ON THE ASYMPTOTIC BEHAVIOUR OF AN ALGEBROID FUNCTION OF ORDER ZERO

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1. Introduction

In this paper we shall investigate a relation between the maximum modulus and the minimum modulus of an algebroid function of order zero. We denote the Nevanlinna deficiency and the Valiron deficiency of value a for $f(z)$ by $\delta(a, f)$ and $\Delta(a, f)$, respectively (see [4, 5, 8, 9, 17]).

We denote the maximum modulus $M(r, f)$ and the minimum modulus $L(r, f)$ of a meromorphic function $f(z)$ by

$$M(r, f) = \sup_{|z|=r} |f(z)|,$$

$$L(r, f) = \inf_{|z|=r} |f(z)|,$$

respectively. We shall assume that $f(z)$ is transcendental, i.e. that

$$\log r = o(T(r, f)) \quad (r \rightarrow \infty).$$

If E is a measurable set on $(0, \infty)$ we define its density as

$$\lim_{r \rightarrow \infty} \frac{m(E \cap (0, r))}{r}$$

if the limit exists, and its upper density by replacing $\lim$ by $\limsup$, where $m(E \cap (0, r))$ denotes the measure of $E \cap (0, r)$.

It is well known that if $f(z)$ is an entire function of order zero, then

$$\log L(r, f) \sim \log M(r, f) \sim T(r, f)$$

in a set of r of upper density 1 [3]. Kubota [7] showed the following striking and elegant theorem which is an analogous result for a meromorphic function of order zero.

THEOREM. Let $f(z)$ be a meromorphic function of order zero. If $\delta(\infty, f) > 0$ then

$$\log L(r, f) \sim \log M(r, f)$$

and

$$\delta(\infty, f) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq \Delta(\infty, f)$$

in a set of r of upper density 1. Hence if $\delta(\infty, f) = \Delta(\infty, f) > 0$, then

$$\log L(r, f) \sim \log M(r, f) \sim \Delta(\infty, f)T(r, f)$$

in a set of r of upper density 1.

The purpose of this paper is to extend Kubota's theorem to n -valued algebroid functions of order zero. Let $f(z)$ be an n -valued algebroid function defined by an irreducible equation

$$F(z, f) \equiv A_0(z)f^n + A_1(z)f^{n-1} + \cdots + A_{n-1}(z)f + A_n(z) = 0, \quad (1)$$

where $A_0, A_1, \dots, A_n$ are entire functions without common zeros. Let $a_1, \dots, a_n$ be complex numbers, different from each other.

We put

$$\begin{aligned} g_v(z) &= F(z, a_v) \quad (v = 1, 2, \dots, n) \\ g(z) &= \max_{1 \leq v \leq n} \{1, |g_v(z)|\} \end{aligned} \quad (2)$$

where $F(z, f) = 0$ is the defining equation of f .

We put

$$f_v(z) = \frac{g_v(z)}{A_0(z)}$$

and set

$$M(r, f) = \sup_{|z|=r} \max_{1 \leq v \leq n} |f_v(z)|, \quad (3)$$

$$L(r, f) = \inf_{|z|=r} \max_{1 \leq v \leq n} |f_v(z)|. \quad (4)$$

We shall prove the following extension of Kubota's theorem which gives a precise meaning to the asymptotic behaviour of an n -valued algebroid function of order zero.

THEOREM 1. Let $f(z)$ be an n -valued transcendental algebroid function of order zero defined by (1).

If $\delta(\infty, f) > 0$, then

$$\log L(r, f) \sim \log M(r, f) \quad (5)$$

and

$$\delta(\infty, f) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \Delta(\infty, f) \quad (6)$$

in a set of r of upper density 1.

Hence, if $\delta(\infty, f) = \Delta(\infty, f) > 0$ and $n = 1$, then

$$\log L(r, f) \sim \log M(r, f) \sim \Delta(\infty, f) T(r, f) \quad (7)$$

in a set of r of upper density 1.

