SOME CLASSICAL AND MATRIX TODA BRACKETS IN THE 13- AND 15-STEMS

Howard J. MARCUM and Nobuyuki ODA

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0. Introduction and results

Matrix Toda brackets have proven useful to settle extension problems when computing homotopy groups of a homotopy pullback (see [1]) and, in the stable range, when computing homotopy groups of Stiefel manifolds (see [7]). In a categorical setting they have been developed extensively in [2]. However, very few explicit computations of matrix Toda brackets seem to have been made.

In a recent paper [3], Hardie, Marcum and Oda have developed a theory of box brackets. A feature of that theory is to enable understanding of relations among various types of brackets, including matrix Toda brackets and classical Toda brackets. In the present paper we rely upon two such relations from [3] and make explicit computations of some matrix Toda brackets in the homotopy groups of spheres. In particular, we see that important generators of the homotopy groups of spheres have representations as matrix Toda brackets. Also several new Toda brackets are determined. The examples of Toda brackets and matrix Toda brackets that we have computed mainly center around the element $\bar{\varepsilon}_n$ ($n \geq 3$) which is defined by Toda in [10] and is an element in the 15-stem of the homotopy groups of spheres. To carry out the computations it is also necessary to consider some Toda brackets in the 13-stem. The results and new relations obtained are summarized by individual section as follows.

In Section 1 we give in Theorem 1 some Toda brackets in the 13-stem and then, using these results, determine some Toda brackets concerning the element $\bar{\varepsilon}_n$.

THEOREM 1.

- (1) $\{\varepsilon_5, \nu_{13}, \eta_{16}\} = \nu_5 \circ \sigma_8 \circ \nu_{15} + \{\nu_5 \circ \eta_8 \circ \mu_9\}$ in π_{18}^5 .
- (2) $\{\varepsilon_9, \nu_{17}, \eta_{20}\} = 0$ in π_{22}^9 ; $\{\varepsilon_{10}, \nu_{18}, \eta_{21}\} = 0$ in π_{23}^{10} .

THEOREM 2. *In the stable range we have the following relations in the 15-stem.*

- (1) $\langle \nu, \varepsilon, \nu \rangle = \langle \varepsilon, \nu, 2\nu \rangle = \langle \varepsilon, 2\nu, \nu \rangle = \langle \varepsilon, 2\iota, \nu^2 \rangle = \bar{\varepsilon}$.
- (2) $\langle \nu, \bar{\nu}, \nu \rangle = \langle \bar{\nu}, \nu, 2\nu \rangle = \langle \bar{\nu}, 2\nu, \nu \rangle = \langle \bar{\nu}, 2\iota, \nu^2 \rangle = \bar{\varepsilon}$.

THEOREM 4. $\{\bar{\nu}_7, 2\nu_{15}, \nu_{18}\} = \{\nu_7, \varepsilon_{10}, \nu_{18}\} = \bar{\varepsilon}_7 + \sigma' \circ \bar{\nu}_{14}$ in π_{22}^7 .

In Section 2 we apply the results in Section 1 to settle the following Toda brackets in the 13-stem of the homotopy groups of spheres.

THEOREM 12. $\{\eta_6, \nu_7, \varepsilon_{10}\} = \{\eta_6, \nu_7, \bar{\nu}_{10}\} = \nu_6 \circ \sigma_9 \circ \nu_{16}$ in π_{19}^6 .

THEOREM 13.

(1) $\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\} = \nu_5 \circ \sigma_8 \circ \nu_{15}$ in π_{18}^5 .

(2) $\{\nu_6^2, 2\iota_{12}, \nu_{12}^2\} = \nu_6 \circ \sigma_9 \circ \nu_{16}$ in π_{19}^6 .

In Section 3 we determine several matrix Toda brackets and more relations of elements in the homotopy groups of spheres. The results in the previous sections are used effectively. Additionally, the following formulae from Corollary 4.8 of [3], which are obtained as special cases of formulae for box brackets in Theorem 4.4 of [3], play a crucial role in determining the matrix Toda brackets in this section:

$$\left\{ \begin{smallmatrix} d \\ s \end{smallmatrix}, \begin{smallmatrix} r \\ b \end{smallmatrix}, g \right\} \circ Ew \subset -d \circ \{r, g, w\} + s \circ \{b, g, w\}$$

and

$$s \circ \left\{ a, \begin{smallmatrix} f \\ u \end{smallmatrix}, \begin{smallmatrix} w \\ h \end{smallmatrix} \right\} \subset \{s, a, f\} \circ Ew - \{s, a, u\} \circ Eh.$$

These formulae are valid when the matrix Toda brackets and Toda brackets involved are all well-defined. They enable us to determine matrix Toda brackets by relating them to ordinary Toda brackets.

Specifically the matrix Toda brackets determined in Section 3 are as follows.

THEOREM 15. $\left\{ \begin{smallmatrix} \nu_6^2 \\ \eta_6 \end{smallmatrix}, \begin{smallmatrix} \nu_{12} \\ \bar{\nu}_7 \end{smallmatrix}, 2\nu_{15} \right\} = \nu_6 \circ \sigma_9 \circ \nu_{16}$ in π_{19}^6 .

THEOREM 16. $\left\{ \begin{smallmatrix} \varepsilon_4 \\ E\nu' \end{smallmatrix}, \begin{smallmatrix} \nu_{12} \\ \bar{\nu}_7 \end{smallmatrix}, 2\nu_{15} \right\} = \bar{\varepsilon}_4$ in π_{19}^4 .

THEOREM 21. $\left\{ \begin{smallmatrix} \bar{\nu}_6 \\ \nu_6 \end{smallmatrix}, \begin{smallmatrix} 2\nu_{14} \\ \varepsilon_9 \end{smallmatrix}, \nu_{17} \right\} = 0$ in π_{21}^6 .

THEOREM 22. $\left\{ \begin{smallmatrix} \nu_6 \circ \sigma_9 \\ \nu_6 \end{smallmatrix}, \begin{smallmatrix} \eta_{16} \\ \varepsilon_9 \end{smallmatrix}, \nu_{17} \right\} = \bar{\varepsilon}_6$ in π_{21}^6 .

By Corollary 4.6 of [3] there is a relation

$$s \circ \left\{ \begin{smallmatrix} b \\ a \end{smallmatrix}, \begin{smallmatrix} g \\ f \end{smallmatrix}, w \right\} = \left\{ s, \begin{smallmatrix} b \\ a \end{smallmatrix}, \begin{smallmatrix} g \\ f \end{smallmatrix} \right\} \circ Ew$$

which is valid whenever the matrix Toda brackets involved are well-defined. By means of this formula we are able to deduce Theorem 23 below from Theorem 22.

THEOREM 23. $\left\{ \begin{smallmatrix} \nu' & \nu_6 \circ \sigma_9 & \eta_{16} \\ & \nu_6 & \varepsilon_9 \end{smallmatrix} \right\} = \bar{\varepsilon}_3 \text{ in } \pi_{18}^3.$

To the best of our knowledge, the result of Theorem 23 is the first post-composition matrix Toda bracket ever computed in the literature. As a general remark we feel methods involving matrix Toda brackets have much to offer in future study of homotopy groups, for the representation of generators by matrix Toda brackets makes accessible the use of properties of matrix Toda brackets in homotopy exact sequences involving homotopy groups of various complexes.

1. Toda brackets in the 15-stem

Our first result determines some Toda brackets in the 13-stem of the homotopy groups of spheres.

THEOREM 1.

(1) $\{\varepsilon_5, \nu_{13}, \eta_{16}\} = \nu_5 \circ \sigma_8 \circ \nu_{15} + \{\nu_5 \circ \eta_8 \circ \mu_9\} \text{ in } \pi_{18}^5.$

(2) $\{\varepsilon_9, \nu_{17}, \eta_{20}\} = 0 \text{ in } \pi_{22}^9; \{\varepsilon_{10}, \nu_{18}, \eta_{21}\} = 0 \text{ in } \pi_{23}^{10}.$

Proof. (1) The indeterminacies of the Toda brackets $\{\varepsilon_5, \nu_{13}, \eta_{16}\}$ and $\{\varepsilon_5, \nu_{13}, \eta_{16}\}_1$ are equal, since

$$\begin{aligned} \varepsilon_5 \circ \pi_{18}^{13} + \pi_{17}^5 \circ \eta_{17} &= \varepsilon_5 \circ E\pi_{17}^{12} + \pi_{17}^5 \circ \eta_{17} \\ &= \{\nu_5^4 \circ \eta_{17}, \nu_5 \circ \mu_8 \circ \eta_{17}, \nu_5 \circ \eta_8 \circ \varepsilon_9 \circ \eta_{17}\} \\ &= \{\nu_5 \circ \eta_8 \circ \mu_9\} \end{aligned}$$

by Proposition 5.9 of [10] and Theorem 7.6 of [10]. We note that $\nu_{14} \circ \eta_{17} = 0$ by (5.9) of [10], $\mu_8 \circ \eta_{17} = \eta_8 \circ \mu_9$ by Proposition 3.1 of [10] and $\nu_5 \circ \eta_8 \circ \varepsilon_9 \circ \eta_{17} = \nu_5 \circ \eta_8^2 \circ \varepsilon_{10} = \nu_5 \circ 4\nu_8 \circ \sigma_{11} = 4\nu_5^2 \circ \sigma_{11} = 0$ by (7.10) of [10].

Then we have

$$H\{\varepsilon_5, \nu_{13}, \eta_{16}\} = H\{\varepsilon_5, \nu_{13}, \eta_{16}\}_1 = -\Delta^{-1}(\varepsilon_4 \circ \nu_{12}) \circ \eta_{17} = \bar{\nu}_9 \circ \eta_{17} = \nu_9^3$$

by Proposition 2.6 of [10], (7.13) of [10] and Lemma 6.3 of [10].

