CANCELLATION LAW FOR INFINITE COMPLEXES

Yoshimi SHITANDA

(Received 28 August 2000 and revised 17 November 2000)

1. Introduction

In this paper we work in the category of based CW -complexes and based maps. We are able to construct a homotopy category as usual. In the homotopy category, there exist wedge sums $\vee$ and product operations $\times$. We propose the following problems. We use the same terminologies and notation of [6] in this paper.

Problem 1. Is a decomposition into wedge sums uniquely determined? That is, if $X_1 \vee X_2 \vee \cdots \vee X_n$ and $Y_1 \vee Y_2 \vee \cdots \vee Y_n$ are homotopy equivalent and X_i and Y_i are prime spaces, then does it hold that X_i and $X_{\sigma(i)}$ are homotopy equivalent for a permutation σ ?

Problem 2. Does a cancellation law hold in the homotopy category? That is, if $X_1 \vee X_2 \vee \cdots \vee X_n$ and $Y_1 \vee Y_2 \vee \cdots \vee Y_n$ are homotopy equivalent and also for X_1 and Y_1 , then does it hold that $X_2 \vee \cdots \vee X_n$ and $Y_2 \vee \cdots \vee Y_n$ are homotopy equivalent?

Generally the answers to these problems are no. For the homotopy category of finite CW -complexes, Freyd [1] and other topologists [2–4, 8] gave some examples and studied them in detail. For the homotopy category of infinite CW -complexes, the author [6] gave some examples and studied the above problems. In this paper, we continue their study and obtain a solution. In this paper, we assume that a phantom map is a map from $\Sigma^i L(I, J)$ to S^{i+3} for $i \geq 0$ where $L(I, J)$ is a space obtained by mixing localized spaces CP_I^∞ and $\Omega(S^3)_J$ over $K(Q, 2)$. Our main theorems are as follows.

THEOREM 1.1. *Let f be a phantom map of infinite order and $f_i = a_i f$ for integers a_i , ($i = 1, 2, \dots, n$). Then, there exist phantom maps g_i ($i = 1, 2, \dots, n$) of infinite order satisfying the following conditions:*

- (1) $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent;

- (2) $g_i = b_i g$ for a phantom map g and integers b_i ($i = 1, 2, \dots, n$) and b_{i+1} is divided by b_i for $i = 1, 2, \dots, n-1$.

THEOREM 1.2. *Let f_i and g_i ($i = 1, 2, \dots, n$) be phantom maps of infinite order and f_1 and g_1 homotopic maps. If two wedge sums $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent, $\bigvee_{j=2}^n C(f_j)$ and $\bigvee_{j=2}^n C(g_j)$ are homotopy equivalent.*

THEOREM 1.3. *Let f_i ($i = 1, 2, \dots, n$) be phantom maps of finite order. Then there exist phantom maps g_i ($i = 1, 2, \dots, n$) of finite order satisfying the following conditions:*

- (1) $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent;
 (2) $\text{ord}(g_i)$ is divided by $\text{ord}(g_{i+1})$.

THEOREM 1.4. *Let f_i and g_i ($i = 1, 2, \dots, n$) be phantom maps of finite order such that $\text{ord}(f_i)$ is divided by $\text{ord}(f_{i+1})$ and also for the g_i . If $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent, it holds that $\text{ord}(f_i) = \text{ord}(g_i)$ for $i = 1, 2, \dots, n$.*

2. Preliminary

We define a space $L(I, J)$ in which to study our problems. Let CP^∞ and $\Omega(S^3)$ be an infinite-dimensional projective space and a loop space of a 3-sphere, respectively, and spaces CP_I^∞ and $\Omega(S^3)_J$ their localizations at sets I and J of prime numbers. Let $L(I, J)$ be a space given by the following pull-back diagram where (I, J) is a partition of all primes, I a non-empty set and the maps from $\Omega(S^3)_J$ and CP_I^∞ to $K(Q, 2)$ are rationalizations:

$$\begin{array}{ccc} L(I, J) & \longrightarrow & \Omega(S^3)_J \\ \downarrow & & \downarrow \\ CP_I^\infty & \longrightarrow & K(Q, 2) \end{array}$$

The homology groups of the space $L(I, J)$ are a free group Z in each even dimension and zero in each odd dimension. Note that the space $L(I, J)$ is an H-space. Although the calculation of a homotopy set $[\Sigma^i L(I, J), S^{i+3}]$ is difficult, it is comparatively easy to calculate the homotopy set $Ph(\Sigma^i L(I, J), S^{i+3})$ of phantom maps. Now we state our results of [6, 7].