THEOREM 2. Let $f(z)$ be an n -valued transcendental algebroid function of order zero defined by (1).

If $\delta(\infty, f) > 0$, then

$$\log L(r, f) \sim \log M(r, f)$$

and

$$\max_v \delta(\infty, f_v) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n^2 \max_v \delta(\infty, f_v) \quad (8)$$

in a set of r of upper density 1.

THEOREM 3. Under the same conditions of Theorem 1, we have

$$\max_v \Delta(\infty, f_v) \leq \limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \max_v \Delta(\infty, f_v) \quad (9)$$

in a set of r of upper density 1.

COROLLARY 1. Under the same conditions of Theorem 1, we have

$$\delta(\infty, f) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \delta(\infty, f) \quad (10)$$

in a set of r of upper density 1.

COROLLARY 2. Under the same conditions of Theorem 1, we have

$$\frac{1}{n} \Delta(\infty, f) \leq \limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \Delta(\infty, f) \quad (11)$$

in a set of r of upper density 1.

2. Preliminaries

Let $f(z)$ be an n -valued transcendental algebroid function defined by an irreducible equation (1). Let $y_j(z)$ be the j th determination of $f(z)$, $1 \leq j \leq n$, and

$$\mu(r, A) = \frac{1}{2\pi n} \int_0^{2\pi} \log A(re^{i\theta}) d\theta,$$

where

$$A(z) = \max_{0 \leq v \leq n} \{1, |A_v(z)|\}.$$

Then, by a theorem of Valiron [16], we have

$$\mu(r, A) = T(r, f) + O(1). \quad (12)$$

Ozawa [10] showed that

$$\mu(r, g) = \mu(r, A) + O(1), \quad (13)$$

and

$$\sum_{j=1}^n \log^+ |y_j(z)| \leq \log \left| \frac{A(z)}{A_0(z)} \right| + O(1). \quad (14)$$

On the other hand, Tsumura [15] showed that

$$\sum_{j=1}^n \log^+ |y_j(z)| \geq \log \left| \frac{A(z)}{A_0(z)} \right| - O(1). \quad (15)$$

In order to prove our theorem, we need the following lemmas. Lemmas 1 and 2 are essentially the same as lemmas in Boas [2] and Cartwright [3], hence we omit their proofs.

LEMMA 1. *If $\varphi(z)$ is a meromorphic function of order less than one with $\varphi(0) = 1$, then for every η ($0 < \eta < \frac{8}{3}e$) we have*

$$\begin{aligned} & |\log |\varphi(z)| - \{N(2R, 0; \varphi) - N(2R, \infty; \varphi)\}| \\ & < \left(1 + \log \frac{4e}{\eta}\right) \{Q(2R, 0; \varphi) + Q(2R, \infty; \varphi)\} \end{aligned} \quad (16)$$

in $|z| < R$, outside a set of circles the sum of whose radii is at most $2\eta R$, where

$$Q(r, a; \varphi) = r \int_r^\infty \frac{n(t, a; \varphi)}{t^2} dt.$$

LEMMA 2. *If $\varphi(z)$ is of order zero, then*

$$\liminf_{r \rightarrow \infty} \frac{Q(r, 0; \varphi) + Q(r, \infty; \varphi)}{N(r)} = 0, \quad (17)$$

where $N(r) = N(r, 0; \varphi) + N(r, \infty; \varphi)$.

The following lemma plays an important role for the proof of our theorem.

LEMMA 3. *Let $f(z)$ be an n -valued transcendental algebroid function defined by (1). Then we have*

$$m(r, f) \leq \frac{1}{n} \sum_{v=1}^n m(r, f_v) + O(1). \quad (18)$$

On the other hand we have

$$m(r, f) \geq \max_v \frac{1}{n} m(r, f_v) - O(1). \quad (19)$$

Proof. First we can consider

$$g_v(z) = \sum_{j=0}^n a_v^j A_{n-j}(z), \quad v = 1, 2, \dots, n \quad (20)$$

as a system of linear equations in $A_1(z), \dots, A_n(z)$.

