CURVES WITH ASYMPTOTIC GEODESICS

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Dedicated to Katsuhiro Shiohama on his sixtieth birthday

0. Introduction

Let $M$ be an $n$-dimensional simply connected complete Riemannian manifold of non-positive sectional curvature. Let $\Sigma$ be a curve of class $C^2$ in $M$ and $x(s)$ be a parameterization of $\Sigma$ by arclength $s$. We assume that $x(s)$ is defined for all real numbers $s$. Let $T$ be a unit tangent vector of $\Sigma$ and let $D$ be the Levi-Civita connection of $M$. Then $k(s) = |D_T T|$ is called the geodesic curvature of $\Sigma$. Let $O$ be a fixed point in $M$ and let $r(s) = d(O, x(s))$, where $d$ is the distance in $M$.

$\Sigma$ is a geodesic if $k(s) = 0$ for all $s$. In particular, in the $n$-dimensional Euclidean space $E^n$, if $k(s) = 0$ for all $s$, $\Sigma$ is a straight line. However, the condition that $\lim_{s \to \infty} k(s) = 0$ does not necessarily imply that $\Sigma$ looks ‘straight’ near infinity, as is seen from such an example as the curve defined by $r = \theta$ in the polar coordinate system $(r, \theta)$ in $E^2$.

In this paper, we give a sufficient condition for $\Sigma$ to have an asymptotic geodesic (asymptotic line) on its end. Here an asymptotic geodesic means a geodesic $\Gamma: y(t)$ which has the property that the function $h(s) := \inf_t d(x(s), y(t))$ tends to zero as $s \to +\infty$ or $s \to -\infty$. We prove that if $\Sigma$ is in $E^n$ and satisfies $\int_{-\infty}^{\infty} k(s) r(s) ds < \infty$, then $\Sigma$ has an asymptotic line on each end (Theorem 3.1). We also show that if $\Sigma$ is in the $n$-dimensional hyperbolic space $H^n$ and satisfies $\int_{-\infty}^{\infty} k(s) ds < \infty$, then $\Sigma$ has an asymptotic geodesic on each end (Theorem 2.1). Theorem 3.1 is a generalization of a result in [E] for curves in $E^2$.

If a curve $\Sigma$ in $E^n$ satisfies the condition that $\lim_{|s| \to \infty} r(s) = \infty$ (i.e. properly immersed) and $k(s) r(s)^{2+\varepsilon}$ is uniformly bounded with some positive constant $\varepsilon$, then $\int_{-\infty}^{\infty} k(s) r(s) ds < \infty$ holds and $\Sigma$ has an asymptotic line on each end (Remark 3.2). In general, of course, $k(s)$ may not converge to zero as $s \to \pm\infty$ under the condition that $\int_{-\infty}^{\infty} k(s) r(s) ds < \infty$.

Although our main results are for curves in $E^n$ or $H^n$, we study some asymptotic behaviors of curves with $\int_{-\infty}^{\infty} k(s) ds < \infty$ in a Riemannian manifold of non-positive sectional curvature (Section 1), which are used to prove our main theorems.
1. Curves in manifolds of non-positive curvature

In this section, we prove the following proposition.

**Proposition 1.1.** Let $\Sigma : x(s) \ (0 < s < 1)$ be a curve in a simply connected complete Riemannian manifold $M$ of non-positive sectional curvature. Let $O$ be a fixed point in $M$ and let $r(s) = d(O, x(s))$. If $\int_{-\infty}^{\infty} k(s) \, ds < \infty$, then

$$\lim_{s \to \pm \infty} \left| \frac{dr}{ds} \right| = 1.$$

In particular, $\lim_{s \to \pm \infty} r(s) = \infty$, i.e. $\Sigma$ is properly immersed.

