A MINIMAL SURFACE OF $\mathbb{R}^3$ WITH PLANAR ENDS

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0. Introduction

In 1867, Riemann [Ri] constructed all minimal surfaces fibered by circles and straight lines in parallel planes, the plane, the catenoid, the helicoid and one-parameter family of minimal surfaces. We call the last one-parameter family of surfaces ‘Riemann’s minimal examples’, and denote them by $\{\mathcal{R}_t \mid t > 0\}$. Recall each $\mathcal{R}_t$ intersects horizontal planes in straight lines at integer heights, with the radius of circles going to infinity near the lines. By virtue of the Schwarz reflection principle for a minimal surface, this implies that $\mathcal{R}_t$ is invariant under a translation by $(t, 0, 2) \in \mathbb{R}^3$ for some $t > 0$. In fact, the quotient surface $\mathcal{R}_t/T$ becomes a properly embedded minimal torus in $\mathbb{R}^3/T$ with finite total curvature, where T is the translation group generated by $(t, 0, 2)$.

Recall, by Osserman’s theorem, that we can use the compact Riemann surface theory for the complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. Notice that many authors have obtained previous results about the uniqueness of Riemann’s minimal examples assuming periodicity, or equivalently, bounding by two straight lines in parallel planes, see [F1, FW, HKR, LRW, P2, T], etc. In particular, recently, Meeks *et al.* [MPR] proved that a properly periodic embedded minimal surface in $\mathbb{R}^3$ of genus zero must be one of Riemann’s minimal examples.

The goal of this paper is to prove the same result together with suitable conditions, without assuming periodicity of the surface itself.

THEOREM. *Let M be a properly embedded minimal surface in $\mathbb{R}^3$ of genus zero, which is the union of planar ends $\{E_n\}_{n \in \mathbb{N}}$ outside of a solid cylinder, at integer heights. Suppose that it is symmetric by the reflection in a plane, and there is a number $k \in \mathbb{Z}$ such that $\text{Torque}(E_{n+2k}) = \text{Torque}(E_n)$ for all $n \in \mathbb{Z}$. Then M must be one of Riemann’s minimal examples.*

In Section 2, we will show the theorem using the approximately periodic property of the set of planar ends. By a planar end we mean a properly embedded minimal

annulus of finite total curvature with compact boundary, which is close to the end of a plane, asymptotically. It is well known that such an end is conformally equivalent to a punctured disk in the plane. Recall that Kusner [K] found an expression for the torque vector associated to such an end.

Let E be a planar end of a minimal surface M in $\mathbb{R}^3$, which is close to a plane Π asymptotically. Additionally, if it has minimum branching order, that is, the Gauss map has the ramification order two at a puncture, then $M \cap \Pi$ is an embedding $\mathbb{R}^1$ outside a large ball. Moreover, $M \cap \Pi$ is asymptotically parallel to a straight line at infinity. In this case, the torque vector of E describes the direction of the line.

Back to Riemann's minimal examples, we know that each of them is symmetric by the reflection with respect to a vertical plane and with respect to a straight line that cuts those planes orthogonally permuting the ends, and hence all the torque vectors of planar ends are the same.

1. Preliminary

A minimal surface in $\mathbb{R}^3$ is a conformal harmonic immersion

$$X : S \hookrightarrow \mathbb{R}^3$$

where S is a two-dimensional smooth manifold, with or without boundary. Let us denote the corresponding holomorphic map by

$$(\phi_1, \phi_2, \phi_3) = 2 \frac{\partial X}{\partial z} : S \hookrightarrow \mathbb{C}^3.$$

Then from the Enneper–Weierstrass representation of a minimal surface we define X by the following multi-valued map,

$$\begin{aligned} X(z) &= \Re \int^z (\phi_1, \phi_2, \phi_3) d\zeta \\ &= \Re \int^z \left(\frac{1}{2} f(1 - g^2), \frac{i}{2} f(1 + g^2), fg \right) d\zeta, \end{aligned} \quad (1.1)$$

where f is a holomorphic function and g is a meromorphic function on S , such that when a pole of order m of g occurs, f has a zero of order $2m$, and this is the only case where f can vanish. Moreover, g is the stereographic projection of the Gauss map of X with respect to the north pole; here we just say that it is *the Gauss map* of X .