Hence we can solve from (20)

$$A_v(z) = \sum_{i=1}^n \alpha_{v,i} g_i(z) + \alpha_{v,0} A_0(z), \quad v = 1, 2, \dots, n$$

where $\alpha_{v,i}$ are constants.

Then we obtain

$$\begin{aligned} \log \left| \frac{A}{A_0} \right| &= \log \left(\max_{1 \leq v \leq n} \left| \frac{A_v}{A_0} \right| \right) \\ &\leq \log \left(\sum_{v=1}^n \left| \frac{g_v}{A_0} \right| \right) + O \left(\log \sum_{v=0}^n |\alpha_{v,i}| \right) \\ &= \log \left(\sum_{v=1}^n |f_v| \right) + O(1) \\ &\leq \sum_{v=1}^n \log^+ |f_v| + O(1). \end{aligned}$$

Thus we have, in view of (14),

$$\begin{aligned} m(r, f) &= \frac{1}{2n\pi} \int_0^{2\pi} \sum_{j=1}^n \log^+ |y_j(re^{i\theta})| d\theta \\ &\leq \frac{1}{2n\pi} \int_0^{2\pi} \sum_{v=1}^n \log^+ |f_v(re^{i\theta})| d\theta + O(1) \\ &= \frac{1}{n} \sum_{v=1}^n m(r, f_v) + O(1) \end{aligned}$$

which gives (18).

Next we use the following inequality:

$$\max_v \log |g_v| \leq \max_v \log \left| \sum_{j=0}^n A_{n-j} a_v^j \right| \leq \log A + O(1),$$

which follows from (20); then we have, by using (15),

$$\begin{aligned} m(r, f) &= \frac{1}{2n\pi} \int_0^{2\pi} \sum_{j=1}^n \log^+ |y_j(re^{i\theta})| d\theta \\ &\geq \frac{1}{2n\pi} \int_0^{2\pi} \log \left| \frac{A(re^{i\theta})}{A_0(re^{i\theta})} \right| d\theta - O(1) \\ &= \frac{1}{2n\pi} \int_0^{2\pi} \log^+ \left| \frac{A(re^{i\theta})}{A_0(re^{i\theta})} \right| d\theta - O(1) \\ &\geq \frac{1}{2n\pi} \int_0^{2\pi} \max_v \log^+ \left| \frac{g_v(re^{i\theta})}{A_0(re^{i\theta})} \right| d\theta - O(1) \\ &\geq \max_{1 \leq v \leq n} \frac{1}{n} m(r, f_v) - O(1) \end{aligned}$$

which gives (19). This completes the proof of Lemma 3. $\square$

By using Lemma 3 we have the following lemma which is analogous to the results of Selberg [14]:

LEMMA 4. *Let $f(z)$ and $f_v(z)$ ($v = 1, 2, \dots, n$) be defined in (1) and (2), respectively. Then we have*

$$\frac{1}{n} T(r, f_v) - O(1) < T(r, f) < \frac{1}{n} \sum_{v=1}^n T(r, f_v) + O(1). \quad (21)$$

Proof. In view of (19), we have

$$\begin{aligned} T(r, f) &= m(r, f) + N(r, f) \\ &\geq \frac{1}{n}m(r, f_v) - O(1) + N(r, 1/A_0) \\ &> \frac{1}{n}\{m(r, f_v) + N(r, f_v)\} - O(1) \\ &= \frac{1}{n}T(r, f_v) - O(1). \end{aligned}$$