THEOREM 1. *$Ph(\Sigma^i L(I, J), S^{i+3})$ is isomorphic to $\hat{Z}_I/Z_I \oplus \bigoplus_{q \in J} Z_{q^\infty}$ where $\hat{Z}_I/Z_I$ is a vector space over Q with uncountable basis and Z_{q^∞} is a torsion group of a component of prime q of Q/Z .*

In the previous papers [5, 6], we classified the homotopy type of mapping cones $C(f)$ of phantom map $f : \Sigma^i L(I, J) \rightarrow S^{i+3}$.

THEOREM 2. *Let f and g be phantom maps from $\Sigma^i L(I, J)$ to S^{i+3} . Then the loop spaces of the mapping cones $\Omega^i C(f)$ and $\Omega^i C(g)$ are homotopy equivalent if and only if f and $\pm g$ are homotopic for $i = 0, 1, \dots, \infty$.*

By making use of these theorems, we studied cancellation and non-cancellation phenomena under the wedge sum operation in the homotopy category. We obtained the partial results for the phenomena in a previous paper [6]. In this paper, we prove some results about the wedge sum decomposition similar to the fundamental theorem of finitely generated abelian groups. From now on, by phantom maps we mean maps from $\Sigma^i L(I, J)$ to S^{i+3} for $i \geq 0$.

THEOREM 3. *Let f_i and f'_i ($i = 1, 2, \dots, m$) be phantom maps of infinite order and g_j and g'_j ($j = 1, 2, \dots, n$) phantom maps of finite order. If two wedge sums $\bigvee_{i=1}^m C(f_i) \vee \bigvee_{j=1}^n C(g_j)$ and $\bigvee_{i=1}^m C(f'_i) \vee \bigvee_{j=1}^n C(g'_j)$ are homotopy equivalent, $\bigvee_{i=1}^m C(f_i)$ and $\bigvee_{i=1}^m C(f'_i)$ are homotopy equivalent and also for $\bigvee_{j=1}^n C(g_j)$ and $\bigvee_{j=1}^n C(g'_j)$.*

Proof. Set $F = \text{Diag}(f_1, \dots, f_m)$ and $F \oplus G = \text{Diag}(f_1, \dots, f_m, g_1, \dots, g_n)$ where the right-hand sides are diagonal matrices with components $f_1, \dots, g_1, \dots$ etc, and also for $F', F' \oplus G'$. A map ϕ is a homotopy equivalence between $C(F \oplus G)$ and $C(F' \oplus G')$.

Consider the following diagram:

$$\begin{array}{ccccccc} (m+n)L(I, J) & \xrightarrow{F \oplus G} & (m+n)S^3 & \xrightarrow{j} & C(F \oplus G) & \xrightarrow{q} & \Sigma(m+n)L(I, J) \xrightarrow{\Sigma(F \oplus G)} \\ & & & & \downarrow \phi & & \\ (m+n)L(I, J) & \xrightarrow{F' \oplus G'} & (m+n)S^3 & \xrightarrow{j'} & C(F' \oplus G') & \xrightarrow{q'} & \Sigma(m+n)L(I, J) \xrightarrow{\Sigma(F' \oplus G')} \end{array}$$

where the horizontal sequences are Puppe sequences induced by $F \oplus G : (m+n)L(I, J) \rightarrow (m+n)S^3$ and $F' \oplus G' : (m+n)L(I, J) \rightarrow (m+n)S^3$, respectively. We will find a map from $(m+n)S^3$ to $(m+n)S^3$ which makes the diagram commutative. A homotopy equivalence $\phi : C(F \oplus G) \rightarrow C(F' \oplus G')$ gives the following matrix which is a homomorphism on three-dimensional homology groups:

$$\begin{aligned} H_3(\phi) &= (A_{ij}) \quad (i, j = 1, 2, 3, 4) \\ A_{ij} : H_3(X_j) &\rightarrow H_3(X_i) \end{aligned}$$

where X_i is mS^3 , nS^3 , $m\Sigma L(I, J)$ and $n\Sigma L(I, J)$ for $i = 1, 2, 3$ and 4 , respectively. Small matrices A_{31} and A_{41} of the matrix are zero by the same method of Theorem 2.3 and Proposition 3.2 in [6], and small matrices A_{13} , A_{14} , A_{23} and A_{24} are zero by (co)homology. Hence, we have obtained the following matrix:

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & A_{43} & A_{44} \end{pmatrix}.$$