On the other hand, $H(\nu_5 \circ \sigma_8 \circ \nu_{15}) = \nu_9^3$ by the result on p. 77 of [10]. Since the group structure of π_{18}^5 is

$$\pi_{18}^5 = Z_2\{\nu_5 \circ \sigma_8 \circ \nu_{15}\} \oplus Z_2\{\nu_5 \circ \eta_8 \circ \mu_9\}$$

by Theorem 7.7 of [10] and $H(v_5 \circ \eta_8 \circ \mu_9) = H(E(v_4 \circ \eta_7 \circ \mu_8)) = 0$, we see that

$$\{\varepsilon_5, v_{13}, \eta_{16}\} \ni v_5 \circ \sigma_8 \circ v_{15} + x v_5 \circ \eta_8 \circ \mu_9$$

for some integer x . Since $\{v_5 \circ \eta_8 \circ \mu_9\}$ is the indeterminacy of the Toda bracket $\{\varepsilon_5, v_{13}, \eta_{16}\}$, we may conclude that

$$\{\varepsilon_5, v_{13}, \eta_{16}\} = v_5 \circ \sigma_8 \circ v_{15} + \{v_5 \circ \eta_8 \circ \mu_9\}.$$

(2) Now $\{\varepsilon_9, v_{17}, \eta_{20}\} \supset E^4\{\varepsilon_5, v_{13}, \eta_{16}\} \ni v_9 \circ \sigma_{12} \circ v_{19} + x v_9 \circ \eta_{12} \circ \mu_{13} = 0$ by (7.20) and (5.9) of [10]. Hence we also have $\{\varepsilon_{10}, v_{18}, \eta_{21}\} \ni 0$. The indeterminacies of these Toda brackets $\{\varepsilon_9, v_{17}, \eta_{20}\}$ and $\{\varepsilon_{10}, v_{18}, \eta_{21}\}$ are respectively

$$\varepsilon_9 \circ \pi_{22}^{17} + \pi_{21}^9 \circ \eta_{21} = 0$$

and

$$\varepsilon_{10} \circ \pi_{23}^{18} + \pi_{22}^{10} \circ \eta_{22} = \{\Delta(v_{21}) \circ \eta_{22}\} = \{\Delta(v_{21} \circ \eta_{24})\} = 0$$

by Proposition 5.9 of [10], Theorem 7.6 of [10] and (5.9) of [10]. Therefore we have $\{\varepsilon_9, v_{17}, \eta_{20}\} = 0$ and $\{\varepsilon_{10}, v_{18}, \eta_{21}\} = 0$. $\square$

THEOREM 2. *In the stable range we have the following relations in the 15-stem.*

- (1) $\langle v, \varepsilon, v \rangle = \langle \varepsilon, v, 2v \rangle = \langle \varepsilon, 2v, v \rangle = \langle \varepsilon, 2\iota, v^2 \rangle = \bar{\varepsilon}$.
- (2) $\langle v, \bar{v}, v \rangle = \langle \bar{v}, v, 2v \rangle = \langle \bar{v}, 2v, v \rangle = \langle \bar{v}, 2\iota, v^2 \rangle = \bar{\varepsilon}$.

Proof. Note that $\bar{\varepsilon}_3 \in \{\varepsilon_3, 2\iota_{11}, v_{11}^2\}_6$ by the definition of $\bar{\varepsilon}_3$ on p. 97 of [10]. So in the stable range $\bar{\varepsilon} \in \langle \varepsilon, 2\iota, v^2 \rangle$. Now the indeterminacies of the eight Toda brackets in the statement above are trivial, since $v \circ (G_{12}; 2) = \varepsilon \circ (G_7; 2) = \bar{v} \circ (G_7; 2) = 0$ (cf. Theorem 14.1 of [10]), where $(G_k; 2)$ is the 2-component of the k th stable homotopy group of spheres. By (3.8) and (3.5) of [10], we have

$$\begin{aligned} \langle \varepsilon, 2\iota, v^2 \rangle + \langle \bar{v}, 2\iota, v^2 \rangle &= \langle \varepsilon + \bar{v}, 2\iota, v^2 \rangle = \langle \sigma \circ \eta, 2\iota, v^2 \rangle \\ &= \sigma \circ \langle \eta, 2\iota, v^2 \rangle \in \sigma \circ (G_8; 2) = 0. \end{aligned}$$

Hence $\langle \varepsilon, 2\iota, v^2 \rangle = \langle \bar{v}, 2\iota, v^2 \rangle$. Similarly we have $\langle \varepsilon, v, 2v \rangle = \langle \bar{v}, v, 2v \rangle$, $\langle \varepsilon, 2v, v \rangle = \langle \bar{v}, 2v, v \rangle$ and $\langle v, \varepsilon, v \rangle = \langle v, \bar{v}, v \rangle$.

Next we see by (3.5) of [10] that

$$\bar{\varepsilon} \in \langle \varepsilon, 2\iota, v^2 \rangle \subset \langle \varepsilon, 2v, v \rangle \supset \langle \varepsilon, v, 2v \rangle.$$

The indeterminacies of these Toda brackets are trivial, and hence we have

$$\bar{\varepsilon} = \langle \varepsilon, 2\iota, v^2 \rangle = \langle \varepsilon, 2v, v \rangle = \langle \varepsilon, v, 2v \rangle.$$

In (3.10) of [10] we may put $\alpha = v \in (G_3; 2)$, $\beta = \varepsilon \in (G_8; 2)$, $k = 3$ and $h = 8$. Then it follows that $\langle v, \varepsilon, v \rangle$ and $\langle \varepsilon, v, 2v \rangle (= \bar{\varepsilon})$ have a common element. Thus we have $\langle v, \varepsilon, v \rangle = \bar{\varepsilon}$. This completes the proof. $\square$

PROPOSITION 3.

- (1) $\{\bar{v}_7, 2v_{15}, v_{18}\} \circ \eta_{22} = 0$ in π_{23}^7 .
- (2) $\{v_7, \varepsilon_{10}, v_{18}\} \circ \eta_{22} = 0$ in π_{23}^7 .
- (3) $v_7 \circ \Delta(v_{21}) = 0$ in π_{22}^7 ; $\Delta(v_{21}) \circ \sigma_{22} = 0$ in π_{29}^{10} .

Proof. (1) Using Proposition 1.4 of [10], we have

$$\{\bar{v}_7, 2v_{15}, v_{18}\} \circ \eta_{22} = -\bar{v}_7 \circ \{2v_{15}, v_{18}, \eta_{21}\} \subset \bar{v}_7 \circ \pi_{23}^{15} = \{\bar{v}_7 \circ \bar{v}_{15}, \bar{v}_7 \circ \varepsilon_{15}\} = 0$$

by Theorem 7.1 of [10] and Proposition 2.8(2) of [9].

(2) For $\{v_7, \varepsilon_{10}, v_{18}\} \circ \eta_{22} = -v_7 \circ \{\varepsilon_{10}, v_{18}, \eta_{21}\} = 0$ by Theorem 1(2).

(3) Since $v_7 \circ \Delta(v_{21}) \in \pi_{22}^7$ and $E : \pi_{22}^7 \rightarrow \pi_{23}^8$ is a monomorphism and $E(v_7 \circ \Delta(v_{21})) = v_8 \circ E\Delta(v_{21}) = 0$ by the exactness of the $EH\Delta$ -sequence, we have $v_7 \circ \Delta(v_{21}) = 0$. Also $\Delta(v_{21}) \circ \sigma_{22} = \Delta(v_{21} \circ \sigma_{24}) = 0$ by Proposition 2.5 of [10] and (7.20) of [10]. $\square$

THEOREM 4. $\{\bar{v}_7, 2v_{15}, v_{18}\} = \{v_7, \varepsilon_{10}, v_{18}\} = \bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14}$ in π_{22}^7 .

Proof. We note that

$$\pi_{22}^7 = Z_8\{\rho''\} \oplus Z_2\{\bar{\varepsilon}_7\} \oplus Z_2\{\sigma' \circ \bar{v}_{14}\} \oplus Z_2\{\sigma' \circ \varepsilon_{14}\}$$

by Theorem 10.5 of [10] and

$$\pi_{23}^7 = Z_2\{\mu_7 \circ \sigma_{16}\} \oplus Z_2\{\eta_7 \circ \bar{\varepsilon}_8\} \oplus Z_2\{\sigma' \circ \mu_{14}\} \oplus \{\sigma' \circ \eta_{14} \circ \varepsilon_{15}\}$$

by Theorem 12.6 of [10] and Lemma 12.1 of [10]. We note the following relations: $\rho'' \circ \eta_{22} = \sigma' \circ \mu_{14}$ by Proposition 2.8(3) of [9]; $\bar{\varepsilon}_7 \circ \eta_{22} = \eta_7 \circ \bar{\varepsilon}_8$ by Lemma 12.10 of [10]; $\sigma' \circ \bar{v}_{14} \circ \eta_{22} = \sigma' \circ v_{14}^3 = v_7 \circ \sigma_{10} \circ v_{17}^2 = \eta_7 \circ \bar{\varepsilon}_8$ by Lemma 6.3 of [10], (7.19) of [10] and Lemma 12.10 of [10]. Thus we see that

$$\text{Ker}\{(\eta_{22})^* : \pi_{22}^7 \rightarrow \pi_{23}^7\} = Z_4\{2\rho''\} \oplus Z_2\{\bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14}\}.$$

So by Proposition 3, parts (1) and (2), we may write

$$\{\bar{v}_7, 2v_{15}, v_{18}\} = x(\bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14}) + 2y\rho'',$$

and

$$\{v_7, \varepsilon_{10}, v_{18}\} = z(\bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14}) + 2w\rho''$$

for some integers x, y, z and w . The indeterminacies of the above Toda brackets are trivial, since $\bar{v}_7 \circ \pi_{22}^{15} + \pi_{19}^7 \circ v_{19} = \{\bar{v}_7 \circ \sigma_{15}\} = 0$ by Proposition 5.15 of [10], Theorem 7.6 of [10] and Lemma 10.7 of [10]; $v_7 \circ \pi_{22}^{10} + \pi_{19}^7 \circ v_{19} = \{v_7 \circ \Delta(v_{21})\} = 0$ by Theorem 7.6 of [10] and Proposition 3(3).

We have equalities $\{\bar{v}_7, 2v_{15}, v_{18}\} = \{\bar{v}_7, 2v_{15}, v_{18}\}_1$ and $\{v_7, \varepsilon_{10}, v_{18}\} = \{v_7, \varepsilon_{10}, v_{18}\}_1$. It follows that

$$\begin{aligned} H\{\bar{v}_7, 2v_{15}, v_{18}\} &= H\{\bar{v}_7, 2v_{15}, v_{18}\}_1 = -\Delta^{-1}(\bar{v}_6 \circ 2v_{14}) \circ v_{19} \\ &= -\Delta^{-1}(2\bar{v}_6 \circ v_{14}) \circ v_{19} = v_{13}^2 \circ v_{19} = v_{13}^3 \end{aligned}$$

by Proposition 2.6 of [10], and (7.18) of [10]. Similarly we have

$$H\{v_7, \varepsilon_{10}, v_{18}\} = H\{v_7, \varepsilon_{10}, v_{18}\}_1 = -\Delta^{-1}(v_6 \circ \varepsilon_9) \circ v_{19} = v_{13}^3.$$

On the other hand, $H(\sigma' \circ \bar{v}_{14}) = H(\sigma') \circ \bar{v}_{14} = \eta_{13} \circ \bar{v}_{14} = v_{13}^3$ by Lemma 5.14 of [10] and Lemma 6.3 of [10]. Moreover, $H(\bar{\varepsilon}_7) = H(E\bar{\varepsilon}_6) = 0$ and $H(2\rho'') = H(E\rho'') = 0$ by the exactness of the $EH\Delta$ -sequence and (10.16) of [10]. Thus we have $x \equiv z \equiv 1 \pmod{2}$. Consequently,

$$\{\bar{v}_7, 2v_{15}, v_{18}\} = \bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14} + 2y\rho''$$

and

$$\{v_7, \varepsilon_{10}, v_{18}\} = \bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14} + 2w\rho''$$

for some integers y and w .

