SOME VARIANTS OF THE CONGRUENT NUMBER PROBLEM II

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0. Introduction

In this paper, we apply Shimura’s theory on modular forms of half-integral weight [ShG2] and the result of Waldspurger [Wal] to \(2\pi/3\)- and \(\pi/3\)-congruent number problems (\(2\pi/3\)-CNP and \(\pi/3\)-CNP), which we have studied in [Yo] (see also Fujiwara [Fuj] and Kan [Kan]). We recall our problems.

\(2\pi/3\)-CNP and \(\pi/3\)-CNP. A squarefree positive integer \(n\) is said to be a \(2\pi/3\)- (respectively \(\pi/3\)-) congruent number if \(n\sqrt{3}\) is the area of a triangle with rational sides, one of whose angles is \(2\pi/3\) (respectively \(\pi/3\)). What is a characterization of \(2\pi/3\)- (respectively \(\pi/3\)-) congruent numbers?

For the basic results of our problems, we refer to [Fuj], [Kan] or [Yo]. According to [Yo], one easily finds that \(2\pi/3\)-CNP and \(\pi/3\)-CNP are reduced to studying a rank of Mordell–Weil group over \(\mathbb{Q}\) of a series of elliptic curves given by

\[ E_{\pm n} : y^2 = x(x \pm n)(x \mp 3n) \quad (n = 1, 2, 3, \ldots). \]

(Note that \(E_{\pm n}\) does not admit complex multiplication.)

Our method is essentially due to Tunnell. In [Tun], Tunnell applies the result of Waldspurger to a series of the elliptic curves \(C_{\pm n}^1 : y^2 = x^3 - n^2x\) (with complex multiplication by \(\mathbb{Z}[\sqrt{-1}]\)) and he uses it to find a characterization of the (classical) congruent number problem.

THEOREM 0.1. [Tun] Let us consider a formal power series in the variable \(q\)

\[
\Phi^{(1,j)} = \left\{ q \prod_{m=1}^{\infty} (1 - q^6m)(1 - q^{6m}) \right\} \sum_{m \in \mathbb{Z}} q^{2jm^2} = \sum_{m=1}^{\infty} a_m(\Phi^{(1,j)})q^m \quad (j = 1, 2)
\]
and let \( n \) be a squarefree positive odd integer. If \( a_n(\Phi^{(1,j)}) \neq 0 \), then the Mordell–Weil group \( C_{1n}(\mathbb{Q}) \) is finite and \( jn \) is not the area of any right-angled triangle with rational sides; that is, \( jn \) is not a congruent number. Moreover, if we assume the conjecture of Birch and Swinnerton-Dyer \([BiSw]\), then the converse is also true.

**Remark.** Frey \([Fre1]\) and Barthel \([Bar]\) apply the same argument to a series of elliptic curves \( C_n^2 : y^2 = x^3 - n^3 \) (with complex multiplication by \( \mathbb{Z}[(-1 + \sqrt{-3})/2] \)).

We also use the same argument. Some of our results are stated as follows.

**Theorem 0.2.** (Theorem 2.1, Corollary 3.4) Let \( \Phi_{3,-3} \) be the formal power series in the variable \( q \) given by

\[
\Phi_{3,-3} = \sum_{x_1,x_2,x_3 \in \mathbb{Z}} q^{x_1^2 + 3x_2^2 + 144x_3^2} - \sum_{x_1,x_2,x_3 \in \mathbb{Z}} q^{x_1^2 + 9x_2^2 + 16x_3^2} - G_2
\]

where \( G_2 \) is a power series of the form \( \sum_{m=1}^{\infty} a_m(G_2)q^m \) which can be written explicitly (see Theorem 2.1). Let \( n \) be a squarefree positive integer such that \( n \equiv 1, 7 \) or \( 13 \pmod{24} \). If \( a_n(\Phi_{3,-3}) \neq 0 \), the group \( E_n(\mathbb{Q}) \) is finite and \( n \) is not a \( 2\pi/3 \)-congruent number. If the conjecture of Birch and Swinnerton-Dyer is true, the converse is also true.

**Theorem 0.3.** (Theorem 3.8) Let \( p \) be a prime such that \( p \equiv 7 \) or \( 13 \pmod{24} \). Then \( a_p(\Phi_{3,-3}) \neq 0 \). In particular, \( p \) is not \( 2\pi/3 \)-congruent. Furthermore, in the case \( p \equiv 13 \pmod{24} \), the Shafarevich–Tate group \( \text{III}(E_p/\mathbb{Q}) \) of \( E_p \) is a non-trivial finite group.

**Remark.** Kan \([Kan]\) has used the two-descent method to prove the same result except for the finiteness of the Shafarevich–Tate group \( \text{III}(E_p/\mathbb{Q}) \).

The paper is organized as follows. In Section 1, we recall some facts on the theory of Shimura on the modular forms of half-integral weight (and on quadratic forms). In Section 2, we construct various cusp forms of weight \( 3/2 \) which correspond to our elliptic curves \( E_{\pm 1} \) and \( E_{\pm 3} \) via the Shimura map. In Section 3, using the results in Section 2, we apply Waldspurger’s theorem to our problems. In Section 4, we give a table on the conjectural order of \( \text{III}(E_{\pm n}/\mathbb{Q}) \).
1. Modular forms

In this section, we recall some facts on modular forms of integral and half-integral weight. For details and proofs, see Shimura [ShG1, ShG2], Knapp [Kna] and Koblitz [Kob].

Following Shimura, we define a symbol \((c/d)\) for \(c, d \in \mathbb{Z}\) by the following conditions:

(i) \((c/d)\) is the Legendre–Kronecker symbol if \(d\) is positive;

(ii) \((c/d) = \begin{cases} -\left(\frac{c}{d}\right) & \text{if } c < 0 \text{ and } d < 0, \\ \left(\frac{c}{d}\right) & \text{otherwise.} \end{cases}\)

For a positive integer \(N\), let \(\Gamma_0(N)\) be the group of matrices in \(SL_2(\mathbb{Z})\) which are upper triangular modulo \(N\). It acts as a discrete group of Möbius transformation on the complex upper half-plane \(H = \{\tau \in \mathbb{C}; \operatorname{Im} \tau > 0\}\) and the cusps \(\mathbb{Q} \cup \{i\infty\}\), \((i = \sqrt{-1})\).