**Proof.** Since $\int_{-\infty}^{\infty} k(s) \, ds < \infty$, for any positive constant $\varepsilon < \pi / 2$, there exists $s_0$ such that

$$\int_{s_0}^{s_1} k(s) \, ds < \varepsilon.$$

holds for any $s_1 > s_0$. Let $\Sigma_1$ be a piecewise $C^2$ closed curve consisting of $\{x(s) : s_0 \leq s \leq s_1\}$ and the geodesic segment $\gamma$ joining $x(s_0)$ and $x(s_1)$. Let $\varphi_0$ be the angle between $T(s_0)$ and $\gamma$ and let $\varphi_1$ be the angle between $T(s_1)$ and $\gamma$. The total absolute curvature of $\Sigma_1$ is given by

$$\int_{s_0}^{s_1} k(s) \, ds + (\pi - \varphi_0) + (\pi - \varphi_1).$$

Since, by a theorem of Szentze [S], the total absolute curvature of any piecewise $C^2$ closed curve in a Riemannian manifold of non-positive sectional curvature is not less than $2\pi$, we have

$$\varphi_0 + \varphi_1 \leq \int_{s_0}^{s_1} k(s) \, ds < \varepsilon.$$

Hence we have

$$0 \leq \varphi_1 < \varepsilon < \frac{\pi}{2}.$$

For $x$ in $M$ let $r_0(x) = d(x(s_0), x)$. Regarding $r_0$ as a function on $M$, we denote the gradient vector field of $r_0$ by $\nabla r_0$. Then we have

$$\left. \frac{d}{ds} r_0(x(s)) \right|_{s=s_1} = (\nabla r_0(x(s_1)), T(s_1))$$

$$= \cos \varphi_1$$

$$> \cos \varepsilon.$$
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Since \( \varepsilon \) is any positive constant, this shows that

\[
\lim_{s \to \infty} \frac{d}{ds} r_0(x(s)) = 1.
\] (1.1)

Since

\[ r_0(x(s)) - r(s_0) \leq r(s) \leq r_0(x(s)) + r(s_0), \]

(1.1) implies that

\[
\lim_{s \to \infty} \frac{dr}{ds} = 1.
\]

Similarly, we have

\[
\lim_{s \to -\infty} \frac{dr}{ds} = 1.
\]

and the proof for Proposition 1.1 is complete.

To state the following lemma, let \( \gamma_s \) be the oriented geodesic of \( M \) from \( O \) to \( x(s) \) and let \( X(s) \) be the unit tangent vector of \( \gamma_s \) at \( O \). \( X(s) \) is regarded as a curve in the unit \((n - 1)\)-sphere consisting of all unit tangent vectors of \( M \) at \( O \).

**Lemma 1.2.** If \( \int_{-\infty}^{\infty} k(s) \, ds < \infty \), then \( X(s) \) converges as \( s \to \pm \infty \).

**Proof.** For \( x \) in \( M \) let \( r(x) = d(O, x) \). Regarding \( r \) as a function on \( M \), the gradient vector \( \nabla r \) is the unit tangent vector of \( \gamma_s \). Let \( \theta(s) \) be the angle between \( \gamma_s \) and \( \Sigma \) at \( x(s) \). Then we have

\[
\frac{d}{ds} r(x(s)) = \langle \nabla r(x(s)), T(s) \rangle = \cos \theta(s).
\] (1.2)

It follows from (1.2) and Proposition 1.1 that

\[
\lim_{s \to \infty} \theta(s) = 0.
\] (1.3)

Let \( \varepsilon \) be any positive constant. Then there exists \( s_0 \) such that for any \( s_1 > s_0 \)

\[
\int_{s_0}^{s_1} k(s) \, ds < \frac{\varepsilon}{3}, \quad \theta(s_0) < \frac{\varepsilon}{3}, \quad \theta(s_1) < \frac{\varepsilon}{3}.
\] (1.4)

