CURVES WITH ASYMPTOTIC GEODESICS

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Dedicated to Katsuhiro Shiohama on his sixtieth birthday

0. Introduction

Let M be an n -dimensional simply connected complete Riemannian manifold of non-positive sectional curvature. Let Σ be a curve of class C^2 in M and $x(s)$ be a parameterization of Σ by arclength s . We assume that $x(s)$ is defined for all real numbers s . Let T be a unit tangent vector of Σ and let D be the Levi-Civita connection of M . Then $k(s) = |D_T T|$ is called the geodesic curvature of Σ . Let O be a fixed point in M and let $r(s) = d(O, x(s))$, where d is the distance in M .

Σ is a geodesic if $k(s) = 0$ for all s . In particular, in the n -dimensional Euclidean space E^n , if $k(s) = 0$ for all s , Σ is a straight line. However, the condition that $\lim_{s \rightarrow \infty} k(s) = 0$ does not necessarily imply that Σ looks ‘straight’ near infinity, as is seen from such an example as the curve defined by $r = \theta$ in the polar coordinate system (r, θ) in E^2 .

In this paper, we give a sufficient condition for Σ to have an asymptotic geodesic (asymptotic line) on its end. Here an asymptotic geodesic means a geodesic $\Gamma : y(t)$ which has the property that the function $h(s) := \inf_t d(x(s), y(t))$ tends to zero as $s \rightarrow +\infty$ or $s \rightarrow -\infty$. We prove that if Σ is in E^n and satisfies $\int_{-\infty}^{\infty} k(s)r(s) ds < \infty$, then Σ has an asymptotic line on each end (Theorem 3.1). We also show that if Σ is in the n -dimensional hyperbolic space H^n and satisfies $\int_{-\infty}^{\infty} k(s) ds < \infty$, then Σ has an asymptotic geodesic on each end (Theorem 2.1). Theorem 3.1 is a generalization of a result in [E] for curves in E^2 .

If a curve Σ in E^n satisfies the condition that $\lim_{s \rightarrow \infty} r(s) = \infty$ (i.e. properly immersed) and $k(s)r(s)^{2+\varepsilon}$ is uniformly bounded with some positive constant ε , then $\int_{-\infty}^{\infty} k(s)r(s) ds < \infty$ holds and Σ has an asymptotic line on each end (Remark 3.2). In general, of course, $k(s)$ may not converge to zero as $s \rightarrow \pm\infty$ under the condition that $\int_{-\infty}^{\infty} k(s)r(s) ds < \infty$.

Although our main results are for curves in E^n or H^n , we study some asymptotic behaviors of curves with $\int_{-\infty}^{\infty} k(s) ds < \infty$ in a Riemannian manifold of non-positive sectional curvature (Section 1), which are used to prove our main theorems.

1. Curves in manifolds of non-positive curvature

In this section, we prove the following proposition.

PROPOSITION 1.1. *Let $\Sigma : x(s)$ ($-\infty < s < \infty$) be a curve in a simply connected complete Riemannian manifold M of non-positive sectional curvature. Let O be a fixed point in M and let $r(s) = d(O, x(s))$. If $\int_{-\infty}^{\infty} k(s) ds < \infty$, then*

$$\lim_{s \rightarrow \pm\infty} \left| \frac{dr}{ds} \right| = 1.$$

In particular, $\lim_{s \rightarrow \pm\infty} r(s) = \infty$, i.e. Σ is properly immersed.