In the stable range, we see that

$$E^\infty\{\bar{v}_7, 2v_{15}, v_{18}\} = \langle \bar{v}, 2v, v \rangle = \bar{\varepsilon}$$

and

$$E^\infty\{v_7, \varepsilon_{10}, v_{18}\} = \langle v, \varepsilon, v \rangle = \bar{\varepsilon}$$

by Theorem 2. On the other hand,

$$E^\infty(\bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14} + 2y\rho'') = \bar{\varepsilon} + 2\sigma \circ \bar{v} + 8y\rho = \bar{\varepsilon} + 8y\rho$$

and

$$E^\infty(\bar{\varepsilon}_7 + \sigma' \circ \bar{v}_{14} + 2w\rho'') = \bar{\varepsilon} + 8w\rho$$

by Lemmas 5.14, 10.6 and 10.9 of [10]. Then we have

$$\bar{\varepsilon} = \bar{\varepsilon} + 8y\rho \quad \text{and} \quad \bar{\varepsilon} = \bar{\varepsilon} + 8w\rho$$

in $(G_{15}; 2) = Z_{32}\{\rho\} \oplus Z_2\{\bar{\varepsilon}\}$. This implies that $y \equiv w \equiv 0 \pmod{4}$, from which it follows that $2y\rho'' = 2w\rho'' = 0$, for ρ'' is a generator of order eight. Therefore, the proof is completed. $\square$

COROLLARY 5. In the 15-stem the following relations hold.

- (1) $\{\bar{v}_n, 2v_{n+8}, v_{n+11}\} = \{v_n, \varepsilon_{n+3}, v_{n+11}\} = \bar{\varepsilon}_n$ in π_{n+15}^n for $n \geq 11$.
- (2) $\{\bar{v}_9, 2v_{17}, v_{20}\} = \bar{\varepsilon}_9$ in π_{24}^9 ;
 $\{v_9, \varepsilon_{12}, v_{20}\} \ni \bar{\varepsilon}_9$ with $v_9 \circ \pi_{24}^{12} + \pi_{21}^9 \circ v_{21} \subset \{\sigma_9 \circ \varepsilon_{16}, \sigma_9 \circ \bar{v}_{16}\}$ in π_{24}^9 .
- (3) $\{\bar{v}_{10}, 2v_{18}, v_{21}\} = \{v_{10}, \varepsilon_{13}, v_{21}\} = \bar{\varepsilon}_{10} + \{\sigma_{10} \circ \bar{v}_{17}\}$ in π_{25}^{10} .

Proof. For $n \geq 9$ we calculate the indeterminacies of the Toda brackets in question: we have $\bar{v}_n \circ \pi_{n+15}^{n+8} = 0$ ($n \geq 9$) by Proposition 5.15 of [10] and Lemma 10.7 of [10]; $\pi_{n+12}^n \circ v_{n+12} = 0$ ($n = 9$ and $n \geq 11$) and $\pi_{22}^{10} \circ v_{22} = \{\Delta(v_{21}) \circ v_{22}\} = \{\Delta(v_{21}^2)\} = \{\sigma_{10} \circ \bar{v}_{17}\}$ by Theorem 7.6 of [10], the relations $\theta' \circ v_{23} = 0, \theta \circ v_{24} = 0$ by Proposition 2.6(1) of [9] and (10.20) of [10]; $v_n \circ \pi_{n+15}^{n+3} = 0$ ($n \geq 11$) and $v_{10} \circ \pi_{25}^{13} = \{v_{10} \circ E\theta\} = \{\sigma_{10} \circ \bar{v}_{17}\}$ and $v_9 \circ \pi_{24}^{12} = \{v_9 \circ \theta, v_9 \circ E\theta'\} \subset \{\sigma_9 \circ \varepsilon_{16}, \sigma_9 \circ \bar{v}_{16}\}$ by Proposition 2.6(2) of [9] and (10.18) of [10]. This establishes the result. $\square$

Remark 6. The following results may be cited from Proposition 2.3(5) on p. 109 of [8]:

$$\begin{aligned} \bar{\varepsilon}_5 &= \{\varepsilon_5, 2\iota_{13}, v_{13}^2\}_1 = \{\varepsilon_5, v_{13}^2, 2\iota_{19}\}_1 = \{\varepsilon_5, 2v_{13}, v_{16}\}_1 = \{\varepsilon_5, v_{13}, 2v_{16}\}_1 \\ &= \{v_5^2, 2\iota_{11}, \varepsilon_{11}\}_1 = \{v_5, 2v_8, \varepsilon_{11}\}_1 = \{2\iota_5, v_5^2, \varepsilon_{11}\}_1 = \{2v_5, v_8, \varepsilon_{11}\}_1. \end{aligned}$$

It follows that

$$\{Ev', v_7, \varepsilon_{10}\} = \bar{\varepsilon}_4.$$

2. Further results on Toda brackets in the 13-stem

In this section we apply the results of the previous section to settle the Toda brackets in Theorems 12 and 13.

PROPOSITION 7.

- (1) $\Delta(\sigma_9 \circ v_{16}) = xv_4^2 \circ \sigma_{10}$ for an odd integer x in π_{17}^4 .
- (2) $v_5^2 \circ \sigma_{11} = 0$ in π_{18}^5 .
- (3) $v_6 \circ \sigma_9^2 = 0$ in π_{23}^6 ; $v_5 \circ \sigma_8^2 \equiv 0 \pmod{2v_5 \circ \kappa_8}$ in π_{22}^5 .

Proof. (1) Using Proposition 2.5 of [10], we have

$$\begin{aligned} \Delta(\sigma_9 \circ v_{16}) &= \Delta(\sigma_9) \circ v_{14} = (x(v_4 \circ \sigma') \pm E\varepsilon') \circ v_{14} \\ &= x(v_4 \circ \sigma') \circ v_{14} \pm E\varepsilon' \circ v_{14} = yv_4 \circ v_7 \circ \sigma_{10} = yv_4^2 \circ \sigma_{10} \end{aligned}$$

for odd integers x and y by (7.16) of [10] and the relations $\varepsilon' \circ \nu_{13} = 0$ by Proposition 2.2(7) of [9] and $\sigma' \circ \nu_{14} = x\nu_7 \circ \sigma_{10}$ for an odd integer x by (7.19) of [10].

(2) This is a direct consequence of (1) and the exactness of the $EH\Delta$ -sequence.

(3) The element $\nu_5 \circ \sigma_8^2$ is in π_{22}^5 . We see that

$$\nu_7 \circ \sigma_{10}^2 = \nu_7 \circ \sigma_{10} \circ \sigma_{17} = x\sigma' \circ \nu_{14} \circ \sigma_{17} = 0$$

for an odd integer x by (7.19) and (7.20) of [10]. Since $E : E\pi_{22}^5 \rightarrow \pi_{24}^7$ is a monomorphism and $\text{Ker}(E : \pi_{22}^5 \rightarrow \pi_{23}^6) = \{2\nu_5 \circ \kappa_8\}$ by Theorem 12.7 of [10], we have the result. We find the last relation in Proposition 2.13(9) of [9]. $\square$

PROPOSITION 8.

(1) $\nu_9 \circ \Delta(\iota_{25}) = \Delta(\nu_{19}^2) = \bar{\nu}_9 \circ \nu_{17}^2 \equiv 2\kappa_9 \pmod{8\sigma_9^2}$ in π_{23}^9 .

(2) $(2x\nu_9) \circ \Delta(\iota_{25}) = 0$ in π_{23}^9 for any integer x .

Proof. (1) The properties of the Whitehead product enable us to calculate the following:

$$\begin{aligned} \nu_9 \circ \Delta(\iota_{25}) &= \nu_9 \circ [\iota_{12}, \iota_{12}] = [\nu_9, \nu_9] = [\iota_9, \iota_9] \circ E(\nu_8 \# \nu_8) \\ &= \Delta(\iota_{19}) \circ \nu_{17}^2 = \Delta(\nu_{19}^2) = \bar{\nu}_9 \circ \nu_{17}^2 \equiv 2\kappa_9 \pmod{8\sigma_9^2} \end{aligned}$$

by Proposition 2.5 of [10], a result on p. 102 of [10], Lemma 10.1 of [10] and (10.7) of [10].

(2) In a similar way we have

$$\begin{aligned} (2x\nu_9) \circ \Delta(\iota_{25}) &= (2x\nu_9) \circ [\iota_{12}, \iota_{12}] = [2x\nu_9, 2x\nu_9] \\ &= [\iota_9, \iota_9] \circ E\{(2x\nu_8) \# (2x\nu_8)\} = \Delta(\iota_{19}) \circ 4y\nu_{17}^2 = 0 \end{aligned}$$

for some integer y . This completes the proof. $\square$

PROPOSITION 9.

(1) $\sigma''' \circ \nu_{12} = \eta_5^2 \circ \varepsilon_7 = 4(\nu_5 \circ \sigma_8) = 4\nu_5 \circ \sigma_8$ in π_{15}^5 .

(2) $\sigma''' \circ \nu_{12}^2 = 0$ in π_{18}^5 ; $\sigma'' \circ \nu_{13}^2 = 0$ in π_{19}^6 ; $\sigma' \circ \nu_{14}^2 = \nu_7 \circ \sigma_{10} \circ \nu_{17}$ in π_{20}^7 .

Proof. (1) By (6.2) of [10], we have $\sigma''' \circ \nu_{12} \equiv \eta_5^2 \circ \varepsilon_7 \pmod{4(\nu_5 \circ E\sigma')}$, where $4(\nu_5 \circ E\sigma') = 8(\nu_5 \circ \sigma_8) = 0$ by (7.16) of [10] and Theorem 7.3 of [10]. It follows that $\sigma''' \circ \nu_{12} = \eta_5^2 \circ \varepsilon_7$.

By (7.10) of [10], we have $\eta_5^2 \circ \varepsilon_7 = 4(\nu_5 \circ \sigma_8)$. We note that $x(\nu_5 \circ \sigma_8) = (x\nu_5) \circ \sigma_8$ for any integer x , since $E : \pi_{15}^5 \rightarrow \pi_{16}^6$ is a monomorphism, and hence $4(\nu_5 \circ \sigma_8) = 4\nu_5 \circ \sigma_8$.

(2) We see that

$$\sigma''' \circ v_{12}^2 = \sigma''' \circ v_{12} \circ v_{15} = \eta_5^2 \circ \varepsilon_7 \circ v_{15} = 0$$

by (1) and (7.13) of [10];

$$\sigma'' \circ v_{13}^2 = \sigma'' \circ v_{13} \circ v_{16} = \pm 2v_6 \circ \sigma_9 \circ v_{16} = 0$$

by Proposition 2.2(1) of [9] and Theorem 7.7 of [10];

$$\sigma' \circ v_{14}^2 = \sigma' \circ v_{14} \circ v_{17} = xv_7 \circ \sigma_{10} \circ v_{17} = v_7 \circ \sigma_{10} \circ v_{17}$$

by (7.19) of [10], where x is an odd integer. □

PROPOSITION 10.

- (1) $\{v_5, 2v_8, v_{11}\}_1 = v_5 \circ \sigma_8 + \{2v_5 \circ \sigma_8\}$ in π_{15}^5 .
- (2) $\{v_n, 2v_{n+3}, v_{n+6}\} = \{\sigma_n \circ v_{n+7}\}$ in π_{n+10}^n for $n = 9, 10, 11$.
- (3) $\{v_n, 2v_{n+3}, v_{n+6}\} = 0$ in π_{n+10}^n for $n \geq 12$.