Let \(\kappa\) be an element of \(\frac{1}{2}\mathbb{Z}\). We always assume that \(4|N\) if \(\kappa \not\in \mathbb{Z}\). For a holomorphic function \(f(\tau)\) on \(H\) and for a character \(\chi\) modulo \(N\), \(f(\tau)\) is called a modular (respectively cusp) form of weight \(\kappa\), level \(N\) and character \(\chi\) if \(f(\tau)\) satisfies:

(i) \(f\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} \chi(d)(c\tau + d)^\kappa f(\tau) & \text{if } \kappa \in \mathbb{Z}, \\ \chi(d)e^{-2\pi i \kappa \left(\frac{c}{d}\right)}((c\tau + d)^{1/2})^{2\kappa} f(\tau) & \text{if } \kappa \not\in \mathbb{Z}, \end{cases}\)

for all \(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in \Gamma_0(N);\) and

(ii) \(f\) is holomorphic (respectively vanishes) at the cusps (see [Kna, Ch. IX]).

Here we use the notation:

- \(e_d = 1\) or \(-1\) according to whether \(d \equiv 1, 3 \pmod{4}\);
- \(z^{1/2}\) is the square root of \(z\) with \(-\pi/2 < \arg(z^{1/2}) \leq \pi/2\) for \(z \in \mathbb{C}\);
- \(\chi_t\) denotes the character corresponding to \(\mathbb{Q}((\sqrt{t})/\mathbb{Q}\) for a non-zero integer \(t\) (so \(\chi_t\) is trivial if \(t\) is a square).

Denote the vector space of all such modular (respectively cusp) forms by \(M_\kappa(N, \chi)\) (respectively \(S_\kappa(N, \chi)\)). For a modular form \(f \in M_\kappa(N, \chi)\), write its
Fourier expansion (at the cusp $i\infty$) as

$$f(\tau) = \sum_{m=0}^{\infty} a_m(f)q^m,$$

where $q = \exp(2\pi i \tau)$.

For any prime $p$ and $\kappa \in \mathbb{Z}$ (respectively $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), let $T_\kappa(p)$ (respectively $T_\kappa(p^2)$) denote the $p$th (respectively the $p^2$th) Hecke operator which acts on $M_\kappa(N,\chi)$ and on $S_\kappa(N,\chi)$ (see [Kna, ShG1, ShG2]). For $\kappa = 3/2$ we use the following formula [ShG2].

**Theorem 1.1.** Let $N$ be an odd positive integer divisible by 4, $f$ an element of $S_{3/2}(N,\chi)$, $p$ a prime. Put

$$f(\tau) = \sum_{m=0}^{\infty} a_m(f)q^m,$$

$$(T_{3/2}(p^2)f)(\tau) = \sum_{m=0}^{\infty} a_m(T_{3/2}(p^2)f)q^m.$$

Then

$$a_m(T_{3/2}(p^2)f) = a_{p^2m}(f) + \chi(p)\chi^{-1}(p)\left(\frac{m}{p}\right)a_m(f) + \chi(p^2)pa_{m/p^2}(f),$$

where we understand that $a_{m/p^2}(f) = 0$ if $m$ is not divisible by $p^2$ and that $\chi(p)\chi^{-1}(p)(m/p) = 0$ if $p = 2$.

If $f_1, f_2 \in S_\kappa(N,\chi)$, we define the Petersson inner product

$$\langle f_1, f_2 \rangle = \int_{\Gamma_0(N),\mathfrak{f}} v^\kappa f_1(\tau)\overline{f_2(\tau)} \frac{du \, dv}{v^2},$$

where $\tau = u + vi$. It is well known (see the argument in [ShG1]) that if $p \nmid N$, then the adjoint of $T_2(p)$ (respectively $T_{3/2}(p^2)$) with respect to $\langle \cdot, \cdot \rangle$ is $\chi(p)T_2(p)$ (respectively $\chi(p)T_{3/2}(p^2)$).

Let $N$ be a positive integer with $4|N$, $\chi$ a character modulo $N$. For any positive integer $c$ and any quadratic character $\psi$ with conductor $r_\psi$ such that $4r_\psi^2c|N$ and $\chi = \psi c \chi^{-1}$, we set

$$\theta_{\psi,c}(\tau) = \sum_{m \in \mathbb{Z}} \psi(m)mq^{cm^2}.$$
Then $\theta_{\psi,c} \in S_{3/2}(N, \chi)$ [ShG2]. By $S_{3/2}^c(N, \chi)$, we denote the subspace of $S_{3/2}(N, \chi)$ generated by such forms $\theta_{\psi,c}$. Let $S_{3/2}^{\perp}(N, \chi)$ be the orthogonal complement of $S_{3/2}(N, \chi)$ with respect to the Petersson inner product $(\cdot, \cdot)$.

The following important result is a special case of the Main Theorem of Shimura ([ShG2], see also [Cip, Fli, Koj, Niw, Shin, Stu]).

**Theorem 1.2.** Let $N$ be a positive integer divisible by 4, $\chi$ a Dirichlet character modulo $N$. Let $f(\tau) = \sum_{m=1}^{\infty} a_m(f) q^m$ be an element of $S_{3/2}^{\perp}(N, \chi)$. Furthermore, let $t$ be a squarefree positive integer.

Define constants $A_t(m)(m = 1, 2, 3, \ldots)$ by the following formula:

$$\sum_{m=1}^{\infty} A_t(m) m^{-s} = \left( \sum_{m=1}^{\infty} \chi(m) \left( \frac{-1}{m} \right) m^{-s} \right) \left( \sum_{m=1}^{\infty} a_{tm^2}(f) m^{-s} \right).$$

Assume that $f$ is a common eigenfunction of $T_{3/2}(p^2)$ for all prime factors $p$ of $N$ not dividing the conductor of $(-1/\cdot)(t/\cdot)\chi$. Then:

1. the function

$$\text{Shim}_t(f)(\tau) = \sum_{m=1}^{\infty} A_t(m) q^m$$

is an element of $S_2(N/2, \chi^2)$;

2. if $f(\tau)$ is an eigenfunction of $T_{3/2}(p^2)$ for a prime $p$ not dividing $N$ with an eigenvalue $\lambda_p$, then $\text{Shim}_t(f)(\tau)$ is also an eigenfunction of $T_2(p)$ with the same eigenvalue $\lambda_p$.

We call Shim$_t$ the $t$-Shimura map.