Proof. Since $\int_{-\infty}^{\infty} k(s) ds < \infty$, for any positive constant $\varepsilon < \pi/2$, there exists s_0 such that

$$\int_{s_0}^{s_1} k(s) ds < \varepsilon.$$

holds for any $s_1 > s_0$. Let Σ_1 be a piecewise C^2 closed curve consisting of $\{x(s) : s_0 \leq s \leq s_1\}$ and the geodesic segment γ joining $x(s_0)$ and $x(s_1)$. Let φ_0 be the angle between $T(s_0)$ and γ and let φ_1 be the angle between $T(s_1)$ and γ . The total absolute curvature of Σ_1 is given by

$$\int_{s_0}^{s_1} k(s) ds + (\pi - \varphi_0) + (\pi - \varphi_1).$$

Since, by a theorem of Szentze [S], the total absolute curvature of any piecewise C^2 closed curve in a Riemannian manifold of non-positive sectional curvature is not less than 2π , we have

$$\varphi_0 + \varphi_1 \leq \int_{s_0}^{s_1} k(s) ds < \varepsilon.$$

Hence we have

$$0 \leq \varphi_1 < \varepsilon < \frac{\pi}{2}.$$

For x in M let $r_0(x) = d(x(s_0), x)$. Regarding r_0 as a function on M , we denote the gradient vector field of r_0 by ∇r_0 . Then we have

$$\begin{aligned} \left. \frac{d}{ds} r_0(x(s)) \right|_{s=s_1} &= \langle \nabla r_0(x(s_1)), T(s_1) \rangle \\ &= \cos \varphi_1 \\ &> \cos \varepsilon. \end{aligned}$$

Since ε is any positive constant, this shows that

$$\lim_{s \rightarrow \infty} \frac{d}{ds} r_0(x(s)) = 1. \quad (1.1)$$

Since

$$r_0(x(s)) - r(s_0) \leq r(s) \leq r_0(x(s)) + r(s_0),$$

(1.1) implies that

$$\lim_{s \rightarrow \infty} \frac{dr}{ds} = 1.$$

Similarly, we have

$$\lim_{s \rightarrow -\infty} \frac{dr}{ds} = 1.$$

and the proof for Proposition 1.1 is complete. $\square$

To state the following lemma, let γ_s be the oriented geodesic of M from O to $x(s)$ and let $X(s)$ be the unit tangent vector of γ_s at O . $X(s)$ is regarded as a curve in the unit $(n-1)$ -sphere consisting of all unit tangent vectors of M at O .

LEMMA 1.2. *If $\int_{-\infty}^{\infty} k(s) ds < \infty$, then $X(s)$ converges as $s \rightarrow \pm\infty$.*

Proof. For x in M let $r(x) = d(O, x)$. Regarding r as a function on M , the gradient vector ∇r is the unit tangent vector of γ_s . Let $\theta(s)$ be the angle between γ_s and Σ at $x(s)$. Then we have

$$\begin{aligned} \frac{d}{ds} r(x(s)) &= \langle \nabla r(x(s)), T(s) \rangle \\ &= \cos \theta(s). \end{aligned} \quad (1.2)$$

It follows from (1.2) and Proposition 1.1 that

$$\lim_{s \rightarrow \infty} \theta(s) = 0. \quad (1.3)$$

Let ε be any positive constant. Then there exists s_0 such that for any $s_1 > s_0$

$$\int_{s_0}^{s_1} k(s) ds < \frac{\varepsilon}{3}, \quad \theta(s_0) < \frac{\varepsilon}{3}, \quad \theta(s_1) < \frac{\varepsilon}{3}. \quad (1.4)$$

Since, by Szenthe's theorem [S], the total absolute curvature of the piecewise C^2 closed curve consisting of γ_{s_0} , γ_{s_1} and $\{x(s) : s_0 \leq s \leq s_1\}$ is not less than 2π , we have

$$\int_{s_0}^{s_1} k(s) ds + \theta(s_0) + \pi - \theta(s_1) + \pi - d_S(X(s_0), X(s_1)) \geq 2\pi, \quad (1.5)$$

where d_S is the distance in the unit $(n - 1)$ -sphere of all unit tangent vectors of M at O . (1.4) and (1.5) give

$$d_S(X(s_0), X(s_1)) \leq \int_{s_0}^{s_1} k(s) ds + \theta(s_0) - \theta(s_1) < \varepsilon.$$

Hence $X(s)$ converges to a unit tangent vector X_∞ of M at O as $s \rightarrow \infty$. The case when $s \rightarrow -\infty$ is similar. $\square$

2. Curves in hyperbolic space

In this section, we prove the following theorem.