Since, by Szenthe’s theorem [S], the total absolute curvature of the piecewise \( C^2 \) closed curve consisting of \( \gamma_{s_0}, \gamma_{s_1} \) and \( \{x(s) : s_0 \leq s \leq s_1\} \) is not less than \( 2\pi \), we have

\[
\int_{s_0}^{s_1} k(s) \, ds + \theta(s_0) + \pi - \theta(s_1) + \pi - d_S(X(s_0), X(s_1)) \geq 2\pi,
\] (1.5)
where $d_S$ is the distance in the unit $(n-1)$-sphere of all unit tangent vectors of $M$ at $O$. (1.4) and (1.5) give

$$d_S(X(s_0), X(s_1)) \leq \int_{s_0}^{s_1} k(s) \, ds + \theta(s_0) - \theta(s_1) < \varepsilon.$$ 

Hence $X(s)$ converges to a unit tangent vector $X_\infty$ of $M$ at $O$ as $s \to \infty$. The case when $s \to -\infty$ is similar. 

\section{Curves in hyperbolic space}

In this section, we prove the following theorem.

**Theorem 2.1.** Let $\Sigma : x(s)$ be a curve of class $C^2$ in $H^n$ which is parameterized by arclength $s$ for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} k(s) \, ds < \infty$, then $\Sigma$ has an asymptotic geodesic on each end.

**Proof.** Let $\Gamma$ be the geodesic of $H^n$ through $O$ which is tangent to $X_\infty$ at $O$, where $X_\infty$ is the unit tangent vector of $H^n$ at $O$ given in Lemma 1.2. Let $y(s)$ be the point on $\Gamma$ such that $d(O, y(s)) = r(s)$. $x(s)$ and $y(s)$ are joined by a curve $\{\exp_O(r(s)X(t)) : s \leq t < \infty\}$, whose length shall be denoted by $\ell(s)$. Let $\tilde{\ell}(s)$ be the length of the curve $\{X(t) : s \leq t < \infty\}$ which lies in the unit sphere consisting of all unit tangent vectors of $H^n$ at $O$. Then we have

$$\ell(s) = \sinh(r(s))\tilde{\ell}(s). \quad (2.1)$$

Let $d\tilde{\ell}$ be the line element of $X(t)$. The Riemannian metric of $H^n$ is written as

$$ds^2 = dr^2 + \sinh^2 r \, d\tilde{\ell}^2. \quad (2.2)$$

It follows from (1.2) and (2.2) that

$$\sin \theta(s) \, ds = \sinh(r(s)) \, d\tilde{\ell}. \quad (2.3)$$

Since $\lim_{s \to \infty} r(s) = \infty$ by Proposition 1.1, we use l’Hospital’s rule together with Proposition 1.1, (1.2), (2.1) and (2.3) to derive

$$\lim_{s \to \infty} \ell(s) = \lim_{s \to \infty} \frac{\tilde{\ell}(s)}{1/\sinh(r(s))}$$

$$= \lim_{s \to \infty} \frac{d\tilde{\ell}/ds}{-(dr/ds) \cosh(r(s))/\sinh^2(r(s))}$$

$$= \lim_{s \to \infty} \frac{-(d\tilde{\ell}/ds) \sinh(r(s))}{-(dr/ds) \cosh(r(s))/\sinh(r(s))}$$
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\[ \lim_{s \to \infty} \frac{d\tilde{s}}{ds} \sinh(r(s)) = \lim_{s \to \infty} \sin \theta(s) = 0. \]  \hspace{1cm} (2.4)

Since \( d(x(s), \Gamma) \leq d(x(s), y(s)) \leq \ell(s) \), (2.4) implies

\[ \lim_{s \to \infty} d(x(s), \Gamma) = 0. \]

Hence \( \Gamma \) is an asymptotic geodesic of \( \Sigma \) for \( s \to \infty \). By a similar argument, one can show that \( \Sigma \) has an asymptotic geodesic for \( s \to -\infty \). \( \square \)

**Remark 2.1.** Asymptotic geodesic on each end is not unique. In fact, in \( H^n \) there are infinitely many geodesics which are asymptotic to one geodesic.