Proof. (1) We have

$$\{v_5, 2v_8, v_{11}\}_1 \subset \pi_{15}^5 = Z_8\{v_5 \circ \sigma_8\} \oplus Z_2\{\eta_5 \circ \mu_6\}$$

by Theorem 7.3 of [10]. So we may put

$$\{v_5, 2v_8, v_{11}\}_1 \ni xv_5 \circ \sigma_8 + y\eta_5 \circ \mu_6$$

for some integers x and y . As remarked in the proof of Proposition 9(1), we have $x(v_5 \circ \sigma_8) = (xv_5) \circ \sigma_8$ for any integer x , since $E : \pi_{15}^5 \rightarrow \pi_{16}^6$ is a monomorphism. Thus we have

$$\{v_5, 2v_8, v_{11}\}_1 \circ \eta_{15} \ni xv_5 \circ \sigma_8 \circ \eta_{15} + y\eta_5 \circ \mu_6 \circ \eta_{15}.$$

We see

$$\{v_5, 2v_8, v_{11}\}_1 \circ \eta_{15} = v_5 \circ E\{2v_7, v_{10}, \eta_{13}\} \ni v_5 \circ \varepsilon_8$$

by (7.6) of [10]. The indeterminacy of the Toda bracket is

$$v_5 \circ E\pi_{14}^7 \circ \eta_{15} = \{v_5 \circ E\sigma' \circ \eta_{15}\} = \{2(v_5 \circ \sigma_8) \circ \eta_{15}\} = 0$$

by Proposition 5.15 of [10] and (7.16) of [10]. Moreover, we see that $v_5 \circ \sigma_8 \circ \eta_{15} = v_5 \circ \varepsilon_8$ by a result on p. 152 of [10] and $\eta_5 \circ \mu_6 \circ \eta_{15} = \eta_5^2 \circ \mu_7 = 4\zeta_5$ by Lemma 6.7 of [10] and Theorem 7.1 of [10]. Thus we have a relation

$$v_5 \circ \varepsilon_8 = xv_5 \circ \varepsilon_8 + 4y\zeta_5$$

in $\pi_{16}^5 = Z_8\{\zeta_5\} \oplus Z_2\{\nu_5 \circ \bar{\nu}_8\} \oplus Z_2\{\nu_5 \circ \varepsilon_8\}$ by Theorem 7.4 of [10]. It follows that $x \equiv 1 \pmod{2}$ and $y \equiv 0 \pmod{2}$. Then we have $\{\nu_5, 2\nu_8, \nu_{11}\}_1 \ni x\nu_5 \circ \sigma_8$ for an odd integer x . The indeterminacy of the Toda bracket is

$$\nu_5 \circ E\pi_{14}^7 + \pi_{12}^5 \circ \nu_{12} = \{\nu_5 \circ E\sigma', \sigma''' \circ \nu_{12}\} = \{2\nu_5 \circ \sigma_8\}$$

by Proposition 5.15 of [10], (7.16) of [10] and the relation $\sigma''' \circ \nu_{12} = 4\nu_5 \circ \sigma_8$ by Proposition 9(1). Consequently, we have the result of (1).

(2) We see that

$$\{\nu_9, 2\nu_{12}, \nu_{15}\} \ni \nu_9 \circ \sigma_{12} = 2x\sigma_9 \circ \nu_{16}$$

for an odd integer x by (1) and (7.19) of [10]. The indeterminacy of the Toda bracket is $\nu_9 \circ \pi_{19}^{12} + \pi_{16}^9 \circ \nu_{16} = \{\sigma_9 \circ \nu_{16}\}$ and hence we have the result of (2) for $n = 9$.

Now the indeterminacy of the Toda bracket $\{\nu_n, 2\nu_{n+3}, \nu_{n+6}\}$ is calculated in a similar way as $\nu_n \circ \pi_{n+10}^{n+3} + \pi_{n+7}^n \circ \nu_{n+7} = \{\sigma_n \circ \nu_{n+7}\}$ for $n = 10, 11$. Thus we have the result for $n = 10, 11$.

(3) Checking the indeterminacy of the Toda bracket with the relation $\sigma_{12} \circ \nu_{19} = 0$ by (7.20) of [10], we have the result. We remark that the relation $\{\nu_{20}, 2\nu_{23}, \nu_{26}\} = 0$ is obtained on p. 185 of Part I of Mimura [4]. His result is equivalent to this relation (3) as a result in the stable range. $\square$

PROPOSITION 11.

- (1) $\{2\nu_9, \nu_{12}, \sigma_{15}\} \subset \{\nu_9, 2\nu_{12}, \sigma_{15}\} = \{\sigma_9^2, 2\kappa_9\}$ in π_{23}^9 ;
 $\{2\nu_8, \nu_{11}, \sigma_{14}\} \subset \{\sigma_8^2, 2\kappa_8, E\sigma' \circ \sigma_{15}\}$ in π_{22}^8 ;
 $\{\nu_7, 2\nu_{10}, \sigma_{13}\} \subset \{2\kappa_7, \sigma' \circ \sigma_{14}\}$ in π_{21}^7 .
- (2) $\{2\nu_n, \nu_{n+3}, \sigma_{n+6}\} = \{\nu_n, 2\nu_{n+3}, \sigma_{n+6}\} = \{\sigma_n^2\}$ in π_{n+14}^n for $n \geq 10$.

Proof. (1) Since we have

$$\{2\nu_9, \nu_{12}, \sigma_{15}\} \subset \pi_{23}^9 = Z_{16}\{\sigma_9^2\} \oplus Z_4\{\kappa_9\}$$

by Theorem 10.3 of [10], we may put

$$\{2\nu_9, \nu_{12}, \sigma_{15}\} \ni x\sigma_9^2 + y\kappa_9$$

for some integers x and y . Composing the above relation with ν_6 , we have

$$\nu_6 \circ \{2\nu_9, \nu_{12}, \sigma_{15}\} = -\{\nu_6, 2\nu_9, \nu_{12}\} \circ \sigma_{16} \ni -\nu_6 \circ \sigma_9^2 = 0$$

by Propositions 10(1) and 7(3), with the indeterminacy of the Toda bracket $\nu_6 \circ \pi_{16}^9 \circ \sigma_{16} = 0$. Thus we have

$$0 = x\nu_6 \circ \sigma_9^2 + y\nu_6 \circ \kappa_9 = y\nu_6 \circ \kappa_9.$$

It follows that $y \equiv 0 \pmod{2}$, since $v_6 \circ \kappa_9$ is an element of order two by Theorem 12.7 of [10].

Now, the indeterminacy of the Toda bracket $\{2v_9, v_{12}, \sigma_{15}\}$ is

$$2v_9 \circ \pi_{23}^{12} + \pi_{16}^9 \circ \sigma_{16} = \{2v_9 \circ \Delta(\iota_{25}), 2v_9 \circ \zeta_{12}, \sigma_9^2\} = \{\sigma_9^2\}$$

by Theorem 7.4 of [10], Proposition 5.15 of [10], Proposition 8(2) and (10.7) of [10]. Then, for some integer z , we have

$$\{2v_9, v_{12}, \sigma_{15}\} = 2z\kappa_9 + \{\sigma_9^2\} \subset \{\sigma_9^2, 2\kappa_9\}.$$

Observe that $\{2v_9, v_{12}, \sigma_{15}\} \subset \{v_9, 2v_{12}, \sigma_{15}\}$. The indeterminacy of $\{v_9, 2v_{12}, \sigma_{15}\}$ is

$$v_9 \circ \pi_{23}^{12} + \pi_{16}^9 \circ \sigma_{16} = \{v_9 \circ \Delta(\iota_{25}), v_9 \circ \zeta_{12}, \sigma_9^2\} = \{2\kappa_9, \sigma_9^2\}$$

by Proposition 8(1). It follows that $\{v_9, 2v_{12}, \sigma_{15}\} = \{\sigma_9^2, 2\kappa_9\}$.

Since

$$E\{2v_8, v_{11}, \sigma_{14}\} \subset -\{2v_9, v_{12}, \sigma_{15}\} \subset \{\sigma_9^2, 2\kappa_9\}$$

and

$$E^2\{v_7, 2v_{10}, \sigma_{13}\} \subset \{v_9, 2v_{12}, \sigma_{15}\} = \{\sigma_9^2, 2\kappa_9\},$$

we have the result by Theorem 10.3 of [10].

(2) We see that

$$\{2v_n, v_{n+3}, \sigma_{n+6}\} \supset \pm E^{n-9}\{2v_9, v_{12}, \sigma_{15}\} \subset E^{n-9}\{\sigma_9^2, 2\kappa_9\} = \{\sigma_n^2\}$$

for $n \geq 10$ by (1) and the relation $2\kappa_{10} \equiv 0 \pmod{8\sigma_{10}^2}$ by Proposition 8(1). Then we may put

$$\{2v_n, v_{n+3}, \sigma_{n+6}\} \ni x\sigma_n^2$$

for some integer x . The indeterminacy of the Toda bracket is

$$2v_n \circ \pi_{n+14}^{n+3} + \pi_{n+7}^n \circ \sigma_{n+7} = \{2v_n \circ \zeta_{n+3}, \sigma_n^2\} = \{\sigma_n^2\}$$

for $n \geq 10$ by Theorem 7.4 of [10], Proposition 5.15 of [10], (10.7) of [10] and Lemma 5.14 of [10]. It follows that $\{2v_n, v_{n+3}, \sigma_{n+6}\} = \{\sigma_n^2\}$ for $n \geq 10$.

Finally, we see that the indeterminacy of $\{v_n, 2v_{n+3}, \sigma_{n+6}\}$ is

$$v_n \circ \pi_{n+14}^{n+3} + \pi_{n+7}^n \circ \sigma_{n+7} = \{v_n \circ \zeta_{n+3}, \sigma_n^2\} = \{\sigma_n^2\}$$

for $n \geq 10$, which is the same as the indeterminacy of $\{2v_n, v_{n+3}, \sigma_{n+6}\}$. Thus the inclusion $\{2v_n, v_{n+3}, \sigma_{n+6}\} \subset \{v_n, 2v_{n+3}, \sigma_{n+6}\}$ implies $\{2v_n, v_{n+3}, \sigma_{n+6}\} = \{v_n, 2v_{n+3}, \sigma_{n+6}\}$ for $n \geq 10$. $\square$

THEOREM 12. $\{\eta_6, \nu_7, \varepsilon_{10}\} = \{\eta_6, \nu_7, \bar{\nu}_{10}\} = \nu_6 \circ \sigma_9 \circ \nu_{16}$ in π_{19}^6 .