To construct cusp forms of weight $3/2$, we recall some facts on quadratic forms and its $\Theta$-series. Let $Q = Q(X_1, \ldots, X_k)$ be a positive definite quadratic form with integer coefficients. Let $A = A_Q$ denote the $k \times k$ matrix

$$A = \begin{bmatrix} \frac{\partial^2 Q}{\partial X_\alpha \partial X_\beta} \end{bmatrix}$$

and $N_Q$ the smallest positive integer so that $N_Q A^{-1}$ has integral entries and even elements on the diagonal. Define the $\Theta$-series corresponding to $Q$ as

$$\Theta(Q)(\tau) = \sum_{x \in \mathbb{Z}^k} q^{Q(x)} \quad (q = \exp(2\pi i \tau), \tau \in \mathbb{H}).$$
Then $\Theta(Q)$ is an element of $M_{k/2}(N_Q, \chi_{d_Q})$, where

$$d_Q = \begin{cases} 
\det(A) & \text{if } k \equiv 0 \pmod{4}, \\
-\det(A) & \text{if } k \equiv 2 \pmod{4}, \\
\det(A)/2 & \text{if } k \text{ is odd}.
\end{cases}$$

(See [ShG2].)

Two integral quadratic forms are in the same genus if they are equivalent over $\mathbb{R}$ and over $\mathbb{Z}_p$ for every prime $p$. If $Q_1$ and $Q_2$ are in the same genus, then one can prove that $N_{Q_1} = N_{Q_2}$, $\chi_{d_{Q_1}} = \chi_{d_{Q_2}}$, and that $\Theta(Q_1) - \Theta(Q_2) \in S_{k/2}(N_{Q_1}, \chi_{d_{Q_1}})$. (See Siegel [Sie] for even $k$ and the proof works for odd $k$ too.)

We will use the following result in the next section (see [Fre2]).

**Proposition 1.3.** Let $f(\tau) = \sum_{m=0}^{\infty} a_m(f)q^m$ be an element of $M_4(N, \chi)$ (4|N if $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, of course). Let $\mu(N) = [SL_2(\mathbb{Z}) : \Gamma_0(N)]$. If $a_m(f) = 0$ for all $m \in \mathbb{Z}$ with $0 \leq m \leq \mu(N)\kappa/12$, then $f = 0$. Hence $f$ is determined by the Fourier coefficients $a_0(f), \ldots, a_{\mu(N)\kappa/12}(f)$.

**Remark.** In [Fre2], Frey gives a proof of above proposition for $\kappa \in \mathbb{Z}$. However, an argument similar to his proof works for $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

2. **Construction of cusp forms of weight 3/2**

Let $\varphi$ be a newform of $S_2(M, \chi_1)$ (see [AtLe, Kna]). In particular, $\varphi$ is a common eigenfunction of $T_2(p)$ for all primes $p$ not dividing the conductor $M$ of $\varphi$ (say $T_2(p)\varphi = \lambda_p\varphi$). Let $N$ be a positive integer divisible by 4 and $\chi$ a quadratic Dirichlet character modulo $N$ such that $\chi(-1) = 1$. By $S_{3/2}(N, \chi, \varphi)$, we denote the subspace of $S_{3/2}(N, \chi)$ which consists of elements $\Phi$ such that $T_{3/2}(p^2)\Phi = \chi_p\Phi$ for all primes $p \mid N$.M.

To state Theorem 2.1, we use the following notation:

- $E_m$ is the (modular) elliptic curve $y^2 = x(x + m)(x - 3m) \ (m \in \mathbb{Z}, m \neq 0)$;
- $\varphi_m$ is the newform corresponding to the elliptic curve $E_m$;
- $\chi_t$ is the Dirichlet character corresponding to $\mathbb{Q}((\sqrt{t})/\mathbb{Q} \ (t \in \mathbb{Z}, t \neq 0)$.

(See Sections 0 and 1.)

We denote a binary quadratic form $b_1X_1^2 + b_2X_1X_2 + b_3X_2^2$ by $Q[b_1, b_2, b_3]$ and a ternary quadratic form $b_1X_1^2 + b_2X_2^2 + b_3X_3^2 + b_4X_2X_3 + b_5X_1X_3 + b_6X_1X_2$ by $Q[b_1 b_2 b_3 b_4 b_5 b_6]$. We also use the abbreviation $\Theta(b_1, b_2, b_3)$ and $\Theta(b_1 b_2 b_3)$ for the $\Theta$-series corresponding to $Q[b_1, b_2, b_3]$ and $Q[b_1 b_2 b_3]$, respectively.
THEOREM 2.1. With the above notation, we have the following:

(1) \[ \Phi_{2,-1} = \Theta \left( \begin{array}{cc} 1 & 6 \\ 0 & 0 \\ 6 & 12 \end{array} \right) - \Theta \left( \begin{array}{cc} 3 & 4 \\ 0 & 0 \\ 4 & 6 \end{array} \right) \in S_{3/2}^\perp(48, \chi_2, \varphi_{-1}), \]

(2) \[ \Phi_{6,-1} = \Theta \left( \begin{array}{cc} 1 & 2 \\ 0 & 0 \\ 2 & 12 \end{array} \right) - \Theta \left( \begin{array}{cc} 2 & 3 \\ 0 & 0 \\ 3 & 4 \end{array} \right) \in S_{3/2}^\perp(48, \chi_6, \varphi_{-1}), \]

(3) \[ \Phi_{1,-1} = \Theta \left( \begin{array}{cc} 2 & 9 \\ 6 & 0 \\ 9 & 9 \end{array} \right) - \Theta \left( \begin{array}{cc} 3 & 6 \\ 0 & 0 \\ 6 & 8 \end{array} \right) - G_1 \in S_{3/2}^\perp(96, \chi_1, \varphi_{-1}), \]

where \( G_1 = G_1(\tau) = \Phi_{2,-1}(2\tau). \)

(4) \[ \Phi_{3,-1} = \Theta \left( \begin{array}{cc} 1 & 8 \\ 0 & 0 \\ 8 & 24 \end{array} \right) - \Theta \left( \begin{array}{cc} 4 & 8 \\ 8 & 4 \\ 8 & 9 \end{array} \right) \in S_{3/2}^\perp(96, \chi_3, \varphi_{-1}), \]

(5) \[ \Phi_{6,3} = \Theta \left( \begin{array}{cc} 1 & 6 \\ 0 & 0 \\ 6 & 36 \end{array} \right) - \Theta \left( \begin{array}{cc} 4 & 6 \\ 0 & 0 \\ 6 & 9 \end{array} \right) \in S_{3/2}^\perp(144, \chi_6, \varphi_3), \]