**Remark 2.2.** It is natural to ask if the above theorem holds for curves in \( M \) whose sectional curvature \( K_M \) satisfies \(-b^2 \leq K_M \leq -a^2 < 0\) with some constants \( a \) and \( b \) \((0 < a < b)\). We do not know the answer, but it seems that the proof given above does not work in this situation, as we explain below. If \(-b^2 \leq K_M \leq -a^2 < 0\), then we have

\[ \sinh(ar(s)) \tilde{\ell}(s) \leq \ell(s) \leq \sinh(br(s)) \tilde{\ell}(s) \]

instead of (2.1), and also

\[ \sinh(ar(s))d\tilde{s} \leq \sin \theta(s) ds \leq \sinh(br(s))d\tilde{s} \]

instead of (2.3). As a result, we have

\[ \lim_{s \to \infty} \ell(s) \leq \lim_{s \to \infty} \frac{\tilde{\ell}(s)}{\frac{1}{\sinh(r(s))}} = \lim_{s \to \infty} \frac{d\tilde{s}}{ds} \sinh(br(s)) \leq \lim_{s \to \infty} \frac{\sinh(br(s))}{\sinh(ar(s))} \sin \theta(s) \]

instead of (2.4). If \( 0 < a < b \), there is no guarantee that

\[ \lim_{s \to \infty} \frac{\sinh(br(s))}{\sinh(ar(s))} \sin \theta(s) = 0 \]

and the existence of asymptotic geodesics cannot be derived from this argument.
3. Curves in Euclidean space

For curves in $E^n$, the condition $\int_{-\infty}^{\infty} k(s) \, ds < \infty$ is not sufficient to have asymptotic lines, as we see from such an example as the graph of $y = x^2$ in $E^2$. In the following theorem, we will give a sufficient condition for $\Sigma$ in $E^n$ to have asymptotic lines. For curves in $E^2$ this theorem is proved in [E].

**Theorem 3.1.** Let $\Sigma : x(s)$ be a curve in $E^n$ which is parameterized by arclength $s$ for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} k(s)r(s) \, ds < \infty$, then $\Sigma$ has an asymptotic line on each end.

**Proof.** Since $\Sigma$ is a curve in the Euclidean space, we have

$$\frac{dx}{ds} = T(s), \quad \frac{d^2x}{ds^2} = \frac{dT}{ds} = k(s)N(s),$$

where $N(s)$ is a unit normal vector of $\Sigma$. Let $x^\perp(s) = x(s) - \langle x(s), T(s) \rangle T(s)$. For any $\varepsilon > 0$, there exists $s_0 > 0$ such that

$$\int_{s_1}^{s_2} k(s)r(s) \, ds < \varepsilon$$

holds for any $s_1, s_2 \geq s_0$. Since

$$\left| \frac{dx^\perp}{ds} \right|^2 = \left| T - \langle T, T \rangle T - \left( x, \frac{dT}{ds} - \langle x, T \rangle \frac{dT}{ds} \right) \right|^2$$

$$= | - \langle x, kN \rangle T - \langle x, T \rangle kN |^2$$

$$= k^2 (\langle x, N \rangle^2 + \langle x, T \rangle^2)$$

$$\leq k(s)^2 |x(s)|^2$$

$$= k(s)^2 r(s)^2,$$

we have

$$|x^\perp(s_2) - x^\perp(s_1)| = \left| \int_{s_1}^{s_2} \frac{dx^\perp}{ds} \, ds \right|$$

$$\leq \int_{s_1}^{s_2} \left| \frac{dx^\perp}{ds} \right| \, ds$$

$$\leq \int_{s_1}^{s_2} k(s)r(s) \, ds$$

$$< \varepsilon.$$
This shows that $\lim_{s \to \infty} x^\perp(s)$ exists. Set $x_\infty^\perp = \lim_{s \to \infty} x^\perp(s)$.