THEOREM 2.1. *Let $\Sigma : x(s)$ be a curve of class C^2 in H^n which is parameterized by arclength s for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} k(s) ds < \infty$, then Σ has an asymptotic geodesic on each end.*

Proof. Let Γ be the geodesic of H^n through O which is tangent to X_∞ at O , where X_∞ is the unit tangent vector of H^n at O given in Lemma 1.2. Let $y(s)$ be the point on Γ such that $d(O, y(s)) = r(s)$. $x(s)$ and $y(s)$ are joined by a curve $\{\exp_O(r(s)X(t)) : s \leq t < \infty\}$, whose length shall be denoted by $\ell(s)$. Let $\tilde{\ell}(s)$ be the length of the curve $\{X(t) : s \leq t < \infty\}$ which lies in the unit sphere consisting of all unit tangent vectors of H^n at O . Then we have

$$\ell(s) = \sinh(r(s))\tilde{\ell}(s). \quad (2.1)$$

Let $d\tilde{s}$ be the line element of $X(t)$. The Riemannian metric of H^n is written as

$$ds^2 = dr^2 + \sinh^2 r d\tilde{s}^2. \quad (2.2)$$

It follows from (1.2) and (2.2) that

$$\sin \theta(s) ds = \sinh(r(s)) d\tilde{s}. \quad (2.3)$$

Since $\lim_{s \rightarrow \infty} r(s) = \infty$ by Proposition 1.1, we use l'Hospital's rule together with Proposition 1.1, (1.2), (2.1) and (2.3) to derive

$$\begin{aligned} \lim_{s \rightarrow \infty} \ell(s) &= \lim_{s \rightarrow \infty} \frac{\tilde{\ell}(s)}{1/\sinh(r(s))} \\ &= \lim_{s \rightarrow \infty} \frac{d\tilde{\ell}/ds}{-(dr/ds) \cosh(r(s))/\sinh^2(r(s))} \\ &= \lim_{s \rightarrow \infty} \frac{-(d\tilde{s}/ds) \sinh(r(s))}{-(dr/ds) \cosh(r(s))/\sinh(r(s))} \end{aligned}$$

$$\begin{aligned}
&= \lim_{s \rightarrow \infty} \frac{d\tilde{s}}{ds} \sinh(r(s)) \\
&= \lim_{s \rightarrow \infty} \sin \theta(s) \\
&= 0.
\end{aligned} \tag{2.4}$$

Since $d(x(s), \Gamma) \leq d(x(s), y(s)) \leq \ell(s)$, (2.4) implies

$$\lim_{s \rightarrow \infty} d(x(s), \Gamma) = 0.$$

Hence Γ is an asymptotic geodesic of Σ for $s \rightarrow \infty$. By a similar argument, one can show that Σ has an asymptotic geodesic for $s \rightarrow -\infty$. $\square$

Remark 2.1. Asymptotic geodesic on each end is not unique. In fact, in H^n there are infinitely many geodesics which are asymptotic to one geodesic.

Remark 2.2. It is natural to ask if the above theorem holds for curves in M whose sectional curvature K_M satisfies $-b^2 \leq K_M \leq -a^2 < 0$ with some constants a and b ($0 < a < b$). We do not know the answer, but it seems that the proof given above does not work in this situation, as we explain below. If $-b^2 \leq K_M \leq -a^2 < 0$, then we have