Proof. By Propositions 1.6 and 1.2 of [10] and Lemmas 6.4 and 5.12 of [10], we have

$$\begin{aligned} \{\eta_6, \nu_7, \varepsilon_{10}\} + \{\eta_6, \nu_7, \bar{\nu}_{10}\} &\supset \{\eta_6, \nu_7, \varepsilon_{10} + \bar{\nu}_{10}\} \\ &= \{\eta_6, \nu_7, \eta_{10} \circ \sigma_{11}\} \ni \{\eta_6, \nu_7, \eta_{10}\} \circ \sigma_{12} = \nu_6^2 \circ \sigma_{12} = 0 \end{aligned}$$

in $\pi_{19}^6 = Z_2\{\nu_6 \circ \sigma_9 \circ \nu_{16}\}$ by Theorem 7.7 of [10]. The indeterminacies of the Toda brackets in the above relation are trivial, since $\eta_6 \circ \pi_{19}^7 = 0$ by Theorem 7.6 of [10], $\pi_{11}^6 \circ \varepsilon_{11} = \{\Delta(\iota_{13}) \circ \varepsilon_{11}\} = \{\Delta(\varepsilon_{13})\} = 0$ and $\pi_{11}^6 \circ \bar{\nu}_{11} = \{\Delta(\bar{\nu}_{13})\} = 0$ by Proposition 5.9 of [10] and (7.27) of [10]. Then we may put

$$\{\eta_6, \nu_7, \varepsilon_{10}\} = \{\eta_6, \nu_7, \bar{\nu}_{10}\} = x \nu_6 \circ \sigma_9 \circ \nu_{16}$$

for some integer x . We note that $\eta_6 \circ \sigma' \circ \bar{\nu}_{14} = 4\bar{\nu}_6 \circ \bar{\nu}_{14} = 0$ by (7.4) of [10]. Then by the result of Theorem 4, we have

$$\begin{aligned} \{\eta_6, \nu_7, \varepsilon_{10}\} \circ \nu_{19} &= -\eta_6 \circ \{\nu_7, \varepsilon_{10}, \nu_{18}\} = \eta_6 \circ (\bar{\varepsilon}_7 + \sigma' \circ \bar{\nu}_{14}) \\ &= \eta_6 \circ \bar{\varepsilon}_7 + \eta_6 \circ \sigma' \circ \bar{\nu}_{14} = \eta_6 \circ \bar{\varepsilon}_7 \end{aligned}$$

with trivial indeterminacy of the Toda bracket. Thus we have

$$\eta_6 \circ \bar{\varepsilon}_7 = x \nu_6 \circ \sigma_9 \circ \nu_{16}^2 = x \eta_6 \circ \bar{\varepsilon}_7$$

in $\pi_{22}^6 = Z_8\{\zeta'\} \oplus Z_2\{\mu_6 \circ \sigma_{15}\} \oplus Z_2\{\eta_6 \circ \bar{\varepsilon}_7\}$ by Lemma 12.10 of [10] and Theorem 12.6 of [10]. It follows that $x \equiv 1 \pmod{2}$. This completes the proof. $\square$

THEOREM 13.

$$(1) \quad \{\nu_5^2, 2\iota_{11}, \nu_{11}^2\} = \nu_5 \circ \sigma_8 \circ \nu_{15} \text{ in } \pi_{18}^5.$$

$$(2) \quad \{\nu_6^2, 2\iota_{12}, \nu_{12}^2\} = \nu_6 \circ \sigma_9 \circ \nu_{16} \text{ in } \pi_{19}^6.$$

Proof. (1) The indeterminacies of the Toda brackets $\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}$ and $\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}_1$ are both trivial since

$$\nu_5^2 \circ \pi_{18}^{11} = \nu_5^2 \circ E\pi_{17}^{10} = \pi_{12}^5 \circ \nu_{12}^2 = 0$$

by Proposition 5.15 of [10] and Propositions 7(2) and 9(2). Then we have

$$\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\} = \{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}_1 \in \pi_{18}^5 = Z_2\{\nu_5 \circ \sigma_8 \circ \nu_{15}\} \oplus Z_2\{\nu_5 \circ \eta_8 \circ \mu_9\}$$

by Theorem 7.7 of [10]. Thus we may put

$$\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\} = x \nu_5 \circ \sigma_8 \circ \nu_{15} + y \nu_5 \circ \eta_8 \circ \mu_9$$

for some integers x and y . Now

$$\begin{aligned} H\{v_5^2, 2\iota_{11}, v_{11}^2\} &= H\{v_5^2, 2\iota_{11}, v_{11}^2\}_1 = -\Delta^{-1}(v_4^2 \circ 2\iota_{10}) \circ v_{12}^2 = -\Delta^{-1}(2v_4^2) \circ v_{12}^2 \\ &= v_9 \circ v_{12}^2 = v_9^3 \end{aligned}$$

by Proposition 2.6 of [10] and (5.13) of [10]. On the other hand, we have $H(v_5 \circ \sigma_8 \circ v_{15}) = v_9^3$ by a result on p. 77 of [10] and $H(v_5 \circ \eta_8 \circ \mu_9) = H(E(v_4 \circ \eta_7 \circ \mu_8)) = 0$. Thus we have a relation $v_9^3 = x v_9^3$, and hence $x \equiv 1 \pmod{2}$ by Theorem 7.2 of [10].

Moreover, we see

$$\{v_5^2, 2\iota_{11}, v_{11}^2\} \circ \sigma_{18} = v_5^2 \circ \{2\iota_{11}, v_{11}^2, \sigma_{17}\} \supset v_5^2 \circ \{2v_{11}, v_{14}, \sigma_{17}\} = \{v_5^2 \circ \sigma_{11}^2\} = 0$$

by Propositions 11(2) and 7(2). Since the indeterminacy of the Toda bracket is $v_5^2 \circ \pi_{11}^{11} \circ \sigma_{18} = \{v_5^2 \circ \sigma_{11}^2\} = 0$, we have

$$\{v_5^2, 2\iota_{11}, v_{11}^2\} \circ \sigma_{18} = 0.$$

Then composing the above relation with σ_{18} , we have

$$0 = \{v_5^2, 2\iota_{11}, v_{11}^2\} \circ \sigma_{18} = (v_5 \circ \sigma_8 \circ v_{15} + y v_5 \circ \eta_8 \circ \mu_9) \circ \sigma_{18} = y v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}.$$

Since $v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}$ is a generator of order two by the group structure of

$$\pi_{25}^5 = Z_2\{v_5^2 \circ \kappa_{11}\} \oplus Z_2\{v_5 \circ \bar{\mu}_8\} \oplus Z_2\{v_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}$$

which is shown on p. 46 of [6], we have $y \equiv 0 \pmod{2}$.

(2) We have $\{v_6^2, 2\iota_{12}, v_{12}^2\} \ni v_6 \circ \sigma_9 \circ v_{16}$ by (1). The indeterminacy of the Toda bracket is

$$v_6^2 \circ \pi_{19}^{12} + \pi_{13}^6 \circ v_{13}^2 = \{v_6^2 \circ \sigma_{12}, \sigma'' \circ v_{13}^2\} = 0$$

by Proposition 5.15 of [10] and Propositions 7(2) and 9(2). Thus we have the result. $\square$

Remark 14. By Theorems 12 and 13(2), we have

$$\{\eta_6, v_7, \varepsilon_{10}\} = \{\eta_6, v_7, \bar{v}_{10}\} = v_6 \circ \sigma_9 \circ v_{16} = \{v_6^2, 2\iota_{12}, v_{12}^2\}.$$

We remark that Proposition 1.5 of [10] implies the relation $\{\eta_6, v_7, \varepsilon_{10}\} = \{v_6^2, 2\iota_{12}, v_{12}^2\}$ directly as follows. Let us put $\alpha = \eta_6, \beta = v_7, \gamma = \eta_{10}, \delta = 2\iota_{11}$ and $\varepsilon = v_{11}^2$ in Proposition 1.5 of [10]. Then we have

$$\{\{\eta_6, v_7, \eta_{10}\}, 2\iota_{12}, v_{12}^2\} + \{\eta_6, \{v_7, \eta_{10}, 2\iota_{11}\}, v_{12}^2\} + \{\eta_6, v_7, \{\eta_{10}, 2\iota_{11}, v_{11}^2\}\} \ni 0.$$

In the first Toda bracket in the above relation, we see $\{\eta_6, v_7, \eta_{10}\} = v_6^2$ by Lemma 5.12 of [10]. The middle Toda bracket is trivial, since $\{v_7, \eta_{10}, 2\iota_{11}\} \subset \pi_{12}^7 = 0$ by Proposition 5.9 of [10], and hence we have

$$\{\eta_6, \{v_7, \eta_{10}, 2\iota_{11}\}, v_{12}^2\} = \eta_6 \circ \pi_{19}^7 + \pi_{13}^6 \circ v_{13}^2 = \{\sigma'' \circ v_{13}^2\} = 0$$

by Theorem 7.6 of [10], Proposition 5.15 of [10] and Proposition 9(2).

Now we consider the third Toda bracket. By (6.1) of [10], $\{\eta_{10}, 2\iota_{11}, \nu_{11}^2\} \ni \varepsilon_{10}$ with the indeterminacy of the Toda bracket being

$$\eta_{10} \circ \pi_{18}^{11} + \pi_{12}^{10} \circ \nu_{12}^2 = \{\eta_{10} \circ \sigma_{11}, \eta_{10}^2 \circ \nu_{12}^2\} = \{\eta_{10} \circ \sigma_{11}\}$$

by Propositions 5.15 and 5.3 of [10] and (5.9) of [10]. Thus we have

$$\{\eta_{10}, 2\iota_{11}, \nu_{11}^2\} = \varepsilon_{10} + \{\eta_{10} \circ \sigma_{11}\}.$$

It follows that

$$\begin{aligned} \{\eta_6, \nu_7, \{\eta_{10}, 2\iota_{11}, \nu_{11}^2\}\} &= \{\eta_6, \nu_7, \varepsilon_{10} + \{\eta_{10} \circ \sigma_{11}\}\} \\ &\subset \{\eta_6, \nu_7, \varepsilon_{10}\} + \{\eta_6, \nu_7, \{\eta_{10} \circ \sigma_{11}\}\} = \{\eta_6, \nu_7, \varepsilon_{10}\} \end{aligned}$$

because

$$\{\eta_6, \nu_7, x\eta_{10} \circ \sigma_{11}\} \supset \{\eta_6, \nu_7, x\eta_{10}\} \circ \sigma_{12} \subset \pi_{12}^6 \circ \sigma_{12} = \{\nu_6^2 \circ \sigma_{12}\} = 0$$

for any integer x , with trivial indeterminacy of the Toda bracket as is shown in the proof of Theorem 12, and hence

$$\{\eta_6, \nu_7, \{\eta_{10} \circ \sigma_{11}\}\} = 0.$$

Therefore, we have

$$\{\nu_6^2, 2\iota_{12}, \nu_{12}^2\} + \{\eta_6, \nu_7, \varepsilon_{10}\} \ni 0.$$

We know that these Toda brackets have trivial indeterminacies by the calculations in the proofs of Theorems 12 and 13, and hence we obtain

$$\{\nu_6^2, 2\iota_{12}, \nu_{12}^2\} = \{\eta_6, \nu_7, \varepsilon_{10}\}.$$

3. Some matrix Toda brackets

Several authors have studied matrix Toda brackets, for example, Part I of [4], [7] and [1]. The definition of the matrix Toda brackets we use here is that given in Definition 5.1 of Hardie [2].

To settle matrix Toda brackets, the following two formulae are very useful. They state that matrix Toda brackets, when composed with suitable elements, can be expressed making use of classical Toda brackets. These formulae are special cases of formulae for box brackets given in Theorem 4.4 of [3] and they are stated in Corollary 4.8 of that paper.