Also because of (18), we obtain

$$\begin{aligned} T(r, f) &\leq \frac{1}{n} \sum_{v=1}^n m(r, f_v) + O(1) + N(r, 1/A_0) \\ &= \frac{1}{n} \sum_{v=1}^n m(r, f_v) + \frac{1}{n} \sum_{v=1}^n N(r, f_v) + O(1) \\ &= \frac{1}{n} \sum_{v=1}^n T(r, f_v) + O(1), \end{aligned}$$

so the lemma is proved. $\square$

3. The relation between $\delta(\infty, f)$ and $\delta(\infty, f_v)$ or $\Delta(\infty, f)$ and $\Delta(\infty, f_v)$

We shall survey a relation between $\delta(\infty, f)$ and $\delta(\infty, f_v)$, and also a relation between $\Delta(\infty, f)$ and $\Delta(\infty, f_v)$.

LEMMA 5. *Let $f(z)$ be an n -valued transcendental algebroid function defined by an irreducible equation (1). Then we have*

$$\delta(\infty, f) \geq \delta(\infty, f_v) \quad (v = 1, 2, \dots, n), \quad (22)$$

$$\Delta(\infty, f) \geq \Delta(\infty, f_v) \quad (v = 1, 2, \dots, n). \quad (23)$$

Proof. By making some mention of the conception of Valiron [16], and using (12), we have

$$\begin{aligned} \delta(\infty, f) &= \liminf_{r \rightarrow \infty} \frac{m(r, f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, f)}{T(r, f)} \right) \\ &= \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, 1/A_0)}{n\mu(r, A)} \right). \end{aligned}$$

In view of (21), we obtain

$$\delta(\infty, f) \geq \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, 1/A_0)}{T(r, f_v)} \right) = \delta(\infty, f_v) \quad (v = 1, 2, \dots, n),$$

which gives (22).

Similarly we can deduce (23), which gives the validity of Lemma 5. $\square$

The following lemma shows an essential relation of deficiencies between $\delta(\infty, f)$ and $\delta(\infty, f_v)$.

LEMMA 6. *Let $f(z)$ be an n -valued algebroid function defined by (1). Then we have some suitable functions $\{f_v\}$ such as*

$$\delta(\infty, f) \leq n\delta(\infty, f_v) \quad (v = 1, 2, \dots, n). \quad (24)$$

Proof. Let $\mathfrak{A} = (A_0(z), \dots, A_n(z))$ be a holomorphic curve. For $B = (\beta_0, \dots, \beta_n) \in \mathbb{P}^n$, we have [18, p 93 Theorem 4.5],

$$N(r, B) + m(r, B) = T(r, \mathfrak{A}) + O(1). \quad (25)$$

Hence we have $N(r, B) \sim T(r, \mathfrak{A})$ for $B \notin \mathcal{B}$, where $\mathcal{B}$ is the set of Valiron deficiencies which is $\mathcal{P}$ -polar[†] in the projective space of hyperplanes in $\mathbb{P}^n$ in view of [6, 12].

Suppose $B_j = (a_j^n, a_j^{n-1}, \dots, a_j, 1) \notin \mathcal{B}$, $1 \leq j \leq n$. Then

$$m(r, B_j) = o(N(r, B_j)) = o(T(r, \mathfrak{A})).$$

Write

$$h_j(z) = a_j^{n-1}A_1(z) + \dots + A_n(z) \equiv g_j(z) - a_j^n A_0(z).$$

Then, writing

$$\mathbb{A}_j = \|B_j\| = \sqrt{\sum_{i=0}^n |a_j^{n-i}|^2},$$

[†]A set $E \subset D$ is called $\mathcal{P}$ -polar if it is the set of singular points of some plurisuperharmonic function on D which is not identically $+\infty$.

$$\begin{aligned}
m(r, B_j) &= \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\mathfrak{A}\|}{|g_j(z)|} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\mathbb{A}_j \sqrt{1 + |A_1(z)/A_0(z)|^2 + \cdots + |A_n(z)/A_0(z)|^2}}{|f_j(z)|} d\theta \\
&\quad + \log \frac{1}{\mathbb{A}_j}.
\end{aligned}$$