$$\sinh(ar(s))\tilde{\ell}(s) \leq \ell(s) \leq \sinh(br(s))\tilde{\ell}(s)$$

instead of (2.1), and also

$$\sinh(ar(s))d\tilde{s} \leq \sin \theta(s) ds \leq \sinh(br(s)) d\tilde{s}$$

instead of (2.3). As a result, we have

$$\begin{aligned}
\lim_{s \rightarrow \infty} \ell(s) &\leq \lim_{s \rightarrow \infty} \frac{\tilde{\ell}(s)}{1/\sinh(r(s))} \\
&= \lim_{s \rightarrow \infty} \frac{d\tilde{s}}{ds} \sinh(br(s)) \\
&\leq \lim_{s \rightarrow \infty} \frac{\sinh(br(s))}{\sinh(ar(s))} \sin \theta(s)
\end{aligned}$$

instead of (2.4). If $0 < a < b$, there is no guarantee that

$$\lim_{s \rightarrow \infty} \frac{\sinh(br(s))}{\sinh(ar(s))} \sin \theta(s) = 0$$

and the existence of asymptotic geodesics cannot be derived from this argument.

3. Curves in Euclidean space

For curves in E^n , the condition $\int_{-\infty}^{\infty} k(s) ds < \infty$ is not sufficient to have asymptotic lines, as we see from such an example as the graph of $y = x^2$ in E^2 . In the following theorem, we will give a sufficient condition for Σ in E^n to have asymptotic lines. For curves in E^2 this theorem is proved in [E].

THEOREM 3.1. *Let $\Sigma : x(s)$ be a curve in E^n which is parameterized by arclength s for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} k(s)r(s) ds < \infty$, then Σ has an asymptotic line on each end.*

Proof. Since Σ is a curve in the Euclidean space, we have

$$\frac{dx}{ds} = T(s), \quad \frac{d^2x}{ds^2} = \frac{dT}{ds} = k(s)N(s),$$

where $N(s)$ is a unit normal vector of Σ . Let $x^\perp(s) = x(s) - \langle x(s), T(s) \rangle T(s)$. For any $\varepsilon > 0$, there exists $s_0 > 0$ such that

$$\int_{s_1}^{s_2} k(s)r(s) ds < \varepsilon$$

holds for any $s_1, s_2 \geq s_0$. Since

$$\begin{aligned} \left| \frac{dx^\perp}{ds} \right|^2 &= \left| T - \langle T, T \rangle T - \left\langle x, \frac{dT}{ds} \right\rangle T - \langle x, T \rangle \frac{dT}{ds} \right|^2 \\ &= | - \langle x, kN \rangle T - \langle x, T \rangle kN |^2 \\ &= k^2 (\langle x, N \rangle^2 + \langle x, T \rangle^2) \\ &\leq k(s)^2 |x(s)|^2 \\ &= k(s)^2 r(s)^2, \end{aligned}$$

we have

$$\begin{aligned} |x^\perp(s_2) - x^\perp(s_1)| &= \left| \int_{s_1}^{s_2} \frac{dx^\perp}{ds} ds \right| \\ &\leq \int_{s_1}^{s_2} \left| \frac{dx^\perp}{ds} \right| ds \\ &\leq \int_{s_1}^{s_2} k(s)r(s) ds \\ &< \varepsilon. \end{aligned}$$

This shows that $\lim_{s \rightarrow \infty} x^\perp(s)$ exists. Set $x_\infty^\perp = \lim_{s \rightarrow \infty} x^\perp(s)$.

We can also show that $\lim_{s \rightarrow \infty} T(s)$ exists. In fact, if s_0 is sufficiently large, we have $r(s) > 1$ for all $s \geq s_0$ and

$$\begin{aligned} |T(s_2) - T(s_1)| &= \left| \int_{s_1}^{s_2} \frac{dT}{ds} ds \right| \\ &\leq \int_{s_1}^{s_2} \left| \frac{dT}{ds} \right| ds \\ &= \int_{s_1}^{s_2} k(s) ds \\ &< \int_{s_1}^{s_2} k(s)r(s) ds \\ &< \varepsilon. \end{aligned}$$

Set $T_\infty = \lim_{s \rightarrow \infty} T(s)$.