Let us call the pair $(g, f dz)$ *the Enneper–Weierstrass data* of X , or *the W-data* of X . In order that X is well defined, the primitive holomorphic map does not have

real periods, that is,

$$\Re \int_{\gamma} f(1 - g^2, i(1 + g^2), 2g) d\zeta = 0 \quad (1.2)$$

for every closed curve γ in S .

It is well known that, by Osserman [O], a complete embedded finite total curvature minimal annulus $A \subset \mathbb{R}^3$ with compact connected boundary must be conformally equivalent to a punctured disk, and its Gauss map extends in a meromorphic way through the puncture. Furthermore, Schoen proved in [S] that there is a compact subset $K \subset A$ such that $A \setminus K$ consists of r components $A_1, \dots, A_r$, i.e.

$$A \setminus K = A_1 \cup \dots \cup A_r$$

where each A_i is the graph of a function u_i with bounded slope over the exterior of a bounded region in some plane $\Pi_i \subset \mathbb{R}^3$. More precisely, if (x_1, x_2) is a coordinate in Π_i then u_i has the following asymptotic behavior:

$$u_i(x_1, x_2) = \beta + \alpha \log r + r^{-2}(\gamma_1 x_1 + \gamma_2 x_2) + O(r^{-2}) \quad (1.3)$$

for $r = (x_1^2 + x_2^2)^{1/2}$ sufficiently large, where β, α, γ_1 and γ_2 are real constants depending on i . We refer to the A_i as the ends of A . Notice that A_i is asymptotic to the end of a catenoid or a plane.

Definition. A complete embedded finite total curvature minimal annulus in $\mathbb{R}^3$ with compact connected boundary, given in (1.3), is called a *planar end* if $\alpha = 0$. Otherwise, we call it a *catenoid-type end*.

On the other hand, let

$$X : D_{\rho}^* \longrightarrow E \subset \mathbb{R}^3$$

be a minimal embedding of such an embedded end E , which has the *W-data* $(g, f dz)$, where $D_{\rho}^* := D_{\rho} \setminus \{0\}$ and $D_{\rho} = \{z \in \mathbb{C} \mid |z| < \rho\}$ for some $\rho > 0$. Then the Gauss map g extends to $z = 0$ meromorphically, and hence we have a well-defined normal vector at this point. It also defines a *limit tangent plane* at the end E . Consider the metric density Λ of X ,

$$\Lambda^2 = \left| \frac{\partial X}{\partial z} \right|^2 = \frac{1}{2} |f|^2 (1 + |g|^2)^2$$

using (1.1). Note that $\Lambda(z) \rightarrow \infty$ as $z \rightarrow 0$, since X is complete.

PROPOSITION 1. [F2] *If X is a minimal embedding of an embedded end of $\mathbb{R}^3$ which is complete and has finite total curvature, then the metric density Λ has order two at the pole, which means that*

$$\Lambda(z) \cong \frac{c}{|z|^2}$$

for some $c > 0$, and vice versa.

Now let E be a planar end, so that after a rotation we may assume that $g(0) = 0$. Let g have a zero of order $k > 0$ at $z = 0$. Since Λ has order two, f must have a pole of order two at $z = 0$, while f cannot have a residue at zero by (1.2). Additionally, the planar end E is asymptotic to a horizontal plane, and hence the third component $\phi_3 = f(z)g(z)$ of the holomorphic map in (1.1) must be bounded near $z = 0$. It follows that the order of g at $z = 0$ is greater than one. Therefore, we have the W -data $(g, f dz)$ of X at near $z = 0$ such that

$$\begin{aligned} g(z) &= z^k h(z), \quad k > 1 \\ f(z) &= \frac{a_{-2}}{z^2} + F(z) \end{aligned} \quad (1.4)$$

where both h and F are holomorphic functions on D_ρ , as well as $a_{-2} \neq 0$ and $h(0) \neq 0$.

Furthermore, if we denote $X = (X^1, X^2, X^3)$ then there are a large $R > 0$ and a small $\tilde{\rho} > 0$ such that

$$X^1 - iX^2 : D_{\tilde{\rho}}^* \longrightarrow \mathbb{C}$$

is onto the unbounded domain $\{\zeta \in \mathbb{C} \mid |\zeta| > R\}$, which is defined by

$$(X^1 - iX^2)(z) = \frac{-a_{-2}}{2z} + O(|z|), \quad a_{-2} \neq 0 \quad (1.5)$$

on the domain $D_{\tilde{\rho}}^*$.