(6) \[ \Phi_{1,1} = \Theta \left( \begin{array}{cc} 1 & 12 \\ 12 & 0 \\ 12 & 15 \end{array} \right) - \Theta \left( \begin{array}{cc} 3 & 4 \\ 4 & 0 \\ 4 & 13 \end{array} \right) \in S_{3/2}^\perp(192, \chi_1, \varphi_1), \]

(7) \[ \Phi_{3,1} = \Theta \left( \begin{array}{cc} 1 & 7 \\ 2 & 0 \\ 7 & 7 \end{array} \right) - \Theta \left( \begin{array}{cc} 3 & 4 \\ 4 & 0 \\ 4 & 5 \end{array} \right) \in S_{3/2}^\perp(192, \chi_3, \varphi_1), \]

(8) \[ \Phi_{3,3} = \{ \Theta(1, 0, 144) - \Theta(9, 0, 16) \} \sum_{m \in \mathbb{Z}} \chi_3(m)q^m \in S_{3/2}^\perp(576, \chi_3, \varphi_3), \]

(9) \[ \Phi_{3,-3} = \Theta \left( \begin{array}{cc} 1 & 3 \\ 0 & 0 \\ 3 & 144 \end{array} \right) - \Theta \left( \begin{array}{cc} 3 & 9 \\ 0 & 0 \\ 9 & 16 \end{array} \right) - G_2 \in S_{3/2}^\perp(576, \chi_3, \varphi_{-3}), \]

where \( G_2 = G_2(\tau) \) is the cusp form of \( S_{3/2}^\perp(576, \chi_{-3}) \) given by

\[ G_2 = \frac{1}{2} \theta_{\chi_{-3},1} + 40\theta_{\chi_{-3},4} + 8\theta_{\chi_{-3},16}. \]
Proof. Each statement of this theorem can be proved similarly, so we give the proof for the parts (1), (3), (7), (8) and (9).

(1) From the table of Brandt and Intrau [BrIn], two ternary quadratic forms $Q[1612]$ and $Q[3465]$ are in the same genus, so $\Phi_{2,-1} = \Theta(1612) - \Theta(3465)$ is in $S_{3/2}(48, \chi_2)$. It is easy to see that $S_{3/2}^0(48, \chi_2) = \{0\}$ and that $\dim C S_{3/2}(48, \chi_2) = 1$ by the formula of [CoOe]. Hence $\Phi_{2,-1}$ is an eigenfunction of $T_{3/2}(p^2)$ for all primes $p$. The main theorem of Shimura (Theorem 1.2) gives that Shim$_1(\Phi_{2,-1}) \in S_2(24, \chi_1)$. Since $S_2(24, \chi_1)$ is a vector space of dimension one and is generated by $\varphi_1$, we have $\Phi_{2,-1} \in S_{3/2}^0(48, \chi_2, \varphi_1)$.

(3) The table of [BrIn] implies that $\Phi_{1,-1} = \Theta(299) - \Theta(368)$ is an element of $S_{3/2}(96, \chi_1)$. From (1) and [ShG2], $G_1(\tau) = \Phi_{2,-1}(2\tau)$ is in $S_{3/2}(96, \chi_1)$ too. By the same argument as in the proof of (1), we have $S_{3/2}(96, \chi_1) = S_{3/2}^0(96, \chi_1)$. Using Theorem 1.1 and Proposition 1.3, we have that $\Phi_{1,-1} = \Phi_{1,-1} - G_1$ is an eigenfunction of $T_{3/2}(p^2)$ for $p = 2, 5, 7$ and 11 with eigenvalues $\lambda_2 = 0$, $\lambda_5 = -2$, $\lambda_7 = 0$ and $\lambda_{11} = 4$, respectively. In fact, by an argument similar to the algorithm of [ABF] and [Fre2, p. 5] (see Remark (2) below), we see that $\Phi_{1,-1}$ is an eigenfunction of $T_{3/2}(p^2)$ for all $p \geq 5$ (and for $p = 2$). Applying Theorem 1.2 to $\Phi_{1,-1}$, we obtain that Shim$_3(\Phi_{1,-1})$ is a non-zero eigenvalue of $S_{2}(48, \chi_1)$ and is an eigenfunction of $T_2(p)$ for all primes $p \geq 5$. Furthermore, Theorem 1.2(2) implies $T_2(p) Shim_1(\Phi_{1,-1}) = \lambda_p Shim_3(\Phi_{1,-1})$ for $p = 5, 7$ and 11. From the table of Cremona on Hecke eigenvalues [Cre], we see that Shim$_1(\Phi_{1,-1})$ is a non-zero constant multiple of the newform $\varphi_1(\in \ S_2(24, \chi_1))$; that is, $\Phi_{1,-1} \in S_{3/2}^0(96, \chi_1, \varphi_1)$.

(7) From the table of Brandt and Intrau [BrIn], we obtain that $\Phi_{3,1} = \Theta(177) - \Theta(445)$ is in $S_{3/2}(192, \chi_3)$. In fact, using the same argument as (3) we obtain that $\Phi_{3,1}$ is an eigenfunction of $T_{3/2}(p^2)$ for all primes $p \neq 3$, with eigenvalues $\lambda_2 = 0$, $\lambda_5 = -2$, $\lambda_7 = 0$ and $\lambda_{11} = -4$ for $T_{3/2}(2^2)$, $T_{3/2}(5^2)$, $T_{3/2}(7^2)$ and $T_{3/2}(11^2)$, respectively. The subspace $S_{3/2}^0(192, \chi_3)$ is generated by one form $\theta_{2,-3,3}$. This form is an eigenfunction for $T_{3/2}(5^2)$ with its eigenvalue 6. Hence it follows from the properties of the Petersson inner product $(\ldots, \ldots)$ (see Section 1) that $\Phi_{3,1}$ is in $S_{3/2}^0(192, \chi_3)$. Then Theorem 1.2 gives that Shim$_1(\Phi_{3,1}) \in S_2(192/2, \chi_1)$ is an eigenfunction of $T_{2}(p)$ for all primes $p \geq 5$ with eigenvalues $\lambda_5 = -2$, $\lambda_7 = 0$, $\lambda_{11} = -4$ for $T_{2}(5)$, $T_{2}(7)$, $T_{2}(11)$, respectively. From the table of Cremona on Hecke eigenvalues [Cre], we obtain that Shim$_1(\Phi_{3,1})$ is a non-zero constant multiple of the newform $\varphi_1(\in \ S_2(48, \chi_1))$. Hence we have $\Phi_{3,1} \in S_{3/2}^0(192, \chi_3, \varphi_1)$. 
From the theory of binary quadratic forms, we see that the quadratic forms $Q[1, 0, 144]$ and $Q[9, 0, 16]$ are in the same genus. So we have $\Theta(1, 0, 144) - \Theta(9, 0, 16) \in S_1(576, \chi_3)$. Since $\sum_{m \in \mathbb{Z}} \chi_3(m)q^m$ is in $M_{1/2}(576, \chi_3)$ by [ShG2], the product $[\Theta(1, 0, 144) - \Theta(9, 0, 16)]\sum_{m \in \mathbb{Z}} \chi_3(m)q^m$ is in $S_{3/2}(576, \chi_3)$ (see [Kob, p. 219]). Then an argument similar to (3) and (7) gives the desired result.