We define a straight line $\Gamma : y(t)$ by $y(t) = x_\infty^\perp + tT_\infty$. Let $\bar{x}(s)$ be a point on Γ which is defined by

$$\bar{x}(s) = x_\infty^\perp + \langle x(s), T_\infty \rangle T_\infty.$$

Γ becomes an asymptotic line of Σ if and only if

$$\lim_{s \rightarrow \infty} |x(s) - \bar{x}(s)| = 0.$$

Since $x(s) = x^\perp(s) + \langle x(s), T(s) \rangle T(s)$, we have

$$\begin{aligned} |x(s) - \bar{x}(s)| &= |(x^\perp(s) - x_\infty^\perp) + \langle x(s), T(s) \rangle T(s) - \langle x(s), T_\infty \rangle T_\infty| \\ &= |(x^\perp(s) - x_\infty^\perp) + \langle x(s), T(s) \rangle (T(s) - T_\infty) + \langle x(s), T(s) - T_\infty \rangle T_\infty| \\ &\leq |x^\perp(s) - x_\infty^\perp| + |\langle x(s), T(s) \rangle| |T(s) - T_\infty| + |\langle x(s), T(s) - T_\infty \rangle| \\ &\leq |x^\perp(s) - x_\infty^\perp| + 2r(s) |T(s) - T_\infty|. \end{aligned}$$

Since $\int_0^\infty k(s)r(s) ds$ converges, for any $\varepsilon > 0$, there exists s_0 such that

$$\int_{s_0}^\infty k(s)r(s) ds < \varepsilon.$$

By Proposition 1.1, we may assume that $dr/ds > 0$ for all $s \geq s_0$. This implies that, if $t \geq s \geq s_0$, then $r(t) > r(s)$. Now we have, for any $s \geq s_0$,

$$\begin{aligned} r(s)|T(s) - T_\infty| &= r(s) \left| \int_s^\infty \frac{dT}{dt} dt \right| \\ &\leq r(s) \int_s^\infty \left| \frac{dT}{dt} \right| dt \\ &= r(s) \int_s^\infty k(t) dt \\ &\leq \int_s^\infty k(t)r(t) dt \\ &< \varepsilon. \end{aligned}$$

Hence

$$\lim_{s \rightarrow \infty} |x(s) - \bar{x}(s)| \leq \lim_{s \rightarrow \infty} (|x^\perp(s) - x_\infty^\perp| + 2r(s)|T(s) - T_\infty|) = 0.$$

This shows that Γ is an asymptotic line of Σ for $s \rightarrow \infty$. Similarly, Σ has an asymptotic line for $s \rightarrow -\infty$. $\square$

Remark 3.1. As we see from the proof, $\lim_{s \rightarrow \infty} T(s)$ exists under the condition that $\int_{-\infty}^\infty k(s) ds < \infty$.

Remark 3.2. If Σ in E^n satisfies the condition that $\lim_{s \rightarrow \infty} r(s) = \infty$ (i.e. properly immersed) and $k(s)r(s)^{2+\varepsilon} \leq A$ for all s with some positive constants ε and A , then by a result in [K] there exist positive constants B and s_0 such that $|dr/ds| \geq B$ for all s with $|s| \geq s_0$. Thus we have

$$\begin{aligned} \int_{-\infty}^\infty k(s)r(s) ds &\leq \int_{-\infty}^\infty \frac{A}{r(s)^{1+\varepsilon}} ds \\ &\leq \int_{-s_0}^{s_0} \frac{A}{r(s)^{1+\varepsilon}} ds + \int_{r(-s_0)}^\infty \frac{A}{Br^{1+\varepsilon}} dr + \int_{r(s_0)}^\infty \frac{A}{Br^{1+\varepsilon}} dr \\ &< \infty. \end{aligned}$$

Hence Σ has an asymptotic line on each end. This shows how fast the geodesic curvature should decay to have asymptotic lines, when one controls the geodesic curvature pointwise. However, our theorem is more general because we do not need pointwise estimates for the geodesic curvature. Our theorems do not need the assumption that Σ is properly immersed, either.

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