Remark 1. If E is a planar end of order $k > 1$, given in (1.3), then

$$\Pi := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = \beta\}$$

is the limit tangent plane of E . Observe that then $E \cap \Pi$ consists of $2(k-1)$ curves which are asymptotic to $2(k-1)$ rays on Π making an equal angle by $\pi/(k-1)$. In particular, if E has minimum branching order, i.e. $k = 2$, then $E \cap \Pi$ is asymptotically parallel to segments of a line $\gamma_1 x_1 + \gamma_2 x_2 = 0$ in Π at infinity, by (1.3). Moreover, if $(g, f dz)$ is the W -data of E as (1.4), then it is easy to show that

$$\gamma_1 x_1 + \gamma_2 x_2 = \frac{1}{2} \Re(-a_{-2}^2 h(0)(x_1 + ix_2)). \quad (1.6)$$

Finally, we compute the flux and the torque of a planar end. From the physical point of view, minimal surfaces in $\mathbb{R}^3$ are objects submitted to a balanced force system, consisting of the forces associated to non-zero one-dimensional homology classes in the surfaces. More precisely, for each closed curve γ in a minimal surface of $\mathbb{R}^3$, we can define the flux and the torque of a closed curve γ of a minimal surface by

$$\begin{aligned}\text{Flux}(\gamma) &= \int_{\gamma} \nu \\ \text{Torque}_0(\gamma) &= \int_{\gamma} X \wedge \nu\end{aligned}$$

where X and ν are the position vector and the conormal of γ , respectively. The first equation provides a tendency of translation, or linear momentum, and the second denotes that of rotation, or angular momentum. These objects have been studied in depth by Kusner in [K], who found the balancing conditions, that is, the global flux and torque of a minimal surface in $\mathbb{R}^3$ are equal to zero.

In general, both the flux and torque are well defined in the homology class of $[\gamma]$, but the torque is dependent on the base point of the position vector X . If we move the base point from 0 to $W \in \mathbb{R}^3$, then the position vector based on W is $X - W$, and the torque is

$$\begin{aligned}\text{Torque}_W(\gamma) &= \text{Torque}_0(\gamma) - W \wedge \int_{\gamma} \nu \\ &= \text{Torque}_0(\gamma) - W \wedge \text{Flux}(\gamma).\end{aligned}$$

PROPOSITION 2. *We define the flux and the torque associated to a planar end E of order $k > 1$ as that of one representative curve for the end. If the W-data of the end is given in (1.4), then we can compute that*

$$\begin{aligned}\text{Flux}(E) &= (0, 0, 0) \\ \text{Torque}(E) &= \begin{cases} -\frac{i\pi}{2}(\overline{a_{-2}^2 h(0)}, 0), & \text{if } k = 2, \\ (0, 0, 0), & \text{if } k > 2. \end{cases}\end{aligned}$$

Proof. Observe that, using the notation in (1.3), we take a representative curve γ for E as the intersection of E and the vertical cylinder $\{x_1^2 + x_2^2 = R^2\}$ for a sufficiently large $R > 0$. Recall that the position vector and the conormal vector of γ are given by

$$\begin{aligned}\gamma(\theta) &= \left(R \cos \theta, R \sin \theta, \frac{1}{R}(\gamma_1 \cos \theta + \gamma_2 \sin \theta) + O(R^{-2}) \right) \\ \nu(\theta) &= (\cos \theta, \sin \theta, 0) + O(R^{-2}),\end{aligned}$$

where $0 \leq \theta \leq 2\pi$, respectively. Then we have

$$\text{Flux}(E) = \int_0^{2\pi} v(\theta) R d\theta = (0, 0, 0),$$

as $R \rightarrow \infty$. It follows that the torque of a planar end does not depend upon a base point. Also, as $R \rightarrow \infty$, we compute the torque

$$\text{Torque}(E) = \int_0^{2\pi} \gamma(\theta) \wedge v(\theta) R d\theta = \pi(-\gamma_2, \gamma_1, 0). \quad (1.7)$$

By (1.6) we complete the proof. $\square$

Note that if E has minimum branching order, then the torque of E is the direction of $E \cap \Pi$ at infinity.