Consider the following composition:

$$(m+n)L(I, J) \xrightarrow{F \oplus G} (m+n)S^3 \xrightarrow{j} C(F \oplus G) \xrightarrow{\phi} C(F' \oplus G') \xrightarrow{r} nS^3$$

where the first three terms are a Puppe sequence of a map $F \oplus G$ and the last map $C(F' \oplus G') \rightarrow nS^3$ is given by Proposition 3.4 in [6]. Since the composed map is homotopic to a constant map, we can easily see $A_{21} = 0$ by Proposition 3.2 in [6]. In the same way as above, we can see that the i th column of A_{32} , A_{42} is a multiple of $\text{ord}(g_i)$ ($1, 2, \dots, n$). By using Propositions 3.4 and 3.5 in [6], we can also construct a map $\psi : C(F \oplus G) \rightarrow C(F \oplus G)$ which induces the following matrix on $H_3(C(F \oplus G))$:

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & E_1 & I & 0 \\ 0 & E_2 & 0 & I \end{pmatrix}.$$

By choosing E_1 and E_2 well, $H_3(\phi\psi)$ becomes a matrix with $A_{32} = 0$, $A_{42} = 0$. Hence we may assume that a matrix A has the following form:

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}.$$

From the above matrix, we obtain a map $\alpha : (m+n)S^3 \rightarrow (m+n)S^3$ and the following commutative diagram:

$$\begin{array}{ccccccc} (m+n)L(I, J) & \xrightarrow{F \oplus G} & (m+n)S^3 & \xrightarrow{j} & C(F \oplus G) & \xrightarrow{q} & \Sigma(m+n)L(I, J) \xrightarrow{\Sigma(F \oplus G)} \\ & & \downarrow & & \downarrow \phi & & \downarrow \\ (m+n)L(I, J) & \xrightarrow{F' \oplus G'} & (m+n)S^3 & \xrightarrow{j'} & C(F' \oplus G') & \xrightarrow{q'} & \Sigma(m+n)L(I, J) \xrightarrow{\Sigma(F' \oplus G')} \end{array}$$

Hence we obtain the following equation from the last square:

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} F' & 0 \\ 0 & G' \end{pmatrix} \begin{pmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{pmatrix}.$$

From this equation, we have $A_{12}G = F'A_{34}$. Since the components of G are of finite order, the components of the left-hand side are of finite order. Since the components of the right-hand side are of infinite order or zero, we see that $A_{34} = 0$. From the result, we obtain $A_{11}F = F'A_{33}$ and $A_{22}G = G'A_{44}$ where A_{11} , A_{22} , A_{33} and A_{44} are unimodular matrices. Hence we obtain the result. $\square$

The following theorem is proved in a way entirely similar to the above theorem and Theorem 2.1 of [6].

THEOREM 4. *Let f_i ($i = 1, 2, \dots, m$) be phantom maps of infinite order which are linearly independent over $\mathcal{Q}$ and also for g_i ($i = 1, 2, \dots, m$). Let $f_{ij} = c_{ij}f_i$ ($j = 1, \dots, p_i$), $g_{ij} = d_{ij}g_i$ ($j = 1, 2, \dots, q_i$) where c_{ij} and d_{ij} are non-zero rational numbers. If two wedge sums $\bigvee_{i,j} C(f_{ij})$ and $\bigvee_{i,j} C(g_{ij})$ are homotopy equivalent, $\bigvee_{j=1}^{p_i} C(f_{ij})$ and $\bigvee_{j=1}^{q_k} C(g_{kj})$ are homotopy equivalent where $k = \sigma(i)$ for a permutation σ and multiples of f_i and $g_{\sigma(i)}$ are homotopic maps for each i .*

3. Main theorems

Let f be a phantom map of infinite order. Then, the following formula is given in section 2 in [6] for suspended maps f and g :

$$C(maf) \vee C(naf) \simeq C(mnaf) \vee C(af) \quad (3.1)$$

where m and n are relatively prime integers and a is an integer. It is proved by the following matrix calculation in the above case that

$$\begin{pmatrix} mf & 0 \\ 0 & nf \end{pmatrix} \begin{pmatrix} 1 & -nt \\ 1 & ms \end{pmatrix} = \begin{pmatrix} m & -t \\ n & s \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & mnf \end{pmatrix}$$

where $ms + nt = 1$. Hence the uniqueness of decomposition fails by the formula (3.1). From the formula, we have a normal form of the decomposition which is a unique decomposition in a sense. At first we prove the formula (3.1) for the general case, that is, non-suspended maps f, g .

Now we have to prepare a lemma to construct a homotopy equivalence for non-suspended spaces.