Let us consider the following maps:

If the matrix Toda bracket $\left\{ \begin{smallmatrix} d & r \\ s & b \end{smallmatrix}, g \right\}$ and the Toda brackets $\{r, g, w\}$ and $\{b, g, w\}$ are all well-defined, then the following inclusion holds,

$$\left\{ \begin{smallmatrix} d & r \\ s & b \end{smallmatrix}, g \right\} \circ Ew \subset -d \circ \{r, g, w\} + s \circ \{b, g, w\}$$

with equality if $d \circ r \circ \pi(EW, B) = 0$. A version of this formula is also obtained in Lemma 3 of [7] in the stable category. Recall from Corollary 5.8 of [1] that if g is a suspension class then the matrix Toda bracket $\left\{ \begin{smallmatrix} d & r \\ s & b \end{smallmatrix}, g \right\}$ has indeterminacy

$$s \circ \pi(EC, X) + \pi(EB, Y) \circ Eg + d \circ \pi(EC, D).$$

Next, let us consider the following maps:

If the matrix Toda bracket $\left\{ \begin{smallmatrix} a & f & w \\ & u & h \end{smallmatrix} \right\}$ and the Toda brackets $\{s, a, f\}$ and $\{s, a, u\}$ are all well-defined, then the following inclusion holds,

$$s \circ \left\{ \begin{smallmatrix} a & f & w \\ & u & h \end{smallmatrix} \right\} \subset \{s, a, f\} \circ Ew - \{s, a, u\} \circ Eh$$

with equality if $\pi(EA, Y) \circ E(u \circ h) = 0$.

We first determine the following matrix Toda bracket.

THEOREM 15. $\left\{ \begin{smallmatrix} v_6^2 & v_{12} \\ \eta_6 & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} = v_6 \circ \sigma_9 \circ v_{16} \text{ in } \pi_{19}^6.$

Proof. We note that $v_6^3 = \eta_6 \circ \bar{v}_7$ by (7.3) of [10] and $2\bar{v}_7 \circ v_{15} = 0$ by (7.18) of [10]. The matrix Toda bracket $\left\{ \begin{smallmatrix} v_6^2 & v_{12} \\ \eta_6 & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\}$ is in $\pi_{19}^6 = Z_2\{v_6 \circ \sigma_9 \circ v_{16}\}$ by Theorem 7.7 of [10]. The indeterminacy of the matrix Toda bracket is

$$\eta_6 \circ \pi_{19}^7 + \pi_{16}^6 \circ 2v_{16} + v_6^2 \circ \pi_{19}^{12} = \{v_6^2 \circ \sigma_{12}\} = 0$$

by Theorems 7.6 and 7.3 of [10], Proposition 5.15 of [10] and Proposition 7(2). Hence the matrix Toda bracket consists of only one element and we may put

$$\left\{ \begin{smallmatrix} v_6^2 & v_{12} \\ \eta_6 & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} = x v_6 \circ \sigma_9 \circ v_{16}$$

for some integer x . We see that $v_6^2 \circ v_{12} \circ \pi_{22}^{15} = \{v_6^3 \circ \sigma_{15}\} = 0$. Hence, composing the matrix Toda bracket with v_{19} , we obtain the equality

$$\begin{aligned} \left\{ \begin{smallmatrix} v_6^2 & v_{12} \\ \eta_6 & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} \circ v_{19} &= -v_6^2 \circ \{v_{12}, 2v_{15}, v_{18}\} + \eta_6 \circ \{\bar{v}_7, 2v_{15}, v_{18}\} \\ &= \eta_6 \circ (\bar{e}_7 + \sigma' \circ \bar{v}_{14}) = \eta_6 \circ \bar{e}_7 + \eta_6 \circ \sigma' \circ \bar{v}_{14} = \eta_6 \circ \bar{e}_7 \end{aligned}$$

by Proposition 10(3) and Theorem 4. We remark that $\eta_6 \circ \sigma' \circ \bar{v}_{14} = 4\bar{v}_6 \circ \bar{v}_{14} = 0$ by (7.4) of [10]. Thus we have a relation

$$\eta_6 \circ \bar{e}_7 = \left\{ \begin{smallmatrix} v_6^2 & v_{12} \\ \eta_6 & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} \circ v_{19} = x v_6 \circ \sigma_9 \circ v_{16}^2 = x \eta_6 \circ \bar{e}_7$$

by Lemma 12.10 of [10]. Since $\eta_6 \circ \bar{e}_7$ is a generator of order two in π_{22}^6 by Theorem 12.6 of [10], we have $x \equiv 1 \pmod{2}$. This completes the proof. $\square$

THEOREM 16. $\left\{ \begin{smallmatrix} \varepsilon_4 & v_{12} \\ E v' & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} = \bar{e}_4$ in π_{19}^4 .

Proof. We note that $\varepsilon_4 \circ v_{12} = E v' \circ \bar{v}_7$ by (7.12) of [10] and $2\bar{v}_7 \circ v_{15} = 0$ by (7.18) of [10]. The indeterminacy of the matrix Toda bracket $\left\{ \begin{smallmatrix} \varepsilon_4 & v_{12} \\ E v' & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\}$ is

$$E v' \circ \pi_{19}^7 + \pi_{16}^4 \circ 2v_{16} + \varepsilon_4 \circ \pi_{19}^{12} = 0$$

by Theorem 7.6 of [10] and Proposition 5.15 of [10] and the relation $\varepsilon_4 \circ \sigma_{12} = 0$ by Lemma 10.7 of [10]. So this matrix Toda bracket consists of a single element in $\pi_{19}^4 = Z_2\{\bar{e}_4\}$ by Theorem 10.5 of [10]. Because $\pi_{20}^{15} = 0$, we have $\varepsilon_4 \circ v_{12} \circ \pi_{20}^{15} = 0$. Hence, composing the matrix Toda bracket with η_{19} yields the following equality:

$$\left\{ \begin{smallmatrix} \varepsilon_4 & v_{12} \\ E v' & \bar{v}_7 \end{smallmatrix}, 2v_{15} \right\} \circ \eta_{19} = -\varepsilon_4 \circ \{v_{12}, 2v_{15}, \eta_{18}\} + E v' \circ \{\bar{v}_7, 2v_{15}, \eta_{18}\}.$$

Note that

$$Ev' \circ \{\bar{v}_7, 2v_{15}, \eta_{18}\} \subset Ev' \circ \pi_{20}^7 = 0$$

by Theorem 7.7 of [10], since $v' \circ v_6 \in \pi_9^3 = 0$ by Proposition 5.11 of [10]. Moreover, we know that

$$\{v_{12}, 2v_{15}, \eta_{18}\} = \varepsilon_{12} + \{\sigma_{12} \circ \eta_{19}\}$$

by (7.6) of [10]. We conclude that

$$\left\{ \begin{array}{c} \varepsilon_4 \\ Ev' \end{array}, \frac{v_{12}}{\bar{v}_7}, 2v_{15} \right\} \circ \eta_{19} = -\varepsilon_4 \circ (\varepsilon_{12} + \{\sigma_{12} \circ \eta_{19}\}) = -\varepsilon_4^2 = \eta_4 \circ \bar{\varepsilon}_5 \neq 0$$

by Lemma 12.10 of [10] and Theorem 12.6 of [10] and hence the matrix Toda bracket

$$\left\{ \begin{array}{c} \varepsilon_4 \\ Ev' \end{array}, \frac{v_{12}}{\bar{v}_7}, 2v_{15} \right\} \text{ must equal } \bar{\varepsilon}_4. \quad \square$$

PROPOSITION 17.

- (1) $\eta_{11} \circ \Delta(\iota_{25}) = 0$ in π_{23}^{11} .
- (2) $v_5^2 \circ \theta' = 0$ in π_{23}^5 .
- (3) $\Delta(\sigma_{13}) \circ v_{18} = 0$ in π_{21}^6 ; $v' \circ \Delta(\sigma_{13}) = 0$ in π_{18}^3 .
- (4) $H(\bar{\varepsilon}_3) = v_5 \circ \sigma_8 \circ v_{15}$ in π_{18}^5 ; $\bar{\varepsilon}_3 \circ \sigma_{18} = 0$ in π_{25}^3 .
- (5) $\eta_3 \circ \mu_4 \circ v_{13} = 0$ in π_{16}^3 .

Proof. (1) We remark that $E : \pi_{23}^{11} \rightarrow \pi_{24}^{12}$ is a monomorphism by Theorem 7.6 of [10]. Since $E(\eta_{11} \circ \Delta(\iota_{25})) = \eta_{12} \circ E(\Delta(\iota_{25})) = 0$ we have the result.

(2) By Proposition 2.6(2) of [9], we see $v_8 \circ \theta' = E\sigma' \circ \varepsilon_{15}$ or $E\sigma' \circ \bar{v}_{15}$. Since $v_5 \circ E\sigma' \circ \varepsilon_{15} = 2(v_5 \circ \sigma_8) \circ \varepsilon_{15} = 0$ and $v_5 \circ E\sigma' \circ \bar{v}_{15} = 2(v_5 \circ \sigma_8) \circ \bar{v}_{15} = 0$ by (7.16) of [10], we have $v_5^2 \circ \theta' = 0$.

(3) By Proposition 2.5 of [10] and (7.20) of [10], we have $\Delta(\sigma_{13}) \circ v_{18} = \Delta(\sigma_{13} \circ v_{20}) = 0$. Since $E(v' \circ \Delta(\sigma_{13})) = 0$ and $E : \pi_{18}^3 \rightarrow \pi_{19}^4$ is a monomorphism, we have the second result $v' \circ \Delta(\sigma_{13}) = 0$.

(4) We see $\bar{\varepsilon}_3 \in \{\varepsilon_3, 2\iota_{11}, v_{11}^2\}_6$ by the definition on p. 97 of [10]. Then we see

$$\begin{aligned} H(\bar{\varepsilon}_3) &\in H\{\varepsilon_3, 2\iota_{11}, v_{11}^2\}_1 \subset \{H(\varepsilon_3), 2\iota_{11}, v_{11}^2\}_1 \\ &= \{v_5^2, 2\iota_{11}, v_{11}^2\}_1 = \{v_5^2, 2\iota_{11}, v_{11}^2\} = v_5 \circ \sigma_8 \circ v_{15} \end{aligned}$$

by Proposition 2.3 of [10], Lemma 6.1 of [10] and Theorem 13(1). Then we have $H(\bar{\varepsilon}_3) = v_5 \circ \sigma_8 \circ v_{15}$. The relation $\bar{\varepsilon}_3 \circ \sigma_{18} = 0$ holds by (2.4) of [5].

(5) We have $\eta_3 \circ \mu_4 \circ v_{13} = \eta_3 \circ Ev' \circ \eta_7 \circ \varepsilon_8 = 0$ by Proposition 2.2(4) of [9] and Lemma 5.7 of [10]. $\square$

Remark 18. Here we give another proof of the first part of Proposition 17(4) and then Theorem 13(1).