2. Proof of the theorem

Recall the embedded minimal surface $M \subset \mathbb{R}^3$ has genus zero and two limit ends. Let us denote the horizontal plane $\Pi_t = \{x_3 = t\}$ where $t \notin \mathbb{Z}$, then $M \cap \Pi_t$ is a compact set by the hypothesis. By Lemma 23.2 in [F2], it implies that M cuts Π_t transversally. Now divide

$$M = M^+ \cup M^-$$

by this plane Π_t and let $M^+ := M \cap \{x_3 \geq t\}$. We know that (see [CKMR]): if a proper minimal surface with non-empty boundary is contained in a closed halfspace of $\mathbb{R}^3$, then it is parabolic as a Riemann surface. Thus M^+ is parabolic. Now attaching the ends to M^+ at infinity, then the resulting surface $\overline{M^+} \subset \mathbb{R}^3 \cup \{\infty\}$ is also parabolic. Additionally, $\overline{M^+}$ is a punctured closed disk $\overline{\mathbb{D}}^*$ topologically, where

$$\overline{\mathbb{D}}^* := \{z \in \mathbb{C} : 0 < z \leq 1\}.$$

Together with parabolicity, it means that $\overline{M^+}$ is also conformally equivalent to $\overline{\mathbb{D}}^*$. Moreover, then $\overline{M^+} \cup \overline{M^-}$ has the conformal type $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and hence

$$M \cong \mathbb{C}^* \setminus K$$

conformally, where K is the set of discrete punctures $p_n \in \mathbb{C}^*$, $n \in \mathbb{Z}$, corresponding to the planar ends E_n at integer height $x_3 = n$, respectively. Now we can define a conformal harmonic embedding of M by

$$X : \mathbb{C}^* \setminus K \hookrightarrow \mathbb{R}^3.$$

Let $X = (X^1, X^2, X^3)$. Then together with a suitable conformal change of the domain of X , we can say that

$$X^3(z) = \frac{1}{\log r} \log |z| \quad (2.1)$$

for some $r > 1$. Moreover, except at integer heights, M meets every horizontal plane in a compact Jordan curve, and hence X^3 can be continuously extended to the whole $\mathbb{C}^*$. In particular, since each E_n is close to the horizontal plane $\{x_3 = n\}$, we have $(1/\log r) \log |p_n| = n$ for all $p_n \in K$. Hence,

$$|p_n| = r^n, \quad n \in \mathbb{Z}. \quad (2.2)$$

Let $(g, f dz)$ be the W -data of X , then from (1.1) we have

$$f(z)g(z) := 2 \frac{\partial X^3}{\partial z} = \frac{d}{dz} \left(\frac{1}{\log r} \log z \right) = \frac{1}{\log r} \frac{1}{z} \quad (2.3)$$

for all $z \in \mathbb{C}^*$. Since both z and dz have no zeros or poles in $\mathbb{C}^*$, every planar end of M has the minimum branching order two.

ASSERTION 1. Let $\mathcal{P} \subset \mathbb{R}^3$ be the reflection symmetric plane of M , then we may assume that

$$\mathcal{P} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$$

up to a rotation around the x_3 -axis of $\mathbb{R}^3$.

Proof. First of all, $\mathcal{P}$ can only be horizontal or vertical, since all ends in M are horizontal. Recall that M cuts any horizontal plane in a compact curve except at integer heights. Since the flux of a planar end vanishes, the fluxes along these curves are the same, and we denote it by $\text{Flux}(M)$.

It is well known see [F2], that a flux is the imaginary period vector of a curve in the minimal surface, i.e.

$$\text{Flux}(\gamma) := \Im \int_{\gamma} \frac{1}{2} f(1 - g^2, i(1 + g^2), 2g) dz$$

where $(g, f dz)$ is the W -data of X . From the period condition (1.2), we can say that

$$\text{Flux}(\gamma) := -i \int_{\gamma} \frac{1}{2} f(1 - g^2, i(1 + g^2), 2g) dz. \quad (2.4)$$

By (1.1) and (2.1), the third primitive coordinate of X has non-zero residue at $z = 0$, and hence $\text{Flux}(M)$ is not a horizontal vector.

Moreover, $\text{Flux}(M)$ cannot be vertical, see the proofs of Lemma 5 and Assertion 1 in [PR]. Consequently, the reflection symmetric plane cannot be horizontal, and so $\mathcal{P}$ must be vertical. $\square$

ASSERTION 2. All the punctures $p_n \in K$ are lying on a line $\ell \subset \mathbb{C}$ containing $z = 0$.