LEMMA 1. *Let f be a phantom map of infinite or finite order from $\Sigma^k L(I, J)$ to S^{k+3} for $k \geq 0$. Then, there exists a codiagonal map:*

$$\Delta : C(f) \rightarrow C(f) \vee C(f).$$

Proof. Since $jf : L(I, J) \rightarrow S^3 \rightarrow C(f)$ is zero homotopic, we have $j_*(f) = 0$ where j_* is given by the following diagram:

$$j_* : Ph(L(I, J), S^3) \rightarrow Ph(L(I, J), C(f)).$$

Since $j_*(f)$ is zero, the composition $(j \vee j)df : L(I, J) \rightarrow S^3 \rightarrow S^3 \vee S^3 \rightarrow C(f) \vee C(f)$ is homotopic to the constant map by using the following diagram (cf. Theorem D of [9]):

$$\begin{array}{ccc} Ph(L(I, J), S^3 \vee S^3) & \xrightarrow{\cong} & Ph(L(I, J), S^3) \oplus Ph(L(I, J), S^3) \\ \downarrow (j \vee j)_* & & \downarrow j_* \oplus j_* \\ Ph(L(I, J), C(f) \vee C(f)) & \xrightarrow{\cong} & Ph(L(I, J), C(f)) \oplus Ph(L(I, J), C(f)) \end{array}$$

where the left homomorphism is induced by $j \vee j : S^3 \vee S^3 \rightarrow C(f) \vee C(f)$.

Consider the following diagram:

$$\begin{array}{ccccccc} L(I, J) & \xrightarrow{f} & S^3 & \xrightarrow{j} & C(f) & \xrightarrow{q} & \Sigma L(I, J) \xrightarrow{\Sigma f} \\ & & \downarrow d & & \downarrow \Delta' & & \downarrow \\ L(I, J) \vee L(I, J) & \xrightarrow{f \vee f} & S^3 \vee S^3 & \xrightarrow{j \vee j} & C(f) \vee C(f) & \xrightarrow{q \vee q} & \Sigma L(I, J) \vee \Sigma L(I, J) \xrightarrow{\Sigma f \vee \Sigma f} \\ & & \downarrow p_i & & \downarrow p_i & & \downarrow \\ L(I, J) & \xrightarrow{f} & S^3 & \xrightarrow{j} & C(f) & \xrightarrow{q} & \Sigma L(I, J) \xrightarrow{\Sigma f} \end{array}$$

Here, three horizontal sequences are Puppe sequences induced by f , $f \vee f$ and f , respectively, $d : S^3 \rightarrow S^3 \vee S^3$ is a codiagonal map and $p_i : X \vee X \rightarrow X$ is a projection for $X = S^3, C(f)$ etc, $i = 1, 2$. We can find out a map $\Delta' : C(f) \rightarrow C(f) \vee C(f)$ by the above consideration. If a composed map $p_i \Delta'$ is not homotopic to id, we have two identity maps $(p_i \Delta', k_i) \mu : C(f) \rightarrow C(f)$ by adding maps $k_i : \Sigma L(I, J) \rightarrow C(f)$ ($i = 1, 2$) where $\mu : C(f) \rightarrow C(f) \vee \Sigma L(I, J)$ is a coaction. Hence we obtain a codiagonal map by adding a map $(k_1 \vee k_2) \nu : \Sigma L(I, J) \rightarrow C(f) \vee C(f)$. Now Δ is given by the following composition:

$$\Delta : C(f) \xrightarrow{\mu} C(f) \vee \Sigma L(I, J) \xrightarrow{\mu \vee \text{id}} C(f) \vee \Sigma L(I, J) \vee \Sigma L(I, J) \longrightarrow C(f) \vee C(f)$$

where the third map is given naturally by Δ' and k_i . $\square$

We can easily construct a map $\alpha_m : C(f) \rightarrow C(mf)$ which is degree m on S^3 and degree one on cone $L(I, J)$ respectively and $\beta_n : C(mnf) \rightarrow C(mf)$ which is degree one on S^3 and degree n on cone $L(I, J)$ respectively:

$$\begin{array}{ccc} L(I, J) & \xrightarrow{f} & S^3 \xrightarrow{j} C(f) \\ \downarrow \text{id} & & \downarrow m \quad \downarrow \alpha_m \\ L(I, J) & \xrightarrow{mf} & S^3 \xrightarrow{j_m} C(mf) \end{array} \quad \begin{array}{ccc} L(I, J) & \xrightarrow{mnf} & S^3 \xrightarrow{j_{mn}} C(mnf) \\ \downarrow n & & \downarrow \text{id} \quad \downarrow \beta_n \\ L(I, J) & \xrightarrow{mf} & S^3 \xrightarrow{j_m} C(mf) \end{array}$$