Consider the following part of the $EH\Delta$ -exact sequence

$$0 \longrightarrow \pi_{18}^3 \xrightarrow{H} \pi_{18}^5 \xrightarrow{\Delta} \pi_{16}^2$$

on p. 104 of [10], where $\pi_{18}^3 = Z_2\{\bar{\varepsilon}_3\}$ and $\pi_{18}^5 = Z_2\{\nu_5 \circ \eta_8 \circ \mu_9\} \oplus Z_2\{\nu_5 \circ \sigma_8 \circ \nu_{15}\}$. It is shown there that $H(\pi_{18}^3)$ has two elements and is generated by $H(\bar{\varepsilon}_3)$.

Now, we may put $\Delta(\nu_5 \circ \sigma_8) = \pm(\eta_2 \circ \varepsilon') + x\eta_2^2 \circ \mu_4$ for some integer x by (7.11) of [10]. It follows that

$$\begin{aligned} \Delta(\nu_5 \circ \sigma_8 \circ \nu_{15}) &= \Delta(\nu_5 \circ \sigma_8) \circ \nu_{13} = \{\pm(\eta_2 \circ \varepsilon') + x\eta_2^2 \circ \mu_4\} \circ \nu_{13} \\ &= \eta_2 \circ \varepsilon' \circ \nu_{13} + x\eta_2^2 \circ \mu_4 \circ \nu_{13} = 0 \end{aligned}$$

by Proposition 2.2(7) of [9] and Proposition 17(5). It follows from the exactness of the $EH\Delta$ -sequence that $H(\bar{\varepsilon}_3) = \nu_5 \circ \sigma_8 \circ \nu_{15}$. Thus we have another proof of the first part of Proposition 17(4).

Now

$$H(\bar{\varepsilon}_3) \in H\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\}_1 \subset \{H(\varepsilon_3), 2\iota_{11}, \nu_{11}^2\}_1 = \{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}_1$$

(see the proof of Proposition 17(4)). Thus we have

$$\nu_5 \circ \sigma_8 \circ \nu_{15} = H(\bar{\varepsilon}_3) \in \{\nu_5^2, 2\iota_{11}, \nu_{11}^2\}.$$

Since the indeterminacy of the Toda bracket is zero, as is shown in the proof of Theorem 13(1), it follows that $\{\nu_5^2, 2\iota_{11}, \nu_{11}^2\} = \nu_5 \circ \sigma_8 \circ \nu_{15}$. This provides another proof of Theorem 13(1).

PROPOSITION 19.

- (1) $\{\bar{\nu}_7, 2\nu_{15}, \eta_{18}\} = 0$ in π_{20}^7 .
- (2) $\{\bar{\nu}_6, 4\nu_{14}, \eta_{17}\} = 0$ in π_{19}^6 .

Proof. (1) The indeterminacy of the Toda bracket is $\bar{\nu}_7 \circ \pi_{20}^{15} + \pi_{19}^7 \circ \eta_{19} = 0$ by Proposition 5.9 of [10] and Theorem 7.6 of [10]. We see that

$$\{\bar{\nu}_7, 2\nu_{15}, \eta_{18}\} \supset \{2\bar{\nu}_7, \nu_{15}, \eta_{18}\} \ni 0$$

since $2\bar{\nu}_7 = 0$ by Theorem 7.1 of [10]. It follows that $\{\bar{\nu}_7, 2\nu_{15}, \eta_{18}\} = 0$.

- (2) We remark that $E : \pi_{19}^6 \rightarrow \pi_{20}^7$ is a monomorphism. We see

$$E\{\bar{\nu}_6, 4\nu_{14}, \eta_{17}\} \subset -\{\bar{\nu}_7, 4\nu_{15}, \eta_{18}\} \supset -\{2\bar{\nu}_7, 2\nu_{15}, \eta_{18}\} \ni 0.$$

The indeterminacy of the Toda bracket is zero by the result of (1). It follows that $E\{\bar{\nu}_6, 4\nu_{14}, \eta_{17}\} = 0$, and hence $\{\bar{\nu}_6, 4\nu_{14}, \eta_{17}\} = 0$. $\square$

PROPOSITION 20.

- (1) $\{\varepsilon_5, \nu_{13}, \sigma_{16}\} = \{\nu_5 \circ \mu_8 \circ \sigma_{17}\}$ in π_{24}^5 .
 (2) $\{\varepsilon_n, \nu_{n+8}, \sigma_{n+11}\} \ni 0$ in π_{n+19}^n for $n \geq 7$;
 $\{\varepsilon_n, \nu_{n+8}, \sigma_{n+11}\} = 0$ in π_{n+19}^n for $n = 7, 8, 9, 10$ and $n \geq 14$.

Proof. (1) We apply Proposition 1.5 of [10] to the case $\alpha = \nu_5^2, \beta = 2\iota_{11}, \gamma = \eta_{11}, \delta = \nu_{12}$ and $\varepsilon = \sigma_{15}$. Thus we have

$$\begin{aligned} & \{\{\nu_5^2, 2\iota_{11}, \eta_{11}\}, \nu_{13}, \sigma_{16}\} + \{\nu_5^2, \{2\iota_{11}, \eta_{11}, \nu_{12}\}, \sigma_{16}\} \\ & + \{\nu_5^2, 2\iota_{11}, \{\eta_{11}, \nu_{12}, \sigma_{15}\}\} \ni 0. \end{aligned}$$

In the first Toda bracket, we see $\{\nu_5^2, 2\iota_{11}, \eta_{11}\} \ni \varepsilon_5$ by (7.6) of [10] with the indeterminacy of the Toda bracket being

$$\nu_5^2 \circ \pi_{13}^{11} + \pi_{12}^5 \circ \eta_{12} = \{\nu_5^2 \circ \eta_{11}^2, \sigma''' \circ \eta_{12}\} = 0$$

by (5.9) and (7.4) of [10]. So we have $\{\nu_5^2, 2\iota_{11}, \eta_{11}\} = \varepsilon_5$. In the second Toda bracket, we see $\{2\iota_{11}, \eta_{11}, \nu_{12}\} \subset \pi_{16}^{11} = 0$ by Proposition 5.9 of [10]. In the third Toda bracket we see

$$\{\eta_{11}, \nu_{12}, \sigma_{15}\} \supset -E\{\eta_{10}, \nu_{11}, \sigma_{14}\} \subset E\pi_{22}^{10} = 0$$

by Theorem 7.6 of [10]. Then we have $\{\eta_{11}, \nu_{12}, \sigma_{15}\} = 0$, since the indeterminacy of the Toda bracket is

$$\eta_{11} \circ \pi_{23}^{12} + \pi_{16}^{11} \circ \sigma_{16} = \{\eta_{11} \circ \Delta(\iota_{25}), \eta_{11} \circ \zeta_{12}\} = 0$$

by Theorem 7.4 of [10], Proposition 5.9 of [10], Proposition 17(1) and Proposition 2.2(5) of [9]. It follows that

$$\{\varepsilon_5, \nu_{13}, \sigma_{16}\} + \nu_5^2 \circ \pi_{24}^{11} + \pi_{17}^5 \circ \sigma_{17} \ni 0.$$

The indeterminacy of the above Toda bracket $\{\varepsilon_5, \nu_{13}, \sigma_{16}\}$ is

$$\begin{aligned} \varepsilon_5 \circ \pi_{24}^{13} + \pi_{17}^5 \circ \sigma_{17} &= \{\varepsilon_5 \circ \zeta_{13}, \nu_5^4 \circ \sigma_{17}, \nu_5 \circ \mu_8 \circ \sigma_{17}, \nu_5 \circ \eta_8 \circ \varepsilon_9 \circ \sigma_{17}\} \\ &= \{\nu_5 \circ \mu_8 \circ \sigma_{17}\} \end{aligned}$$

by Theorems 7.4 and 7.6 of [10], Proposition 2.20(6) of [9], (7.20) of [10] and Lemma 10.7 of [10]. Moreover, we see that

$$\begin{aligned} & \nu_5^2 \circ \pi_{24}^{11} + \pi_{17}^5 \circ \sigma_{17} \\ &= \{\nu_5^2 \circ \theta' \circ \eta_{23}, \nu_5^2 \circ \sigma_{11} \circ \nu_{18}^2, \nu_5^4 \circ \sigma_{17}, \nu_5 \circ \mu_8 \circ \sigma_{17}, \nu_5 \circ \eta_8 \circ \varepsilon_9 \circ \sigma_{17}\} \\ &= \{\nu_5 \circ \mu_8 \circ \sigma_{17}\} \end{aligned}$$

by Theorems 7.7 and 7.6 of [10], Proposition 17(2), Proposition 7(2) and the above calculation. Thus we have $\{\varepsilon_5, \nu_{13}, \sigma_{16}\} = \{\nu_5 \circ \mu_8 \circ \sigma_{17}\}$.

(2) By (1), we see that

$$\{\varepsilon_n, \nu_{n+8}, \sigma_{n+11}\} \supset \{\nu_n \circ \mu_{n+3} \circ \sigma_{n+12}\} = 0$$

for $n \geq 7$ by (7.25) of [10]. The indeterminacy of the Toda bracket is

$$\varepsilon_n \circ \pi_{n+19}^{n+8} + \pi_{n+12}^n \circ \sigma_{n+12} = \{\varepsilon_n \circ \zeta_{n+8}\} = 0$$

for $n = 7, 8, 9, 10$ and $n \geq 14$ by Theorems 7.4 and 7.6 of [10], Proposition 3(3) and Proposition 2.20(6) of [9]. Then the result follows. $\square$

THEOREM 21. $\left\{ \begin{smallmatrix} \bar{\nu}_6 & 2\nu_{14} \\ \nu_6 & \varepsilon_9 \end{smallmatrix} , \nu_{17} \right\} = 0$ in π_{21}^6 .

Proof. We remark that $\bar{\nu}_6 \circ 2\nu_{14} = \nu_6 \circ \varepsilon_9$ by (7.18) of [10] and $\varepsilon_9 \circ \nu_{17} = 0$ by (7.13) of [10]. The indeterminacy of the matrix Toda bracket is

$$\nu_6 \circ \pi_{21}^9 + \pi_{18}^6 \circ \nu_{18} + \bar{\nu}_6 \circ \pi_{21}^{14} = \{\Delta(\sigma_{13}) \circ \nu_{18}, \bar{\nu}_6 \circ \sigma_{14}\} = 0$$

by Theorem 7.6 of [10], Proposition 5.15 of [10], Proposition 17(3) and Lemma 10.7 of [10]. Since the matrix Toda bracket is in $\pi_{21}^6 = Z_4\{\rho'''\} \oplus Z_2\{\bar{\varepsilon}_6\}$ by Theorem 10.5 of [10], we may put

$$\left\{ \begin{smallmatrix} \bar{\nu}_6 & 2\nu_{14} \\ \nu_6 & \varepsilon_9 \end{smallmatrix} , \nu_{17} \right\} = x\rho''' + y\bar{\varepsilon}_6$$

for some integers x and y . We first compose the above relation with σ_{21} . The left-hand side becomes

$$\begin{aligned} \left\{ \begin{smallmatrix} \bar{\nu}_6 & 2\nu_{14} \\ \nu_6 & \varepsilon_9 \end{smallmatrix} , \nu_{17} \right\} \circ \sigma_{21} &\subset -\bar{\nu}_6 \circ \{2\nu_{14}, \nu_{17}, \sigma_{20}\} + \nu_6 \circ \{\varepsilon_9, \nu_{17}, \sigma_{20}\} \\ &= \{\bar{\nu}_6 \circ \sigma_{14}^2\} = 0 \end{aligned}$$