Proof. Let $P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the reflection under the plane $\mathcal{P} = \{x_2 = 0\}$, then it produces an antiholomorphic involution B of $\mathbb{C}^*$ such that

$$X(B(z)) = P(X(z)) := (X^1(z), -X^2(z), X^3(z)).$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^* & \xrightarrow{B} & \mathbb{C}^* \\ \downarrow X & & \downarrow X \\ \mathbb{R}^3 \cup \{\infty\} & \xrightarrow{P} & \mathbb{R}^3 \cup \{\infty\} \end{array}$$

We know that $X^3(z) = (1/\log r) \log |z|$ on $\mathbb{C}^*$, and hence

$$\log |B(z)| = \log |z|, \quad \text{for all } z \in \mathbb{C}^*$$

in the above diagram. It follows that

$$B(z) = e^{i2\theta} \bar{z}$$

for some $0 \leq \theta \leq 2\pi$. In other words, B is the reflection along the line ℓ ,

$$\ell := \{z \in \mathbb{C} \mid \arg z = \theta \text{ or } \theta + \pi\} \cup \{0\}.$$

It is clear that each puncture p_n must be fixed by the involution B , and hence $p_n \in \ell$ for all $n \in \mathbb{Z}$. Furthermore,

$$X^{-1}(M \cap \mathcal{P}) = \ell \setminus \{0, p_n \mid n \in \mathbb{Z}\}. \quad \square$$

Now we will show that

$$\arg p_{n+2k} = \arg p_n$$

for all $n \in \mathbb{Z}$. First, denote by $\Pi_n := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = n\}$ the limit tangent plane of E_n . Since E_n has the minimum branching order, $M \cap \Pi_n$ is an embedding of $\mathbb{R}^1$. Additionally, from the fact that $P(M) = M$, it is asymptotically parallel to the x_2 -axis of $\mathbb{R}^3$. More precisely, see (1.3), E_n is the graph of a function u_n which has asymptotic behavior,

$$u_n(x_1, x_2) = n + \frac{\alpha_n x_1}{r^2} + O(r^{-2}) \quad (2.5)$$

for $r = (x_1^2 + x_2^2)^{1/2}$ sufficiently large, where α_n is a real constant and (x_1, x_2) is a coordinate of Π_n . We then have from (1.7),

$$\text{Torque}(E_n) = \pi(0, \alpha_n, 0).$$

Moreover, embeddedness ensures that consecutive ends of M have reversed vertical limit normal vectors. In particular,

$$g(p_n) = g(p_{n+2k})$$

and hence we can take the same coordinates for Π_n and Π_{n+2k} after a parallel translation along the horizontal plane. Recall that both E_n and E_{n+2k} have the same torque, and so

$$\alpha_n = \alpha_{n+2k}. \quad (2.6)$$

Observe that $E_n \cap \mathcal{P}$ consists of two divergent curves Γ_n^+ and Γ_n^- ,

$$\Gamma_n^+ := E_n \cap \mathcal{P} \cap \{x_3 > n\}, \quad \Gamma_n^- := E_n \cap \mathcal{P} \cap \{x_3 < n\}$$

each of which lies in the different component of $\mathcal{P} \setminus \{x_1 = 0\}$ outside of a compact subset. We say that the planar end E_n has *the right position* if Γ_n^+ is contained in the right component of $\mathcal{P} \setminus \{x_1 > 0\}$ outside of a compact subset, and in *the left position* otherwise.

ASSERTION 3. *For all $n \in \mathbb{Z}$, the planar ends E_n and E_{n+2k} have the same position by (2.5) and (2.6).*

ASSERTION 4. *If $\arg p_n = \arg p_{n+1}$, then E_n and E_{n+1} have different positions, and vice versa. In other words, there is a position change of ends if the consecutive punctures have the same arguments.*

Proof. We may assume that $\arg p_n = \theta$. Define the segment $\gamma \subset \ell$ connecting p_n and p_{n+1} , and $-\gamma$ by

$$\begin{aligned} \gamma &:= \{z \in \ell \mid \arg z = \theta, |p_n| < |z| < |p_{n+1}|\} \\ -\gamma &:= \{z \in \ell \mid -z \in \gamma\}. \end{aligned}$$

Since $X(\gamma), X(-\gamma) \subset \mathcal{P}$ and $X(\gamma) \cap X(-\gamma) = \emptyset$, we can say that $X(\gamma)$ is contained in a component of $\mathcal{P} \setminus \{x_1 = 0\}$ outside of a compact subset. See Figure 1.

Moreover, notice that Γ_n^+ and Γ_{n+1}^- are contained in $X(\gamma)$. It follows that they both have the same position, and hence Γ_n^+ and Γ_{n+1}^+ have different positions. $\square$

FIGURE 1.