Before giving a homotopy equivalence (3.1), we have to prepare the following. Define a map $\gamma_k : C(f) \rightarrow C(f)$ by the following diagram:

$$\begin{array}{ccc} L(I, J) & \xrightarrow{f} & S^3 \xrightarrow{j} C(f) \\ \downarrow k & & \downarrow k \quad \downarrow \gamma_k \\ L(I, J) & \xrightarrow{f} & S^3 \xrightarrow{j} C(f) \end{array}$$

where the first vertical arrow is a map induced by the H-space structure of degree k on the two-dimensional (co)homology group, and the second one is a map of degree k . A map $\gamma'_k : C(f) \rightarrow C(f)$ is defined as follows:

$$\gamma'_k : C(f) \rightarrow C(f) \vee \cdots \vee C(f) \rightarrow C(f)$$

where the first arrow is a k -fold codiagonal map and the second one is a folding map. The homomorphisms on homology groups are given as follows:

$$(\gamma)_*(u) = ku, \quad (\gamma)_*(v_i) = k^i v_i, \quad (\gamma')_*(u) = ku, \quad (\gamma')_*(v_i) = kv_i$$

where u and v_i are three-dimensional and $(2i + 1)$ -dimensional generators of $H_3(S^3)$ and $H_3(\Sigma L(I, J))$, respectively. By the exactness of the Puppe sequence, we obtain a relation $(\gamma_k, \delta)\mu = \gamma'_k$ for a map $\delta : \Sigma L(I, J) \rightarrow C(f)$.

We define β'_n and β'_m by adding maps δ_1 and δ_2 to β_n and β_m as above, respectively. The maps β'_n and β'_m satisfy the relations

$$(\beta'_n)_*(u) = u, \quad (\beta'_n)_*(v_i) = nv_i, \quad (\beta'_m)_*(u) = u, \quad (\beta'_m)_*(v_i) = mv_i.$$

A homotopy equivalence for (3.1) is given by the following composition:

$$C(f) \vee C(mnf) \rightarrow C(f) \vee C(f) \vee C(mnf) \vee C(mnf) \rightarrow C(mf) \vee C(nf)$$

where the first map is $\Delta \vee \Delta$ and the second map is given by the following map:

$$\begin{pmatrix} \alpha_m & 0 & -t\beta'_n & 0 \\ 0 & \alpha_n & 0 & s\beta'_m \end{pmatrix}.$$

Since it induces an isomorphism on homology groups, it is a homotopy equivalence. From the consideration above, we have the following lemma.

LEMMA 2. *Let f be a phantom map of infinite or finite order from $\Sigma^k L(I, J)$ to S^{k+3} for $k \geq 0$. Then there exists a homotopy equivalence*

$$C(af) \vee C(mnaf) \simeq C(maf) \vee C(naf)$$

where m and n are relatively prime integers and a is an integer.

From the lemma, we have the following theorem.

THEOREM 5. *Let f be a phantom map of infinite order and $f_i = a_i f$ for integers a_i , ($i = 1, 2, \dots, n$). Then there exist phantom maps g_i ($i = 1, 2, \dots, n$) of infinite order satisfying the following conditions:*

- (1) $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent;
- (2) $g_i = b_i g$ for a phantom map g and integers b_i ($i = 1, 2, \dots, n$) and b_{i+1} divided by b_i for $i = 1, 2, \dots, n-1$.

Although the uniqueness of decomposition fails, it holds for the case of phantom maps of infinite order in the sense of normal form. Moreover, we have the following cancellation theorem.

THEOREM 6. *Let f_i and g_i ($i = 1, 2, \dots, n$) be phantom maps of infinite order and f_1 and g_1 homotopic maps. If two wedge sums $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent, $\bigvee_{j=2}^n C(f_j)$ and $\bigvee_{j=2}^n C(g_j)$ are homotopy equivalent.*

Proof. By Theorem 4, it is sufficient to prove the case for $f_i = a_i f$, $g_i = b_i f$ and $a_1 = b_1$ where a_i, b_i are integers and f a phantom map. As in the proof of Theorem 3, we may translate the theorem into a matrix problem. Now we may assume a_{i+1} is divided by a_i ($i = 2, \dots, n$) and also for b_{i+1} and b_i . We set

- (a) $a_1 = \alpha_1, a_i = \alpha_2 \cdots \alpha_i$ ($i = 2, \dots, n$);
- (b) $p_1 = (a_1, \alpha_2), p_i = (a_1/p_1 \cdots p_{i-1}, \alpha_{i+1})$ $i = 2, \dots, n-1, p_n = a_1/p_1 \cdots p_{n-1}$.

Here (a, b) means the greatest common divisor of integers a and b . By making use of Lemma 2, a diagonal matrix $\text{Diag}(a_1, a_2, \dots, a_n)$ can be transformed to $\text{Diag}(c_1, c_2, \dots, c_n)$ by elementary unimodular matrices. We obtain the following conditions inductively by the above conditions:

- (1) $a_1 = p_1 p_2 \cdots p_n$;
- (2) $a_i = p_1 p_2 \cdots p_{i-1} m_2 \cdots m_i, \alpha_i = p_{i-1} m_i$ for $i > 1$;
- (3) $c_i = p_1 p_2 \cdots p_i m_2 \cdots m_i$ for $i > 1, c_1 = p_1$;

- (4) m_i is prime to $p_i \cdots p_n$ for $i > 1$.