We can also show that $\lim_{s \to \infty} T(s)$ exists. In fact, if $s_0$ is sufficiently large, we have $r(s) > 1$ for all $s \geq s_0$ and

$$
[T(s_2) - T(s_1)] = \left| \int_{s_1}^{s_2} \frac{dT}{ds} \, ds \right|
\leq \int_{s_1}^{s_2} \left| \frac{dT}{ds} \right| \, ds
= \int_{s_1}^{s_2} k(s) \, ds
< \int_{s_1}^{s_2} k(s)r(s) \, ds
< \varepsilon.
$$

Set $T_\infty = \lim_{s \to \infty} T(s)$.

We define a straight line $\Gamma : y(t) = x_\infty^\perp + tT_\infty$. Let $\tilde{x}(s)$ be a point on $\Gamma$ which is defined by

$$
\tilde{x}(s) = x_\infty^\perp + \langle x(s), T_\infty \rangle T_\infty.
$$

$\Gamma$ becomes an asymptotic line of $\Sigma$ if and only if

$$
\lim_{s \to \infty} |x(s) - \tilde{x}(s)| = 0.
$$

Since $x(s) = x^\perp(s) + \langle x(s), T(s) \rangle T(s)$, we have

$$
|x(s) - \tilde{x}(s)| = |(x^\perp(s) - x_\infty^\perp) + \langle x(s), T(s) \rangle T(s) - \langle x(s), T_\infty \rangle T_\infty|
\leq |x^\perp(s) - x_\infty^\perp| + |\langle x(s), T(s) \rangle| |T(s) - T_\infty| + |\langle x(s), T(s) - T_\infty \rangle|
\leq |x^\perp(s) - x_\infty^\perp| + 2r(s)|T(s) - T_\infty|.
$$

Since $\int_0^\infty k(s)r(s) \, ds$ converges, for any $\varepsilon > 0$, there exists $s_0$ such that

$$
\int_{s_0}^\infty k(s)r(s) \, ds < \varepsilon.
$$
By Proposition 1.1, we may assume that $dr/ds > 0$ for all $s \geq s_0$. This implies that, if $t \geq s \geq s_0$, then $r(t) > r(s)$. Now we have, for any $s \geq s_0$,

$$r(s)|T(s) - T_\infty| = r(s) \left| \int_s^\infty \frac{dT}{dt} \, dt \right|$$

$$\leq r(s) \int_s^\infty \left| \frac{dT}{dt} \right| \, dt$$

$$= r(s) \int_s^\infty k(t) \, dt$$

$$\leq \int_s^\infty k(t)r(t) \, dt$$

$$< \varepsilon.$$

Hence

$$\lim_{s \to \infty} |x(s) - \bar{x}(s)| \leq \lim_{s \to \infty} \left( |x^\perp(s) - x^\perp_\infty| + 2r(s)|T(s) - T_\infty| \right) = 0.$$

This shows that $\Gamma$ is an asymptotic line of $\Sigma$ for $s \to \infty$. Similarly, $\Sigma$ has an asymptotic line for $s \to -\infty$. \hfill $\Box$

**Remark 3.1.** As we see from the proof, $\lim_{s \to -\infty} T(s)$ exists under the condition that $\int_{-\infty}^{\infty} k(s) \, ds < \infty$.

**Remark 3.2.** If $\Sigma$ in $E^n$ satisfies the condition that $\lim_{t \to -\infty} r(s) = \infty$ (i.e. properly immersed) and $k(s) r(s)^{2+\varepsilon} \leq A$ for all $s$ with some positive constants $\varepsilon$ and $A$, then by a result in [K] there exist positive constants $B$ and $s_0$ such that $|dr/ds| \geq B$ for all $s$ with $|s| \geq s_0$. Thus we have

$$\int_{-\infty}^{\infty} k(s)r(s) \, ds \leq \int_{-\infty}^{\infty} \frac{A}{r(s)^{1+\varepsilon}} \, ds$$

$$\leq \int_{-s_0}^{s_0} \frac{A}{r(s)^{1+\varepsilon}} \, ds + \int_{r(s_0)}^{\infty} \frac{A}{Br^{1+\varepsilon}} \, dr + \int_{r(s_0)}^{\infty} \frac{A}{Br^{1+\varepsilon}} \, dr$$

$$< \infty.$$
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