(9) An argument similar to (8) gives that $Q_{3144000} - Q_{3916000}$ is in $S_{3/2}(576, \chi_3)$. However, this cusp form is not an eigenfunction. Let $G_2 = G_2(\tau)$ be the cusp form given in the statement of the theorem. The function $G_2$ is an element of $S_{3/2}(576, \chi_3)$. Then, by an argument similar to (3) and (7), we see that $Q_{3144000} - Q_{3916000} = G_2$ is an element of $S_{3/2}(576, \chi_3, \varphi_3)$. 

Remark. (1) Some of cusp forms ($Q_{3144000}, Q_{6144001}, Q_{3333001},$ etc.) are also calculated by Lehman [Leh1, Leh2] and Ono [Ono].

(2) In the algorithm of [ABF, p. 5], they have used the 1-Shimura map only, and we can see that $\text{Shim}_1(Q_{3144000}) = \text{Shim}_1(Q_{3333001}) = 0$. So we cannot apply the algorithm of [ABF] to the cusp forms $Q_{3144000}$ and $Q_{3333001}$ directly. However, if we use the $t$-Shimura maps for various $t$ in the proof of Proposition 1.3 of [ABF], then we see that the algorithm can be applied to our cases.

3. Applications

First, we state the theorem of Waldspurger [Wal].

THEOREM 3.1. [Wal] Let $E$ be a modular elliptic curve defined over $\mathbb{Q}$ and $\varphi_E$ the corresponding cusp form. Let $\Phi$ be a non-zero element of $S_{3/2}(N, \chi, \varphi_E)$. Write

$$\Phi(\tau) = \sum_{m=1}^{\infty} a_m(\Phi)q^m.$$ 

Let $d$ and $d_0$ be positive squarefree integers. Assume that $d \equiv d_0 \bmod \prod_{l|N} \mathbb{Q}^{x^2}$ and $d \cdot d_0$ is prime to $N$. Then

$$L(E \otimes \chi_{-td}, 1)\sqrt{d} a_{d_0}(\Phi)^2 = L(E \otimes \chi_{-td_0}, 1)\sqrt{d_0} a_d(\Phi)^2,$$

where, for each integer $m \neq 0$, let $E \otimes \chi_m$ denote the twist of $E$ over $\mathbb{Q}(\sqrt{m})$ and $L(E \otimes \chi_m, s)$ denotes its $L$-series. In particular, if $L(E \otimes \chi_{-td_0}, 1)a_{d_0}(\Phi) \neq 0$ for some $d_0$, then $L(E \otimes \chi_{-td}, 1) = 0$ if and only if $a_d(\Phi) = 0$. 


For a positive squarefree integer $n$, we define $c_{2\pi/3}(n)$ and $c_{\pi/3}(n)$ as:

$$
c_{2\pi/3}(n) = \begin{cases} 
  a_n(\Phi_{1,-1}) & \text{if } n \equiv 11 \pmod{24}, \\
  a_n(\Phi_{3,-1}) & \text{if } n \equiv 1, 7, 13 \pmod{24}, \\
  a_n/2(\Phi_{2,-1}) & \text{if } n \equiv 2 \pmod{12}, \\
  a_n/3(\Phi_{3,-1}) & \text{if } n \equiv 3 \pmod{24}, \\
  a_n/6(\Phi_{6,-1}) & \text{if } n \equiv 6 \pmod{12}, \\
  0 & \text{otherwise,}
\end{cases}
$$

$$
c_{\pi/3}(n) = \begin{cases} 
  a_n(\Phi_{1,1}) & \text{if } n \equiv 1, 7, 19 \pmod{24}, \\
  a_n(\Phi_{3,3}) & \text{if } n \equiv 5 \pmod{24}, \\
  a_n/2(\Phi_{6,3}) & \text{if } n \equiv 2 \pmod{12}, \\
  a_n/3(\Phi_{3,1}) & \text{if } n \equiv 3, 9, 15 \pmod{24}, \\
  0 & \text{otherwise.}
\end{cases}
$$

Using a computer, we can check that:

$$
L(E_{d_0}, 1)c_{2\pi/3}(d_0) \neq 0 \text{ for } d_0 = 1, 7, 11 and 13;
L(E_{-d_0}, 1)c_{\pi/3}(d_0) \neq 0 \text{ for } d_0 = 1, 5, 7 and 19;
L(E_{2d_0}, 1)c_{2\pi/3}(2d_0) \neq 0 \text{ for } d_0 = 1, 13, 19 and 31;
L(E_{-2d_0}, 1)c_{\pi/3}(2d_0) \neq 0 \text{ for } d_0 = 1, 7, 13 and 19;
L(E_{3d_0}, 1)c_{2\pi/3}(3d_0) \neq 0 \text{ for } d_0 = 1 and 17;
L(E_{-3d_0}, 1)c_{\pi/3}(3d_0) \neq 0 \text{ for } d_0 = 1, 5, 11, 17, 19 and 37;
L(E_{6d_0}, 1)c_{2\pi/3}(6d_0) \neq 0 \text{ for } d_0 = 1, 5, 7, 13, 17, 23, 35 and 67.
$$

For any squarefree positive integer $n$, it follows from [Yo, Table 3], that

$$
L(E_n, 1) = 0 \text{ if } n \equiv 5, 9, 10, 15, 17, 19, 21, 22, 23 \pmod{24},
L(E_{-n}, 1) = 0 \text{ if } n \equiv 6, 8, 11, 13, 17, 18, 21, 22, 23 \pmod{24}.
$$

So Theorem 3.1 gives the following.