Similarly it holds for $\text{Diag}(b_1, b_2, \dots, b_n)$ satisfying the following conditions:

- (5) $b_1 = q_1 q_2 \cdots q_n$;
 (6) $b_i = q_1 q_2 \cdots q_{i-1} l_2 \cdots l_i$, $\beta_i = q_{i-1} l_i$ for $i > 1$;
 (7) $d_i = q_1 q_2 \cdots q_i l_2 \cdots l_i$ for $i > 1$, $d_1 = q_1$;
 (8) l_i is prime to $q_i \cdots q_n$ for $i > 1$.

By the uniqueness of the normal form of a diagonal matrix, we see $c_i = d_i$ and $p_i m_i = q_i l_i$ by $c_i/c_{i-1} = d_i/d_{i-1}$ for $i = 1, 2, \dots, n$. By $c_n = d_n$ and $a_1 = b_1$, we see $m_2 \cdots m_n = l_2 \cdots l_n$. Hence we have the following equation:

$$\frac{p_n m_n}{m_2 \cdots m_n} = \frac{q_n l_n}{l_2 \cdots l_n}.$$

By cancelling m_n and l_n , we see $p_n = q_n$, $m_n = l_n$ and $m_2 \cdots m_{n-1} = l_2 \cdots l_{n-1}$ by the above conditions (4) and (8). By using a descending induction, we see $p_i = q_i$, $m_i = l_i$ and $a_i = b_i$. Hence we obtain the result. $\square$

From the above results, we obtain the cancellation law and the uniqueness of decomposition in the sense of the normal forms for mapping cones of phantom maps of infinite order under the wedge sum operation. Hereafter, we examine the case for phantom maps of finite order. Now we prepare some lemmas of elementary group theory.

LEMMA 3. *Let m, n and a be integers such that m and n are relatively prime, and f and g are generators of cyclic groups Z/ma and Z/na , respectively. Then, there exist integers c and d such that $cf + dg$ is a generator of Z/mna and c and d are relatively prime. Here Z/ma and Z/na are considered as subgroups of Z/mna .*

Proof. We choose integers $\overline{f}$ and $\overline{g}$ which represent f and g , respectively. By assumption, there exist integers x, y, z, s, t and w satisfying $\overline{f}x + may = 1$, $\overline{g}z + naw = 1$ and $ms + nt = 1$. Set $\overline{F} = nxt\overline{f} + mzs\overline{g}$. This represents a generator of Z/mna by $nt(x\overline{f} + may) + ms(z\overline{g} + naw) = 1$. Moreover, we may assume $\overline{F} = cn\overline{f} + dm\overline{g} = 1$ by taking $(n\overline{f}, m\overline{g}) = 1$ because of a theorem of Dirichlet. Hence $F = cf + dg$ is a generator. $\square$

LEMMA 4. *Let f and g be generators of Z/ma and Z/na , respectively, and m and n relatively prime. There exist unimodular matrices P and Q satisfying the following:*

- (1) $P \text{Diag}(f, g) = \text{Diag}(F, G)Q$;
 (2) $\text{ord}(F) = amn$ and $\text{ord}(G) = a$.

Proof. By Lemma 3, we find a generator F of Z/mna which is represented by $\overline{F} = cn\overline{f} + dm\overline{g} = 1$. Set $\overline{G} = mn\overline{g}\overline{f}$ which is a representation of G . Since it holds $F = cf + dg$ and $G = m\overline{g}f = n\overline{f}g$, we see $\text{ord}(F) = amn$ and $\text{ord}(G) = a$ where $\overline{f}nF = f$, $\overline{g}mF = g$. We obtain the result by defining matrices P and Q as follows:

$$P = \begin{pmatrix} 1 & d \\ -\overline{g}m & cn\overline{f} \end{pmatrix}, \quad Q = \begin{pmatrix} n\overline{f} & dm\overline{g} \\ -1 & c \end{pmatrix}. \quad \square$$

From this lemma, we obtain a normal form of wedge sum decomposition for mapping cones of phantom maps of finite order. From the above equation, we have the following maps:

$$\begin{aligned} \alpha : C(f) &\rightarrow C(F), & \beta : C(g) &\rightarrow C(F) \\ \gamma : C(f) &\rightarrow C(G), & \delta : C(g) &\rightarrow C(G). \end{aligned}$$

For example, α is defined by the following diagram:

$$\begin{array}{ccccc} L(I, J) & \xrightarrow{f} & S^3 & \xrightarrow{j} & C(f) \\ \downarrow n\overline{f} & & \downarrow \text{id} & & \downarrow \alpha \\ L(I, J) & \xrightarrow{F} & S^3 & \xrightarrow{J} & C(F) \end{array}$$

where the first vertical arrow is a map of degree $n\overline{f}$ and the second one is an identity map. The other maps are defined similarly. As in the proof of Lemma 2, we have the following homotopy equivalence:

$$C(f) \vee C(g) \simeq C(F) \vee C(G) \quad (3.2)$$

From this result, we have the following theorem.

