frac{\lambda_1 \bar{\lambda}_2}{|\beta_2|^2}(b_1 \bar{h}_2 - b_2 \bar{h}_1) + \frac{\bar{\lambda}_1 \lambda_2}{|\beta_2|^2}(\bar{b}_1 h_2 - \bar{b}_2 h_1) \\ &= \frac{|\lambda_1|^2}{|\beta_2|^2}(\bar{b}_1 b_2 - b_1 \bar{b}_2) + \frac{|\lambda_2|^2}{|\beta_2|^2}(\bar{h}_2 h_1 - h_2 \bar{h}_1) + \frac{\bar{g}_2 g_1 - g_2 \bar{g}_1}{|G'(F)|^2} + \frac{\bar{b}_2 b_1 - b_2 \bar{b}_1}{|F|^2}. \end{aligned} \quad (4.23)$$

The left-hand side of (4.23) is real but the right-hand side is purely imaginary, so

$$\frac{\lambda_1 \bar{\lambda}_2}{|\beta_2|^2}(b_1 \bar{h}_2 - b_2 \bar{h}_1) \in i\mathbb{R}. \quad (4.24)$$

Similarly, from (4.20) and (4.21) we can get

$$\frac{\lambda_1 \bar{\lambda}_2}{|\beta_2|^2}(b_1 \bar{h}_3 - b_3 \bar{h}_1) \in i\mathbb{R}, \quad (4.25)$$

$$\frac{\lambda_1 \bar{\lambda}_2}{|\beta_2|^2}(b_2 \bar{h}_3 - b_3 \bar{h}_2) \in i\mathbb{R}. \quad (4.26)$$

Set

$$\xi := \frac{\lambda_1 \bar{\lambda}_2}{|\beta_2|^2} (b_2 \bar{h}_3 - b_3 \bar{h}_2, b_3 \bar{h}_1 - b_1 \bar{h}_3, b_1 \bar{h}_2 - b_2 \bar{h}_1).$$

By direct computation we obtain that

$$\xi \perp G^{(2)}(F), \quad \bar{\xi} \perp F \quad (4.27)$$

with respect to the standard Hermitian inner product on $\mathbb{C}^3$. From (4.24)–(4.26) we have $\bar{\xi} = -\xi$, thus

$$\xi \perp F. \quad (4.28)$$

Since λ_1 and λ_2 are non-identically-zero functions on S^2 and $|\beta_2|^2$ is positive, we can define a map

$$[\xi] : S^2 \rightarrow \mathbb{R}P^2 \subset \mathbb{C}P^2.$$

On the other hand, $F, G'(F), G^{(2)}(F)$ form a Hermitian orthogonal basis in $\underline{\mathbb{C}}^3$. Comparing with (4.27) and (4.28) we obtain

$$G'(f) = [\xi] : S^2 \rightarrow \mathbb{R}P^2 \subset \mathbb{C}P^2.$$

Thus $f : S^2 \rightarrow \mathbb{C}P^2$ is a totally isotropic holomorphic map (cf. [8]). As a consequence, in this subcase $\phi : S^2 \rightarrow SU(3)$ factors through $SU(3)/SO(3)$ if and only if $f : S^2 \rightarrow \mathbb{C}P^2$ is a full totally isotropic holomorphic map.

4.4. Let $\phi : S^2 \rightarrow SU(3)$ be of type (iii) given by Corollary 2.2. It is obvious that ϕ factors through $SU(3)/SO(3)$ if and only if ϕ is the composition of conjugation $z \mapsto \bar{z}$ of $S^2 = \mathbb{C} \cup \{\infty\}$ and a map of case (b) of type (ii) obtained from a totally isotropic holomorphic map into $\mathbb{C}P^2$.

We now combine Corollary 2.2 and the above analysis to get our main result.

THEOREM 4.1. *Let $\psi : S^2 \rightarrow SU(3)/SO(3)$ be a smooth map. Then ψ is harmonic if and only if there exists a harmonic map $\phi : S^2 \rightarrow SU(3)$ such that $\psi = i^{-1} \circ \phi$, where i^{-1} is the inverse of the Cartan embedding of $SU(3)/SO(3)$ into $SU(3)$ given by Section 3 and ϕ is one of the following cases.*

- (1) The map ϕ is a Grassmannian solution obtained from a full harmonic map $f : S^2 \rightarrow \mathbb{R}P^2 \subset \mathbb{C}P^2$ (see Theorem 2.1).
- (2) The map ϕ is of the form $\phi = -\phi_0(\pi_{\beta_1} - \pi_{\beta_1}^\perp)(\pi_{\beta_2} - \pi_{\beta_2}^\perp)$, where $\phi_0 : S^2 \rightarrow SU(3)$ is a constant map, β_1 is a Frenet pair associated to a full totally isotropic holomorphic map $f : S^2 \rightarrow \mathbb{C}P^2$ and $\underline{\beta}_2$ is a rank-1 holomorphic subbundle of the mixed pair $\underline{f} \oplus G^{(2)}(f) = G'(f)^\perp(\beta_2 \neq f)$.

- (3) The map is the composition of conjugation $z \mapsto \bar{z}$ of $S^2 = \mathbb{C} \cup \{\infty\}$ and a map of type (2).

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REFERENCES

- [1] A. L. Besse. Einstein Manifolds (Ergebnisse der Mathematik und ihrer Grenzgebiete:3.Folge, Bd.10). Springer, World Publishing Corporation, Beijing, 1989.
- [2] F. E. Burstall, D. Ferus, F. Pedit and U. Pinkall. Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras. *Ann. Math.* **138** (1993), 173–212.
- [3] F. E. Burstall and M. A. Guest. Harmonic two-spheres in compact symmetric spaces, revisited. *Math. Ann.* **309** (1997), 541–572.
- [4] F. E. Burstall and J. H. Rawnsley. Twistor Theory for Riemannian Symmetric Spaces with Applications to Harmonic Maps of Riemann Surfaces (Lecture Notes in Mathematics, 1424). Springer, 1990.
- [5] F. E. Burstall and J. C. Wood. The construction of harmonic maps into complex Grassmannians. *J. Diff. Geom.* **23** (1986), 255–297.
- [6] J. Cheeger and D. Ebin. Comparison Theorems in Riemannian Geometry. North-Holland, Amsterdam, 1975.
- [7] S. Helgason. Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York, San Francisco, London, 1978.
- [8] J. Eells and J. C. Wood. Harmonic maps from surfaces into projective spaces. *Adv. Math.* **49** (1983), 217–263.
- [9] K. Uhlenbeck. Harmonic maps into Lie groups (classical solutions of the chiral model). *J. Diff. Geom.* **30** (1989), 1–50.
- [10] J. G. Wolfson. Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds. *J. Diff. Geom.* **27** (1988), 161–178.
- [11] J. C. Wood. The explicit construction and parameterization of all harmonic maps from the two-sphere to a complex Grassmannian. *J. Reine Angew. Math.* **386** (1988), 1–31.
- [12] J. C. Wood. Explicit construction and parameterization of harmonic two-spheres in the unitary group. *Proc. London Math. Soc.* **58**(3) (1989), 608–624.

Hui Ma
Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
People's Republic of China
(E-mail: hma@math.tsinghua.edu.cn)