by Propositions 11(2) and 20(2) and Lemma 10.7 of [10]. Then we have

$$0 = \left\{ \begin{smallmatrix} \bar{\nu}_6 & 2\nu_{14} \\ \nu_6 & \varepsilon_9 \end{smallmatrix} , \nu_{17} \right\} \circ \sigma_{21} = x\rho''' \circ \sigma_{21} + y\bar{\varepsilon}_6 \circ \sigma_{21} = x\rho''' \circ \sigma_{21}$$

by Proposition 17(4). Since $\rho''' \circ \sigma_{21}$ is a generator of order four in π_{28}^6 by Theorem B of [4], we have $x \equiv 0 \pmod{4}$. Then we conclude that

$$\left\{ \begin{smallmatrix} \bar{\nu}_6 & 2\nu_{14} \\ \nu_6 & \varepsilon_9 \end{smallmatrix} , \nu_{17} \right\} = y\bar{\varepsilon}_6$$

for some integer y . Now composing the above relation with η_{21} , we have

$$\left\{ \begin{array}{c} \bar{\nu}_6, 2\nu_{14}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} \circ \eta_{21} \subset -\bar{\nu}_6 \circ \{2\nu_{14}, \nu_{17}, \eta_{20}\} + \nu_6 \circ \{\varepsilon_9, \nu_{17}, \eta_{20}\} \\ \subset \bar{\nu}_6 \circ \pi_{22}^{14} = \{\bar{\nu}_6 \circ \varepsilon_{14}, \bar{\nu}_6^2\} = 0$$

by Theorem 1(2) and Theorem 7.1 of [10] and Proposition 2.8(2) of [9]. Then we have

$$0 = \left\{ \begin{array}{c} \bar{\nu}_6, 2\nu_{14}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} \circ \eta_{21} = y\bar{\varepsilon}_6 \circ \eta_{21} = y\eta_6 \circ \bar{\varepsilon}_7$$

by Lemma 12.10 of [10]. Since $\eta_6 \circ \bar{\varepsilon}_7$ is a generator of order two by the group structure of π_{22}^6 given by Theorem 12.6 of [10], we have $y \equiv 0 \pmod{2}$. It follows that

$$\left\{ \begin{array}{c} \bar{\nu}_6, 2\nu_{14}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} = 0.$$

This completes the proof. $\square$

THEOREM 22. $\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} = \bar{\varepsilon}_6$ in π_{21}^6 .

Proof. We have $\nu_6 \circ \sigma_9 \circ \eta_{16} = \nu_6 \circ \varepsilon_9$ by a result on p. 152 of [10]. Moreover, $\eta_{16} \circ \nu_{17} = 0$ by (5.9) of [10] and $\varepsilon_9 \circ \nu_{17} = 0$ by (7.13) of [10]. This shows that the matrix Toda bracket is well-defined. Its indeterminacy is

$$\nu_6 \circ \pi_{21}^9 + \pi_{18}^6 \circ \nu_{18} + \nu_6 \circ \sigma_9 \circ \pi_{21}^{16} = \{\Delta(\sigma_{13}) \circ \nu_{18}\} = 0$$

for $\pi_{21}^{16} = 0$ by Proposition 5.9 of [10], $\pi_{21}^9 = 0$ and $\pi_{18}^6 = Z_{16}\{\Delta(\sigma_{13})\}$ by Theorem 7.6 of [10] and $\Delta(\sigma_{13}) \circ \nu_{18} = 0$ by Proposition 17(3). Hence

$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\}$ consists of a single element in $\pi_{21}^6 = Z_4\{\rho'''\} \oplus Z_2\{\bar{\varepsilon}_6\}$ by Theorem 10.5 of [10]. Since the matrix Toda bracket is in π_{21}^6 , we may put

$$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} = x\rho''' + y\bar{\varepsilon}_6$$

for some integers x and y as in the case of Theorem 21. If we compose this relation with σ_{21} , the left-hand side becomes

$$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} \circ \sigma_{21} \subset -\nu_6 \circ \sigma_9 \circ \{\eta_{16}, \nu_{17}, \sigma_{20}\} + \nu_6 \circ \{\varepsilon_9, \nu_{17}, \sigma_{20}\} = 0$$

by (7.20) of [10] and Proposition 20(2). We remark that $\{\eta_{16}, \nu_{17}, \sigma_{20}\} \subset \pi_{28}^{16} = 0$ by Theorem 7.6 of [10]. Then we have

$$0 = \left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \\ \nu_6, \varepsilon_9, \nu_{17} \end{array} \right\} \circ \sigma_{21} = x\rho''' \circ \sigma_{21} + y\bar{\varepsilon}_6 \circ \sigma_{21} = x\rho''' \circ \sigma_{21}$$

by Proposition 17(4). Since $\rho''' \circ \sigma_{21}$ is a generator of order four in π_{28}^6 by Theorem B of [4], we have $x \equiv 0 \pmod{4}$. We conclude that

$$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \nu_{17} \\ \nu_6, \varepsilon_9 \end{array} \right\} = y\bar{\varepsilon}_6$$

for some integer y . Next we compose the matrix Toda bracket with η_{21} . This yields the equality

$$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \nu_{17} \\ \nu_6, \varepsilon_9 \end{array} \right\} \circ \eta_{21} = -\nu_6 \circ \sigma_9 \circ \{\eta_{16}, \nu_{17}, \eta_{20}\} + \nu_6 \circ \{\varepsilon_9, \nu_{17}, \eta_{20}\}.$$

Now $\{\eta_{16}, \nu_{17}, \eta_{20}\} = \nu_{16}^2$ by Lemma 5.12 of [10] and $\{\varepsilon_9, \nu_{17}, \eta_{20}\} = 0$ by Theorem 1(2). Thus

$$\left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \nu_{17} \\ \nu_6, \varepsilon_9 \end{array} \right\} \circ \eta_{21} = \nu_6 \circ \sigma_9 \circ \nu_{16}^2 = \eta_6 \circ \bar{\varepsilon}_7.$$

We remark that $\bar{\varepsilon}_6 \circ \eta_{21} = \eta_6 \circ \bar{\varepsilon}_7 = \nu_6 \circ \sigma_9 \circ \nu_{16}^2$ by Lemma 12.10 of [10]. Hence

$$\eta_6 \circ \bar{\varepsilon}_7 = \left\{ \begin{array}{c} \nu_6 \circ \sigma_9, \eta_{16}, \nu_{17} \\ \nu_6, \varepsilon_9 \end{array} \right\} \circ \eta_{21} = y\bar{\varepsilon}_6 \circ \eta_{21} = y\eta_6 \circ \bar{\varepsilon}_7.$$

Since $\pi_{22}^6 = Z_8\{\zeta'\} \oplus Z_2\{\mu_6 \circ \sigma_{15}\} \oplus Z_2\{\eta_6 \circ \bar{\varepsilon}_7\}$ by Theorem 12.6 of [10], it follows that $y \equiv 1 \pmod{2}$. This completes the proof. $\square$

In the homotopy commutative diagram

$$\begin{array}{ccccccc} & & & B & & & \\ & & g \nearrow & & b \searrow & & \\ W & \xrightarrow{w} & C & & X & \xrightarrow{s} & Y \\ & & f \searrow & & a \nearrow & & \\ & & A & & & & \end{array}$$

suppose that $g \circ w = 0$, $f \circ w = 0$, $s \circ a = 0$ and $s \circ b = 0$. Then, by Corollary 4.6 of [3], the equation

$$s \circ \left\{ \begin{array}{c} b, g, w \\ a, f \end{array} \right\} = \left\{ s, b, g \right\} \circ Ew$$

is valid. We shall use this equation to deduce the following theorem from Theorem 22.

THEOREM 23. $\left\{ \begin{array}{c} \nu', \nu_6 \circ \sigma_9, \eta_{16} \\ \nu_6, \varepsilon_9 \end{array} \right\} = \bar{\varepsilon}_3$ in π_{18}^3 .

Proof. The matrix Toda bracket is well-defined, since $\nu' \circ \nu_6 \in \pi_9^3 = 0$ by Proposition 5.11 of [10] and $\nu_6 \circ \sigma_9 \circ \eta_{16} = \nu_6 \circ \varepsilon_9$ as is remarked at the beginning of the proof of Theorem 22.

We consider the following homotopy commutative diagram:

$$\begin{array}{ccccccc}
 & & & & S^{16} & & \\
 & & \nearrow \eta_{16} & & \searrow \nu_6 \circ \sigma_9 & & \\
 S^{20} & \xrightarrow{\nu_{17}} & S^{17} & & & & S^6 \xrightarrow{\nu'} S^3 \\
 & & \searrow \varepsilon_9 & & \nearrow \nu_6 & & \\
 & & & & S^9 & &
 \end{array}$$

We observe that $\eta_{16} \circ \nu_{17} = 0$, $\varepsilon_9 \circ \nu_{17} = 0$ and $\nu' \circ \nu_6 = 0$ by (5.9), (7.13) and Proposition 5.11 respectively of [10]. Thus the equation which is mentioned just before this theorem is applicable. In view of Theorem 22, this yields the equality

$$\nu' \circ \bar{\varepsilon}_6 = \nu' \circ \left\{ \begin{array}{c} \nu_6 \circ \sigma_9 \\ \nu_6 \end{array}, \begin{array}{c} \eta_{16} \\ \varepsilon_9 \end{array}, \nu_{17} \right\} = \left\{ \nu', \begin{array}{c} \nu_6 \circ \sigma_9 \\ \nu_6 \end{array}, \begin{array}{c} \eta_{16} \\ \varepsilon_9 \end{array} \right\} \circ \nu_{18}$$

in $\pi_{21}^3 = Z_4\{\mu' \circ \sigma_{14}\} \oplus Z_2\{\nu' \circ \bar{\varepsilon}_6\} \oplus Z_2\{\eta_3 \circ \bar{\mu}_4\}$ by Theorem 12.8 of [10]. We have $\nu' \circ \bar{\varepsilon}_6 = \bar{\varepsilon}_3 \circ \nu_{18} \neq 0$ by Proposition 2.17(4) of [9]. Hence, because

$$\left\{ \nu', \begin{array}{c} \nu_6 \circ \sigma_9 \\ \nu_6 \end{array}, \begin{array}{c} \eta_{16} \\ \varepsilon_9 \end{array} \right\} \subset \pi_{18}^3 = Z_2\{\bar{\varepsilon}_3\}$$

by Theorem 10.5 of [10], $\left\{ \nu', \begin{array}{c} \nu_6 \circ \sigma_9 \\ \nu_6 \end{array}, \begin{array}{c} \eta_{16} \\ \varepsilon_9 \end{array} \right\}$ must consist of the single element $\bar{\varepsilon}_3$ and the theorem is established. $\square$

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Howard J. Marcum
The Ohio State University at Newark
1179 University Drive
Newark, Ohio 43055
USA
(E-mail: marcum@math.ohio-state.edu)

Nobuyuki Oda
Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka 814-0180
Japan
(E-mail: odanobu@cis.fukuoka-u.ac.jp)