THEOREM 7. *Let f_i ($i = 1, 2, \dots, n$) be phantom maps of finite order. There exist phantom maps g_i ($i = 1, 2, \dots, n$) of finite order satisfying the following conditions:*

- (1) $\bigvee_{i=1}^n C(f_i) \simeq \bigvee_{i=1}^n C(g_i)$;
- (2) $\text{ord}(g_i)$ is divided by $\text{ord}(g_{i+1})$.

We study Problem 2. That is, does a cancellation law hold for phantom maps of finite order? We can easily give counterexamples.

Example 1. Let f be a phantom map of order mn and integers a, n satisfy $ax + mny = 1$ for integers x, y . Set $g = mf$. If g satisfies $ag = g$, the following equation holds:

$$\begin{pmatrix} af & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} x & m \\ -ny & a \end{pmatrix} = \begin{pmatrix} xa & 1 \\ -mny & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}.$$

Hence, we see that $C(af) \vee C(g) \simeq C(f) \vee C(g)$. Here $\text{ord}(f) = \text{ord}(af) = mn$, $\text{ord}(g) = n$. This shows that a cancellation law does not hold.

Example 2. Let f be phantom maps of order mn and $g = mf$. Then, we choose integers a, s such that $as + mnt = 1$ and $sx + ny = 1$. We have the following equation:

$$\begin{pmatrix} a & -tn \\ m & s \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} af & 0 \\ 0 & sg \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

From this, we see that $C(f) \vee C(g) \simeq C(af) \vee C(sg)$. Generally, $C(f)$ and $C(af)$ are not homotopy equivalent and also for $C(g)$ and $C(sg)$.

Although a cancellation law dose not hold for the case of phantom maps of finite order, the uniqueness of decomposition holds in some sense. That is, we have the following lemma.

LEMMA 5. *Let f_i, g_i ($i = 1, 2, \dots, n$) be phantom maps of finite order such that $\text{ord}(f_i)$ is divided by $\text{ord}(f_{i+1})$ and also for g_i 's. If there exist unimodular matrices A, B such that*

$$A \text{Diag}(f_1, f_2, \dots, f_n) = \text{Diag}(g_1, g_2, \dots, g_n)B \quad (*)$$

then $\text{ord}(f_i) = \text{ord}(g_i)$ for $i = 1, 2, \dots, n$.

Proof. Note that $\text{ord}(f)$ is equal to 1 for a trivial phantom map f . We set $\text{ord}(f_i) = m_i \cdots m_n$, $\text{ord}(g_i) = l_i \cdots l_n$ for $i = 1, 2, \dots, n$. From equation (*), we have

(a)

$$a_{ij}f_j = g_ib_{ij} \quad (i, j = 1, 2, \dots, n).$$

Since $\det A = +1, -1$,

(b)

$$\begin{aligned} f_k &= \pm(\det A)f_k \\ &= \pm \sum a_{\sigma(1)1} \cdots a_{\sigma(k)k} \cdots a_{\sigma(n)n} f_k \\ &= \pm \sum a_{\sigma(1)1} \cdots a_{\sigma(k-1)k-1} b_{\sigma(k)k} a_{\sigma(k+1)k+1} \cdots a_{\sigma(n)n} g_{\sigma(k)} \\ &= \text{Sum of type 1} + \text{Sum of type 2.} \end{aligned}$$

Here, type 1 is the summation satisfying $\sigma(k)$ greater than $k-1$ and type 2 is that with $\sigma(k)$ smaller than k . Clearly the order of type 1 is equal to the factor of $\text{ord}(g_k)$. We prove the theorem by induction on k . Since for the case $k = 1$, there is no term of type 2, $\text{ord}(f_1)$ divides $\text{ord}(g_1)$, that is, $m_1 m_2 \cdots m_n$ divides $l_1 l_2 \cdots l_n$. Conversely, $\text{ord}(g_1)$

divides $\text{ord}(f_1)$, hence $\text{ord}(f_1) = \text{ord}(g_1)$. Now we assume $\text{ord}(f_i) = \text{ord}(g_i)$ for $i < k$. From (a) and the hypothesis of induction, $a_{\sigma(i)i}$ is divided by $l_i l_{i+1} \cdots l_{\sigma(i)-1}$ for the case when $\sigma(i)$ is greater than i , $i = 1, 2, \dots, k-1$. Define N_σ for a permutation σ as follows:

$$N_\sigma = \prod_{\substack{\sigma(i) > i, \\ i=1, \dots, k-1}} (l_i l_{i+1} \cdots l_{\sigma(i)-1}).$$