Thus

$$m\left(r, \frac{1}{f_j}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f_j(z)} \right| d\theta \leq m(r, B_j) + O(1).$$

Suppose, for each j , $1 \leq j \leq n$, $A_0(z)$ and $h_j(z)$ have no common zeros.

$$\begin{aligned}
N(r, 1/f_j) + m(r, 1/f_j) + O(1) &= N(r, f_j) + m(r, f_j) = T(r, f_j), \\
N(r, 1/f_j) &= N(r, B_j) \sim T(r, \mathfrak{A}), \\
m(r, 1/f_j) &\leq m(r, B_j) = o(N(r, B_j)), \\
N(r, f_j) &= N(r, 1/A_0).
\end{aligned}$$

Thus

$$m(r, f_j) \sim T(r, \mathfrak{A}) - N(r, 1/A_0) \sim m(r, f_{j'}),$$

for any j, j' .

On the other hand, we have

$$\begin{aligned}
m(r, f) &\leq \frac{1}{n} \sum_{j=1}^n m(r, f_j) + O(1), \\
T(r, f_j) &\leq nT(r, f) + O(1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{m(r, f)}{T(r, f)} &\leq \sum_{j=1}^n \frac{m(r, f_j)}{nT(r, f)} \\
&\sim \frac{nm(r, f_j)}{nT(r, f)} (1 + o(1)) \leq \frac{nm(r, f_j)}{T(r, f_j)} \quad (1 \leq j \leq n).
\end{aligned}$$

Hence we deduce

$$\delta(\infty, f) \leq n\delta(\infty, f_v)$$

for all v .

□

LEMMA 7. *Let $f(z)$ be an n -valued algebroid function defined by (1). Then we have*

$$\Delta(\infty, f) \leq n \max_v \Delta(\infty, f_v). \quad (26)$$

Proof. In view of (21) and (18), we have

$$\begin{aligned} \Delta(\infty, f) &= \limsup_{r \rightarrow \infty} \frac{m(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\frac{1}{n} \sum_{v=1}^n m(r, f_v) + O(1)}{T(r, f)} \\ &= \limsup_{r \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n \frac{m(r, f_v)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{1}{n} \sum_{v=1}^n \frac{m(r, f_v)}{(1/n)T(r, f_v) - O(1)} \\ &\leq \sum_{v=1}^n \limsup_{r \rightarrow \infty} \frac{m(r, f_v)}{T(r, f_v)} = \sum_{v=1}^n \Delta(\infty, f_v) \\ &\leq n \max_v \Delta(\infty, f_v). \quad \square \end{aligned}$$

The following lemma announces that our definitions of the maximum modulus $M(r, f)$ and the minimum modulus $L(r, f)$ of an algebroid function would seem to be independent of the selection of $\{a_v\}$ in view of (2)–(4).

LEMMA 8. *Let $\{a_v\}, \{b_v\}$ be distinct complex numbers of two sets. We put*

$$g_v(z) = F(z, a_v), \quad g_v^*(z) = F(z, b_v) \quad (v = 1, 2, \dots, n)$$

where $F(z, f) = 0$ is the defining equation of f .

We put

$$f_v(z) = \frac{g_v(z)}{A_0(z)}, \quad f_v^*(z) = \frac{g_v^*(z)}{A_0(z)}.$$

Then we have

$$\begin{aligned} \log M(r, f^*) &= \log M(r, f) + O(1), \\ \log L(r, f^*) &= \log L(r, f) + O(1). \end{aligned}$$

Proof. By the definition we consider

$$g_v(z) = \sum_{j=0}^n a_v^j A_{n-j}(z) \quad (v = 1, 2, \dots, n),$$

so we can solve

$$A_v(z) = \sum_{i=1}^n \alpha_{v,i} g_i(z) + \alpha_{v,0} A_0(z) \quad (v = 1, 2, \dots, n)$$

where $\alpha_{v,i}$ are constants. Similarly we have

$$g_v^*(z) = \sum_{j=0}^n b_v^j A_{n-j}(z) \quad (v = 1, 2, \dots, n),$$

and

$$A_v(z) = \sum_{i=1}^n \beta_{v,i} g_i^*(z) + \beta_{v,0} A_0(z) \quad (v = 1, 2, \dots, n)$$

where $\beta_{v,i}$ are constants.