FIGURE 2. $\arg p_n = \arg p_{n+2}$.

Using this result, we can prove the following assertion.

ASSERTION 5. *For all $i = 1, 2, \dots$, the following argument holds.*

If $\arg p_n = \arg p_{n+2i}$, then there are an even number of position changes in the set of ends $\{E_n, E_{n+1}, \dots, E_{n+2i}\}$, and vice versa. Or equivalently, E_n and E_{n+2i} have the same position.

Proof. We use the induction over $i = 1, 2, \dots$

First, from Figure 2 we have two or zero position changes among the three ends $\{E_n, E_{n+1}, E_{n+2}\}$ if $\arg p_n = \arg p_{n+2}$, while there is only one position change among these ends otherwise, see Figure 3. This implies that the argument is true for $i = 1$.

FIGURE 3. $\arg p_n \neq \arg p_{n+2}$.

FIGURE 4. $\arg p_n = \arg p_{n+2(i+1)} = \arg p_{n+2i}$.

Next, we assume that both $\mathcal{E}1$ and $\mathcal{E}2$ are true, where:

- $\mathcal{E}1$: if $\arg p_n = \arg p_{n+2i}$, then there are $2m_i$ position changes in the set of ends $E(i) := \{E_n, E_{n+1}, \dots, E_{n+2i}\}$ for some $m_i = 0, 1, \dots$;
- $\mathcal{E}2$: if $\arg p_n \neq \arg p_{n+2i}$, then we have $(2\tilde{m}_i + 1)$ position changes in $E(i)$ for some $\tilde{m}_i = 0, 1, \dots$.

Now consider the four cases of $i + 1$.

Case 1. Let $\arg p_n = \arg p_{n+2(i+1)}$ and $\arg p_n = \arg p_{n+2i}$. Then from the argument $\mathcal{E}1$ and Figure 4, we can say that there are $(2m_i) + 2$ or $2m_i$ position changes in $E(i + 1)$.

Case 2. Let $\arg p_n = \arg p_{n+2(i+1)}$ and $\arg p_n \neq \arg p_{n+2i}$. Then we have $(2\tilde{m}_i + 1) + 1$ position changes in $E(i + 1)$, see Figure 5 and $\mathcal{E}2$.

FIGURE 5. $\arg p_n = \arg p_{n+2(i+1)} \neq \arg p_{n+2i}$. FIGURE 6. $\arg p_n \neq \arg p_{n+2(i+1)} = \arg p_{n+2i}$.

Case 3. Let $\arg p_n \neq \arg p_{n+2(i+1)}$ and $\arg p_n \neq \arg p_{n+2i}$. Then there are $(2\tilde{m}_i + 1)$ or $(2\tilde{m}_i + 1) + 2$ position changes in $E(i + 1)$, see Figure 6 and $\mathcal{E}2$.

Case 4. Let $\arg p_n \neq \arg p_{n+2(i+1)}$ and $\arg p_n = \arg p_{n+2i}$. Then there are $(2m_i) + 1$ position changes in $E(i + 1)$, see Figure 7 and $\mathcal{E}1$.

Therefore for $i + 1$, this argument also holds, and hence we have proved the assertion. $\square$

Recall that both E_n and E_{n+2k} have the same positions, so

$$\arg p_n = \arg p_{n+2k}$$

FIGURE 7. $\arg p_n \neq \arg p_{n+2(i+1)} \neq \arg p_{n+2i}$.

for all $n \in \mathbb{Z}$. Therefore, together with (2.2) and Assertion 2, we have the set of punctures by

$$K = \{p_n \in \mathbb{C}^* \mid \arg p_n = \theta \text{ or } \theta + \pi, |p_n| = r^n, \\ \arg p_n = \arg p_{n+2k} \text{ for all } n \in \mathbb{Z}, \text{ respectively}\}.$$

Observe that if we take $\zeta : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\zeta(z) = r^{2k}z,$$

then it is a conformal mapping of $\mathbb{C}^* \setminus K$ onto itself.