**Theorem 3.2.** Let $n$ be a squarefree positive integer. Then:

1. $L(E_n, 1) = 0$ if and only if $c_{2\pi/3}(n) = 0$;
2. $L(E_{-n}, 1) = 0$ if and only if $c_{\pi/3}(n) = 0$.

Theorem 3.2 can be used to prove that certain numbers are not $2\pi/3$- or $\pi/3$-congruent numbers by invoking the following theorem.
THEOREM 3.3. \cite{Kol1, Kol2} Let $E$ be a modular elliptic curve over $\mathbb{Q}$. If $L(E, 1) \neq 0$, then the Mordell–Weil group $E(\mathbb{Q})$ and the Shafarevich–Tate group $\Sha(E/\mathbb{Q})$ of $E$ are finite.

From this and Theorem 3.2 (and \cite{Yo, Corollary 3.2}), we have the following.

COROLLARY 3.4. (A criteria for non-$2\pi/3$- and non-$\pi/3$-congruentness) Let $n$ be a squarefree positive integer. Then:

1. if $c_{2\pi/3}(n) \neq 0$, then $E_n(\mathbb{Q})$ and $\Sha(E_n/\mathbb{Q})$ are finite and $n$ is not $2\pi/3$-congruent;
2. if $c_{\pi/3}(n) \neq 0$ and $n \neq 1$, then $E_{-n}(\mathbb{Q})$ and $\Sha(E_{-n}/\mathbb{Q})$ are finite and $n$ is not $\pi/3$-congruent.

Remark. Ono \cite{Ono} has obtained some cases of Corollary 3.4 using $\Phi_{6,-1}$, $\Phi_{3,1}$.

From Birch and Swinnerton-Dyer \cite{BiSw} and Yo \cite[Tables 2 and 3]{Yo}, we expect the following.

CONJECTURE 3.5. (The Birch and Swinnerton-Dyer conjecture for $E_{\pm n}$) Let $n$ be a squarefree positive integer. Let $\Omega_{\pm n}$ be the real period of $E_{\pm n}$ (see \cite{Si}). Define $u(\pm n)$ for $\pm = 0, \pm$ as follows.

\[
\begin{align*}
u_0(n) &= \sharp\{\text{primes } \ell \geq 5 \text{ dividing } n\} \\
u_+(n) &= \begin{cases} 0 & \text{if } 6 \mid n, \\
 & \text{2 if } n \equiv 1, 11, 13 \pmod{24}, \\
 & \text{1 otherwise.} \end{cases} \\
u_-(n) &= \begin{cases} 0 & \text{if } 3 \mid n \text{ and } n/3 \equiv 11, 19 \pmod{24}, \\
 & \text{2 if } n \equiv 5, 7, 19 \pmod{24}, \\
 & \text{3 if } n = 1, \\
 & \text{1 otherwise.} \end{cases}
\end{align*}
\]

Then:

(BSD1) $L(E_{\mp n}, 1) \neq 0$ if and only if $E_{\mp n}(\mathbb{Q})$ is finite, respectively;
(BSD2+) if $E_n(\mathbb{Q})$ is finite, then
\[
\frac{L(E_n, 1)}{\Omega_n} = \sharp\Sha(E_n/\mathbb{Q})2^{\nu_0(n) - \nu_+(n)};
\]
(BSD2−) if $E_{-n}(\mathbb{Q})$ is finite, then
\[
\frac{L(E_{-n}, 1)}{\Omega_{-n}} = \sharp\Sha(E_{-n}/\mathbb{Q})2^{\nu_0(n) - \nu_-(n)}.
\]
Applying Corollary 3.4, we give some criteria for non-$2\pi/3$-congruence and for non-$\pi/3$-congruence. Some of them have already been obtained in [Yo]. However, (in fact) Corollary 3.4 gives stronger results.

**Proposition 3.6.** Let $p$ be a prime $\equiv 11 \pmod{24}$. Then $c_{2\pi/3}(p) \equiv 4 \pmod{8}$. In particular, we obtain that $L(E_p, 1) \neq 0$ and $p$ is not $2\pi/3$-congruent.

**Proof.** By the definition of $c_{2\pi/3}(p)$, we have

$$c_{2\pi/3}(p) = \sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 2x_1^2 + 9x_2^2 + 9x_3^2 + 6x_2x_3 = p\} - \sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 6x_2^2 + 8x_3^2 = p\}.$$

First, we consider the set $\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 2x_1^2 + 9x_2^2 + 9x_3^2 + 6x_2x_3 = p\}$. Since $p \equiv 11 \pmod{24}$, there exists a unique pair of positive integers $a, b$ such that $2a^2 + 9b^2$. So we have

$$\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 2x_1^2 + 9x_2^2 + 9x_3^2 + 6x_2x_3 = p\} \cap \{x_1x_2x_3 = 0\} = 8.$$

It is easy to see that

$$\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 2x_1^2 + 9x_2^2 + 9x_3^2 + 6x_2x_3 = p\} \cap \{x_1x_2x_3 \neq 0\}$$

is divisible by 8, since the ternary quadratic form $2X_1^2 + 9X_2^2 + 9X_3^2 + 6X_2X_3$ is stable under transformations

$$(X_1, X_2, X_3) \mapsto (\pm X_1, X_2, X_3), (\pm X_1, -X_2, -X_3), (\pm X_1, X_3, X_2)$$

and $(\pm X_1, -X_3, -X_2)$.

Next, we consider the set $\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 6x_2^2 + 8x_3^2 = p\}$. Since $p \equiv 11 \pmod{24}$, it follows from the theory of binary quadratic forms that there exists a unique pair of positive integers $a, b$ such that $2a^2 + 3b^2$. Furthermore, we see that $a$ is even, so we have

$$\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 6x_2^2 + 8x_3^2 = p\} \cap \{x_1x_2x_3 = 0\} = 4.$$

Considering the signs of $x_1, x_2, x_3$, we see that

$$\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 6x_2^2 + 8x_3^2 = p\} \cap \{x_1x_2x_3 \neq 0\}$$

is divisible by 8.

Hence we obtain that $c_{2\pi/3}(p) \equiv 0 - 4 \equiv 4 \pmod{8}$, which gives the desired result. \qed
**Proposition 3.7.** If \( p \) is a prime such that \( p \equiv 5 \pmod{24} \), then \( c_{\pi/3}(p) \equiv 4 \pmod{8} \). In particular, we obtain \( L(E_p, 1) \neq 0 \) and \( p \) is not a \( \pi/3 \)-congruent number.