Thus we have from (3)

$$\begin{aligned} \log M(r, f) &= \sup_{|z|=r} \max_v \log |f_v| \\ &= \sup_{|z|=r} \max_v \log \left| \frac{g_v}{A_0} \right| \\ &\leq \sup_{|z|=r} \max_v \log \left(\sum_{v=0}^n \left| \frac{A_v}{A_0} \right| \right) + O \left(\log \sum_{v=1}^n |a_v^j| \right) \\ &\leq \sup_{|z|=r} \log \left(\max_v \left| \frac{A_v}{A_0} \right| \right) + O(1) \\ &= \sup_{|z|=r} \log \left(\max_v \left| \frac{\sum_{i=1}^n \beta_{v,i} g_i^*(z) + \beta_{v,0} A_0(z)}{A_0} \right| \right) + O(1) \\ &\leq \sup_{|z|=r} \log \left(\max_v \left| \frac{g_v^*}{A_0} \right| \right) + O(1) \\ &= \sup_{|z|=r} \log \left(\max_v |f_v^*| \right) + O(1) \\ &= \sup_{|z|=r} \max_v \log |f_v^*| + O(1) \\ &= \log M(r, f^*) + O(1). \end{aligned}$$

The same reasoning yields

$$\log M(r, f^*) \leq \log M(r, f) + O(1).$$

Thus we obtain

$$\log M(r, f^*) = \log M(r, f) + O(1).$$

Similarly we have

$$\log L(r, f^*) = \log L(r, f) + O(1).$$

□

Further, we shall show a relation between the maximum modulus of $M(r, f)$ of ours and the $M(r, y)$ introduced by Rauch [11] (see [13]) in the following.

LEMMA 9. *Let*

$$\log M(r, y) = \sup_{|z|=r} \max_v \log |y_v(z)|,$$

$$\log L(r, y) = \inf_{|z|=r} \max_v \log |y_v(z)|,$$

which are both definitions of Valiron [16], are independent of $\{a_v\}$ entirely.

Then we have

$$\log M(r, y) - O(1) \leq \log M(r, f) \leq n \log M(r, y) + O(1), \quad (27)$$

$$\log L(r, y) - O(1) \leq \log L(r, f) \leq n \log L(r, y) + O(1). \quad (28)$$

Proof. First we have, in view of (14), (15) and (3),

$$\begin{aligned} \log M(r, y) &= \sup_{|z|=r} \max_v \log |y_v(z)| \\ &\leq \sup_{|z|=r} \sum_{v=1}^n \log |y_v(z)| \\ &\leq \sup_{|z|=r} \sum_{v=1}^n \log^+ |y_v(z)| \leq \sup_{|z|=r} \log \left| \frac{A(z)}{A_0(z)} \right| + O(1) \\ &= \sup_{|z|=r} \log \left(\max_v \left| \frac{A_v}{A_0} \right| \right) + O(1) \\ &\leq \sup_{|z|=r} \log \left(\sum_{v=1}^n \left| \frac{g_v}{A_0} \right| \right) + O \left(\log \sum_{v=0}^n |\alpha_{v,i}| \right) \\ &= \sup_{|z|=r} \log \left(\sum_{v=1}^n |f_v| \right) + O(1) \\ &\leq \sup_{|z|=r} \log(n \max_v |f_v|) + O(1) \\ &\leq \sup_{|z|=r} \log(\max_v |f_v|) + O(1) \\ &= \sup_{|z|=r} \max_v \log |f_v| + O(1) = \log M(r, f) + O(1). \end{aligned}$$