Let $C_R \subset \mathbb{R}^3$ be a solid cylinder such that M is the union of planar ends in the complement of C_R , i.e.

$$M \setminus C_R \subset \bigcup_{n \in \mathbb{Z}} E_n$$

where the axis of C_R has the direction vector $(s, 0, k) \in \mathbb{R}^3$ for a given $k \in \mathbb{Z}$ and some $s > 0$, and sufficiently large radius $R > 0$. Now let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the translation by the vector $2(s, 0, k)$, i.e.

$$T(x_1, x_2, x_3) = (x_1 + 2s, x_2, x_3 + 2k)$$

and denote by $\tilde{M}$ the translation of M along T , that is,

$$\tilde{M} := T(M) = \{\mathbf{x} + 2(s, 0, k) \mid \mathbf{x} \in M\}.$$

Then, with the above coordinate ζ , we have a minimal embedding

$$\tilde{X} : \mathbb{C}^* \setminus K \longrightarrow \mathbb{R}^3$$

of $\tilde{M}$ such that

$$\tilde{X} \circ \zeta = T \circ X.$$

Consider the commutative diagram,

$$\begin{array}{ccc} \mathbb{C}^* \setminus K & \xrightarrow{\zeta} & \mathbb{C}^* \setminus K \\ \downarrow X & & \downarrow \tilde{X} \\ M & \xrightarrow{T} & \tilde{M} \end{array}$$

Then, we have

$$\begin{aligned} \tilde{X}^3(\zeta) &= X^3(z) + 2k = \frac{1}{\log r} \log |z| + 2k \\ &= \frac{1}{\log r} \log |r^{2k} z| = \frac{1}{\log r} \log |\zeta|. \end{aligned}$$

Moreover, it extends to the whole $\mathbb{C}^*$ continuously. Therefore,

$$X^3 \equiv \tilde{X}^3 \quad \text{on } \mathbb{C}^*.$$

Now let $(g, f dz)$ and $(\tilde{g}, \tilde{f} dz)$ be the Weierstrass data of X and $\tilde{X}$, respectively. Then, similar to (2.3), we have

$$f(z)g(z) = \tilde{f}(z)\tilde{g}(z) = 2 \frac{\partial \tilde{X}^3}{\partial z} = \frac{1}{\log r} \frac{1}{z}$$

for all $z \in \mathbb{C}^*$. Recall that every planar end of M and $\tilde{M}$ has minimum branching order two. Therefore, from (1.4) we may assume that

$$g(z) = (z - p_n)^2 h(z), \quad \tilde{g}(z) = (z - p_n)^2 \tilde{h}(z)$$

on a sufficiently small neighborhood $D_n \subset \mathbb{C}^*$ of p_n , where $p_m \notin D_n$ if $m \neq n$, and h and $\tilde{h}$ are the holomorphic functions on D_n with $h(p_n) \neq 0$ and $\tilde{h}(p_n) \neq 0$. Therefore,

$$\begin{aligned} f(z) &= \frac{1}{\log r} \frac{1}{zg(z)} = \frac{1}{\log r} \frac{1}{p_n h(p_n)} \frac{1}{(z - p_n)^2} + F(z) \\ \tilde{f}(z) &= \frac{1}{\log r} \frac{1}{p_n \tilde{h}(p_n)} \frac{1}{(z - p_n)^2} + \tilde{F}(z) \end{aligned}$$

where F and $\tilde{F}$ are holomorphic functions on D_n , and hence from (1.5),

$$\begin{aligned}(X^1 - iX^2)(z) &= \frac{-1}{2 \log r} \frac{1}{p_n h(p_n)} \frac{1}{z - p_n} + O(|z - p_n|) \\ (\tilde{X}^1 - i\tilde{X}^2)(z) &= \frac{-1}{2 \log r} \frac{1}{p_n \tilde{h}(p_n)} \frac{1}{z - p_n} + O(|z - p_n|).\end{aligned}$$

Together with the result of Proposition 2, we then have

$$\begin{aligned}\text{Torque}(E_n) &= \frac{-i\pi}{2(\log r)^2} ((\overline{p_n^2 h(p_n)})^{-1}, 0) \\ \text{Torque}(\tilde{E}_n) &= \frac{-i\pi}{2(\log r)^2} ((\overline{p_n^2 \tilde{h}(p_n)})^{-1}, 0)\end{aligned}$$

where $\tilde{E}_n$ is the end of $\tilde{M}$ at the height $x_3 = n$. Notice that $\tilde{E}_n$ is the only translation of $E_{n-2k} \subset M$, and hence