Since N_σ is divided by $l_{\sigma(k)} \cdots l_{k-1}$ for the case of $\sigma(k) < k$ in type 2, $N_\sigma g_{\sigma(k)}$ has an order $l_k l_{k+1} \cdots l_n$ at most. Since $a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(k-1)k-1}$ is divided by N_σ , type 2 has an order $l_k l_{k+1} \cdots l_n$ at most. Since the order of type 1 divides $l_k l_{k+1} \cdots l_n$, we see that the order of f_k divides $l_k l_{k+1} \cdots l_n$. Conversely the order of g_k divides $m_k m_{k+1} \cdots m_n$. Hence $\text{ord}(f_i) = \text{ord}(g_i)$. $\square$

From the above lemma and the proof as in Theorem 3, we obtain the following theorem.

THEOREM 8. *Let f_i ($i = 1, 2, \dots, n$) be phantom maps of finite order such that $\text{ord}(f_i)$ is divided by $\text{ord}(f_{i+1})$ and also for g_i 's. If $\bigvee_{i=1}^n C(f_i)$ and $\bigvee_{i=1}^n C(g_i)$ are homotopy equivalent, it holds that $\text{ord}(f_i) = \text{ord}(g_i)$ for $i = 1, 2, \dots, n$.*

Proof. Let $\phi : \bigvee_{j=1}^n C(f_j) \rightarrow \bigvee_{j=1}^n C(g_j)$ be a homotopy equivalence. As in the proof of Theorem 3, we consider

$$H_3(\phi) = (A_{ij}) \quad (i, j = 1, 2)$$

$$A_{ij} : H_3(X_j) \rightarrow H_3(X_i),$$

where X_i is nS^3 and $n\Sigma L(I, J)$ for $i = 1$ and 2 , respectively. We see that the small matrix A_{12} is zero and the i th column of A_{21} is a multiple of $\text{ord}(f_i)$ by the same method of Section 3 in [6]. By Proposition 3.5 of [6], we can also construct a map of homotopy equivalence:

$$\psi : \bigvee_{j=1}^n C(f_j) \rightarrow \bigvee_{j=1}^n C(g_j)$$

which is represented by the following matrix:

$$\begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{pmatrix}.$$

The composition map $\phi\psi$ gives a homotopy equivalence:

$$\bigvee_{i=1}^n S^3 \rightarrow \bigvee_{i=1}^n S^3.$$

As in the proof of Theorem 3, we obtain the equation:

$$A_{11} \operatorname{Diag}(f_1, f_2, \dots, f_n) = \operatorname{Diag}(g_1, g_2, \dots, g_n) A_{22}.$$

Hence we have the result by Lemma 5. $\square$

Acknowledgement. The author would like to thank a referee for his useful comments.

REFERENCES

- [1] P. Freyd. Stable homotopy 1. Proc. Conf. Categorical Algebra (La Jolla, 1965). Springer, New York, 1966.
- [2] P. Freyd. Stable homotopy 2. Amer. Math. Soc. Proc. Symp. Pure Math. **18** (1970), 161–183.
- [3] C. A. McGibbon. The Mislin genus of a space. Centre de Recherches Mathematiques, CRM Proc. Lecture Note. **6** (1994), 75–102.
- [4] G. Mislin. Cancellation properties of H-spaces. Comment. Math. Helv. **49** (1974), 195–200.
- [5] Y. Shitanda. Uncountably many loop spaces of the same n -type for all n , 1, 2. Yokohama Math. J. **41** (1993), 17–24; Yokohama Math. J. **41** (1994), 85–93.
- [6] Y. Shitanda. Cancellation and non cancellation phenomena for infinite complexes. Publ. RIMS, Kyoto Univ. **32** (1996), 383–399.
- [7] Y. Shitanda. Spaces of the same clone type. J. Math. Soc. Japan **48**(4) (1996), 705–714.
- [8] C. Wilkerson. Genus and cancellation. Topology **14** (1975), 29–36.
- [9] A. Zabrodsky. On phantom maps and a theorem of H. Miller. Israel J. Math. **58**(2) (1987), 129–143.

Yoshimi Shitanda
Meiji University
Izumi Campus
Eifuku 1-9-1, Suginami-ku
Tokyo 168, Japan
(E-mail: shitanda@isc.meiji.ac.jp)