**Proof.** Let \( g_1 \) and \( g_2 \) be quadratic forms

\[
g_1(X_1, X_2, X_3) = X_1^2 + 144X_2^2 + X_3^2, \quad g_2(X_1, X_2, X_3) = 9X_1^2 + 16X_2^2 + X_3^2.
\]

For \( j \in \mathbb{Z}/12\) and \( \alpha \in \{1, 2\} \), let \( V(p, g_\alpha)_j \) be

\[
\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid g_\alpha(x_1, x_2, x_3) = p, x_3 > 0, x_3 \equiv j \pmod{12}\}.
\]

Then from the definition of \( c_{\pi/3}(p) \), we have

\[
c_{\pi/3}(p)/2 = a_p(\Phi_{3,3})/2 = \sharp V(p, g_1)_1 + \sharp V(p, g_1)_11 - \sharp V(p, g_1)5 - \sharp V(p, g_1)7 - \sharp V(p, g_2)_1 - \sharp V(p, g_2)_{11} + \sharp V(p, g_2)_5 + \sharp V(p, g_2)_7.
\]

It is easy to show that \( \sharp (V(p, g_\alpha)_j \cap \{X_1X_2 \neq 0\}) \) is divisible by 4 and that \( V(p, g_\alpha)_j \cap \{X_1 = 0\} \) and \( V(p, g_\alpha)_j \cap \{X_2 = 0\} \) are empty.

To prove \( c_{\pi/3}(p)/2 \equiv 2 \pmod{4} \), it suffices to show that there exists a \( j_0 \in (\mathbb{Z}/12)\) such that \( \sharp (V(p, g_1)_{j_0} \cap \{X_2 = 0\}) = 2 \) and \( \sharp (V(p, g_1)_{j_0} \cap \{X_2 = 0\}) = 0 \) for \( j \neq j_0 \). Since \( p \equiv 5 \pmod{24} \), we can find positive integers \( x_1 \) and \( x_3 \) (uniquely) such that \( x_1^2 + x_3^2 = p \) and \( (x_1, x_3, 6) = 1 \). Hence if we take \( j_0 \equiv x_3 \pmod{12} \), then we have the desired result.

The following result is more interesting to the author.

**Theorem 3.8.** Let \( p \) be a prime with \( p \equiv 7, 13 \pmod{24} \). Then \( c_{2\pi/3}(p) \equiv 4 \pmod{8} \), so \( L(E_p, 1) \neq 0 \). In particular, the groups \( E_p(\mathbb{Q}) \) and \( \text{III}(E_p/\mathbb{Q}) \) are finite and \( p \) is not \( 2\pi/3 \)-congruent. Furthermore, \( \text{III}(E_p/\mathbb{Q}) \) is non-trivial in the case \( p \equiv 13 \pmod{24} \).

**Proof.** Since \( p \equiv 7 \) or \( 13 \pmod{24} \), it is easy to see from the definitions of \( c_{2\pi/3}(p) \) and \( \Phi_{3,3} \) in Theorem 2.1 that

\[
c_{2\pi/3}(p) = a_p(\Phi_{3,3}) = \sharp \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 3x_2^2 + 144x_3^2 = p\}
= \sharp \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 9x_2^2 + 16x_3^2 = p\}.
\]

Since \( p \equiv 1 \pmod{3} \), we obtain that

\[
\sharp \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 3x_2^2 + 144x_3^2 = p\} \cap \{x_3 = 0\} = 4.
\]
It follows from \( p \equiv 7 \) or \( 13 \) (mod 24) that
\[
\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 3x_2^2 + 144x_3^2 = p\} \cap \{x_1x_2 = 0\} = \emptyset.
\]
Hence we have
\[
\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 3x_2^2 + 144x_3^2 = p\} \equiv 4 \pmod{8}
\]
since \( \sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1^2 + 3x_2^2 + 144x_3^2 = p\} \cap \{x_1x_2x_3 \neq 0\} \) is divisible by 8.

By the fact that \( p \not\equiv 1, 3 \pmod{8} \), we have
\[
\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 9x_2^2 + 16x_3^2 = p\} \cap \{x_1x_2x_3 = 0\} = \emptyset.
\]
So we obtain
\[
\sharp\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid 3x_1^2 + 9x_2^2 + 16x_3^2 = p\} \equiv 0 \pmod{8}.
\]
Hence we see that
\[
c_{2\pi/3}(p) \equiv 4 - 0 \equiv 4 \pmod{8}.
\]

If \( p \equiv 13 \pmod{24} \), then the last statement follows from the two-descent method (see [Yo]) that
\[
1 \leq \dim_{\mathbb{F}_2} \text{III}(E_p/\mathbb{Q})[2] \leq 2
\]
where \( \text{III}(E_p/\mathbb{Q})[2] \) means the two-torsion subgroup of \( \text{III}(E_p/\mathbb{Q}) \). (In fact, from the fact that \( \text{III}(E_p/\mathbb{Q}) \) is finite, it follows that \( \dim_{\mathbb{F}_2} \text{III}(E_p/\mathbb{Q})[2] = 2 \) by the property of the Cassels–Tate pairing [Cas, Tat].) So \( \text{III}(E_p/\mathbb{Q}) \) is non-trivial. \( \square \)

An argument similar to the above results gives a generalization of Theorem 3.4 of [Yo] (and some results of [Fuj] and [Kan]).

**Theorem 3.9.** Let \( p \geq 5 \) be a prime.

1. \( c_{2\pi/3}(n) \neq 0 \) for each integer \( n \) of the following types:
   - \( n = p \) if \( p \equiv 7, 11, 13 \) (mod 24);
   - \( n = 2p \) if \( p \equiv 13, 19 \) (mod 24);
   - \( n = 3p \) if \( p \equiv 17 \) (mod 24);
   - \( n = 6p \) if \( p \equiv 5, 7, 13, 17 \) (mod 24).

In particular, both \( E_n(\mathbb{Q}) \) and \( \text{III}(E_n/\mathbb{Q}) \) are finite and \( n \) is not \( 2\pi/3 \)-congruent.
(2) $c_{\pi/3}(n) \neq 0$ for each integer $n$ of the following types:

\[ n = p \quad \text{if } p \equiv 5, 7, 19 \pmod{24}; \]
\[ n = p \quad \text{if } p \equiv 7, 19 \pmod{24}; \]
\[ n = 2p \quad \text{if } p \equiv 7, 13 \pmod{24}; \]
\[ n = 3p \quad \text{if } p \equiv 5, 17 \pmod{24}. \]

So $E_n(\mathbb{Q})$ and $\text{III}(E_n/\mathbb{Q})$ are finite and $n$ is not $\pi/3$-congruent.