$$\text{Torque}(\tilde{E}_n) = \text{Torque}(E_{n-2k}) = \text{Torque}(E_n)$$

by the hypothesis. Thus

$$\tilde{h}(p_n) = h(p_n).$$

This leads us to

$$(X^1 - iX^2)(z) - (\tilde{X}^1 - i\tilde{X}^2)(z) = O(|z - p_n|)$$

on the small neighborhood of p_n . From the fact that $X^3 \equiv \tilde{X}^3$, this shows that each $p_n \in K$ is the removable singularity of $X - \tilde{X}$. Hence, we can obtain the extended harmonic map $Y : \mathbb{C}^* \hookrightarrow \mathbb{R}^3$ such that

$$Y \equiv \begin{cases} X - \tilde{X}, & \text{on } \mathbb{C}^* \setminus K, \\ 0, & \text{on } K. \end{cases} \quad (2.7)$$

We will now show that Y is a bounded harmonic map on $\mathbb{C}^*$. First, we take a connected neighborhood $U_n \subset \mathbb{C}^*$ of p_n , such that

$$\begin{cases} X^{-1}(E_n) \subset U_n, & \tilde{X}^{-1}(\tilde{E}_n) \subset U_n \\ X^{-1}(E_m) \cap U_n = \phi, & \tilde{X}^{-1}(\tilde{E}_m) \cap U_n = \phi, \quad \text{if } m \neq n. \end{cases}$$

Recall that, in the complement of the solid cylinder C_R both M and $\tilde{M}$ are contained in the union of their planar ends. Thus

$$X(\partial U_n) \subset C_R \quad \text{and} \quad \tilde{X}(\partial U_n) \subset C_R$$

FIGURE 8.

since $X(\partial U_n)$ and $\tilde{X}(\partial U_n)$ are disjoint from the planar ends E_m and $\tilde{E}_m$ for all $m \in \mathbb{Z}$, respectively. See Figure 8.

Moreover, since $\tilde{X}^3 \equiv X^3$, we have

$$\|Y(z)\| = \|(X^1 - iX^2)(z) - (\tilde{X}^1 - i\tilde{X}^2)(z)\| \leq 2\tilde{R} \quad \text{on } \partial U_n$$

where $\tilde{R} := R\sqrt{1 + (s/k)^2}$ and $(s, 0, k)$ is the direction of C_R . By the maximum principle of the harmonic map, this implies that

$$\|Y\| \leq 2\tilde{R} \quad \text{on } U_n \quad (2.8)$$

for all $n \in \mathbb{Z}$.

On the other hand, by the definition of U_n , it is clear that

$$\begin{cases} (\bigcup_{n \in \mathbb{Z}} E_n) \cap X(\mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} U_n) = \emptyset \\ (\bigcup_{n \in \mathbb{Z}} \tilde{E}_n) \cap \tilde{X}(\mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} U_n) = \emptyset \end{cases}$$

from which it follows that both $X(\mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} U_n)$ and $\tilde{X}(\mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} U_n)$ are contained in the solid cylinder C_R . See Figure 8 again. Therefore, we have that

$$\|Y\| \leq 2\tilde{R} \quad \text{on } \mathbb{C}^* \setminus \bigcup_{n \in \mathbb{Z}} U_n. \quad (2.9)$$

Together with (2.8) and (2.9), Y is a bounded harmonic map on the punctured plane $\mathbb{C}^*$. By virtue of Liouville's theorem, this implies that Y is a constant map. In fact,

$Y \equiv 0$ from (2.7), and hence

$$M = \tilde{M}.$$

This means that M is periodic under the translation. However, we know the result of [MPR]: a properly periodic embedded minimal annulus in $\mathbb{R}^3$ must be one of Riemann's minimal examples. Thus we have proved the theorem. $\square$

Remark 2. This result holds even when the limit tangent planes have periodic heights rather than integer heights. The proofs are essentially the same as the proof of the theorem. More precisely, if M is a properly embedded minimal surface in $\mathbb{R}^3$ of genus zero such that:

- M is symmetric by the reflection in a plane,
- in the complement of a solid cylinder, M is the union of planar ends $\{E_n\}_{n \in \mathbb{N}}$ close to the horizontal planes $\{x_3 = t_n\}$, $t_n < t_{n+1}$, respectively, and
- there is a number $k \in \mathbb{Z}$ and a constant $s > 0$, such that for all $n \in \mathbb{Z}$,

$$\begin{aligned} t_{n+2k} &= t_n + s \\ \text{Torque}(E_{n+2k}) &= \text{Torque}(E_n), \end{aligned}$$

then M must be one of Riemann's minimal examples.

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