**Remark.** Let $p$ be a prime. Using the modular forms $\Phi_{i,j}$, the author cannot obtain that $c_{\pi/3}(3p) \neq 0$ in the case $p \equiv 11, 19 \pmod{24}$ (Compare with Theorem 3.4 of [Yo].) If we wish to prove this by a similar method, we must show that $c_{\pi/3}(3p) \equiv 8 \pmod{16}$. However, in the case where $p \equiv 11 \pmod{24}$, we can prove $L(E_{-3p}, 1) \neq 0$ by using another cusp form

\[ \left( \Theta(9, 0, 32) - \Theta(17, 2, 17) \right) \sum_{m \in \mathbb{Z}} \chi_3(m)q^{2m^2} \in S_{3/2}(1152, \chi_3, \psi_1). \]

4. **Table**

For a squarefree integer $m$, let $\text{III}_m$ be the conjectural order of the Shafarevich–Tate group $\text{III}(E_m/\mathbb{Q})$ of $E_m$. (Hence $\text{III}_m = 2\text{III}(E_m/\mathbb{Q})$ if Conjecture 3.5 is correct.)

The method to compute $\text{III}_m$ is the following. Let $m$ be a squarefree integer, $\varphi_m$ the newform corresponding to $E_m$. Let $E_{\varphi_m}/\mathbb{Q}$ be the elliptic curve constructed from $\varphi_m$ by Eichler–Shimura theory (see [Kna, Theorem 11.74], or [Yo, Section 4]). So $E_{\varphi_m}$ is isogenous (over $\mathbb{Q}$) to $E_m$ and $L(E_{\varphi_m}, s) = L(E_m, s)$. Let $\Omega(\varphi_m)$ be the real period of $E_{\varphi_m}$. Consider the identity

\[ \frac{L(E_m, 1)}{\Omega_m} = \frac{L(E_{\varphi_m}, 1)}{\Omega(\varphi_m)} \frac{\Omega(\varphi_m)}{\Omega_m}. \]

Using formula (2.8.10) of [Cre], we can compute an integer $\alpha_m$ such that $\alpha_m L(E_{\varphi_m}, 1)/\Omega(\varphi_m) \in \mathbb{Z}$. Since the twist of $E_{\varphi_m}$ over $\mathbb{Q}(\sqrt{m})$ is isogenous to the curve $E_1$ (= the twist of $E_m$ over $\mathbb{Q}(\sqrt{m})$) and the table of [Cre], we know that there are six possibilities for $E_{\varphi_m}$ and all the possible curves can be written explicitly. Then using the arithmetic–geometric mean of Gauss (see [Cre, p. 97]), we can compute all possibilities for $\Omega(\varphi_m)$ and the ratio $\Omega(\varphi_m)/\Omega_m$. (Indeed, we have $\Omega(\varphi_m)/\Omega_m = 1, 1/2$ or $1/4$.) So we obtain that

\[ 4\alpha_m \frac{L(E_m, 1)}{\Omega_m} \in \mathbb{Z}. \]
Using [Cre, Proposition 2.11.1], we calculate the value $4\alpha_m L(E_m, 1)/\Omega_m$ approximately if the sign $\omega_{E_m}$ of the functional equation for $L(E_m, s)$ is $+1$ (see Table 3 of [Yo]). (If $\omega_{E_m} = -1$, then $L(E_m, 1) = 0$.) Then using the fact that $4\alpha_m L(E_m, 1)/\Omega_m$ is an integer, we have the correct value $L(E_m, 1)/\Omega_m \in \mathbb{Q}$.

By computing $L(E_m, 1)/\Omega_m$ for $m = 1, 7, 11, 13; -1, -5, -7, -19; 2, 26, 38, 62; -2, -14, -26, -38; 3, 51; -3, -15, -33, -51, -57, -111; 6, 30, 42, 78, 102, 138, 210, 402 and using Theorem 3.1 (and Conjecture 3.5) we can calculate the values $\text{III}_m$ for every $m$ with $L(E_m, 1) \neq 0$.

For example, we obtain that if $n$ is a positive squarefree integer and $c_{2\pi/3}(n) \neq 0$ (see Theorem 3.2), then

$$\text{III}_n = \left(\frac{c_{2\pi/3}(n)}{2\omega(n)}\right)^2 \times \begin{cases} 1 & \text{if } n \equiv 1, 13 \pmod{24}, \\ 1/4 & \text{if } n \equiv 2, 3, 6, 7, 11, 14, 18 \pmod{24}. \end{cases}$$

In the following, we compute $\text{III}_m$ for all squarefree integers $m$ such that $L(E_m, 1) \neq 0$ and $|m| < 1000$. Note that in the case $L(E_m, 1) \neq 0$, $\text{III}(E_m/\mathbb{Q})$ is always finite by [Kol1, Kol2].


III \_ m = 4 \text{ for } m = 97, 221, 295, 313, 337, 341, 355, 413, 433, 559, 577, 583, 667, 673, 697, 793, 895, 937, 955; 38, 86, 194, 326, 386, 410, 434, 470, 482, 626, 638, 710, 758, 818, 866, 902, 914; 111, 183, 219, 291, 471, 723, 903, 987.

III \_ m = 9 \text{ for } m = 83, 127, 151, 223, 227, 271, 439, 481, 683, 703, 727, 827, 899, 919, 923; 362, 422; 123, 699, 843; 618, 678.


III \_ m = 16 \text{ for } m = 73, 193, 241, 313, 337, 577, 601, 769, 817, 913, 973; 878, 974; 579; 978; −917; −422, −566; −543, −687.

III \_ m = 25 \text{ for } m = 347, 463, 587, 823; 662, 998; −293, −317, −509, −557, −787, −797, −907, −941; −842.

III \_ m = 36 \text{ for } m = 277, 349, 397, 613, 733, 877, 797.

III \_ m = 49 \text{ for } m = 967; −773.

III \_ m = 64 \text{ for } m = 457, 673, 937.

Note added in proof. Very recently, we have used a result of Gauss to obtain a relation between the values \( c_{2 \pi/3}(n), c_{\pi/3}(n) \) and the class numbers of imaginary quadratic fields. Using this relation, we can prove that \( c_{2 \pi/3}(2p) \) does not vanish (and \( \text{III}(E_{2p}/Q) \) is non-trivial) for any prime number \( p \) congruent to 31 (mod 48), for example (see \([Yo2]\)).

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