Next we have, by using (20),

$$\begin{aligned}
 \log M(r, f) &= \sup_{|z|=r} \max_v \log |f_v| = \sup_{|z|=r} \max_v \log \left| \frac{g_v}{A_0} \right| \\
 &\leq \sup_{|z|=r} \log \left| \sum_0^n \frac{A_{n-j} a_v^j}{A_0} \right| \leq \sup_{|z|=r} \log \left| \frac{A}{A_0} \right| + O(1) \\
 &\leq \sup_{|z|=r} \sum_{v=1}^n \log^+ |y_v| + O(1) \\
 &\leq \sup_{|z|=r} \left(n \max_v \log |y_v| \right) + O(1) \\
 &= n \log M(r, y) + O(1).
 \end{aligned}$$

Thus we deduce the inequality (27).

Similarly we can deduce (28), which gives the validity of Lemma 8. $\square$

4. Proof of Theorem 1

Applying Lemma 1 for $\varphi(z) = f_v(z)$ and putting the right-hand side of (17) as $Q_0^{(v)}(2R)$, we have

$$Q_0^{(v)}(2R) = Q(2R, 0, f_v) + Q(2R, \infty, f_v).$$

First we assume that each meromorphic function $f_v(z)$ ($v = 1, 2, \dots, n$) satisfies

$$T(r, f_v) \sim N_v(r, 0, f_v), \quad f_v(0) = 1. \quad (29)$$

Suppose $\delta(\infty, f) > 0$, then we can say that there exists some suitable functions f_v ($1 \leq v \leq n$), such that $\delta(\infty, f_v) > 0$ in view of (24), so that there is for some positive numbers ρ_v ($v = 1, \dots, n$), $0 < \rho_v < \delta(\infty, f_v)$, and $R_0^{(v)}$, by the definition of deficiency

$$N(2R, 0, f_v) - N(2R, \infty, f_v) > \frac{\rho_v}{2 - \rho_v} N_v(2R) \quad (R \geq R_0^{(v)})$$

where

$$N_v(r) = N(r, 0, f_v) + N(r, \infty, f_v) \quad (v = 1, \dots, n).$$

Thus, by putting $\rho = \min_v \rho_v$, we have

$$N(2R, 0, f_v) - N(2R, \infty, f_v) > \frac{\rho}{2 - \rho} N_v(2R) \quad (R \geq R_0^{(v)}). \quad (30)$$

Applying Lemma 1 for $\varphi(z) = f_v(z)$ and in view of (16), we have

$$\left| \frac{\log |f_v(z)|}{N(2R, 0, f_v) - N(2R, \infty, f_v)} - 1 \right| \leq \left(1 + \log \frac{4e}{\eta_v} \right) \frac{2 - \rho}{\rho} \frac{Q_0^{(v)}(2R)}{N_v(2R)} \quad (31)$$

in $|z| < R$, outside a set of circles the sum of whose radii is at most $2\eta_v R$ provided $R \geq R_0^{(v)}$ and $0 < \eta_v < \frac{8}{3}e$.

Let $\varepsilon (>0)$ be given. By Lemma 2 it is possible to choose an arbitrarily large number $R_\varepsilon^{(v)}$ such that

$$\left(1 + \log \frac{4e}{\eta_v} \right) \frac{2 - \rho}{\rho} \frac{Q_0^{(v)}(r)}{N_v(r)} < \varepsilon \quad (32)$$

for each v ($1 \leq v \leq n$) in a set $E(\eta_v, \varepsilon)$ of r ($< R_\varepsilon^{(v)}$) of measure at least $(1 - 2\eta_v)R_\varepsilon^{(v)}$.

Then we have, by setting $R_\varepsilon = \min_v R_\varepsilon^{(v)}$, in view of (29)–(32), and using the first fundamental theorem,

$$\left| \frac{\log |f_v(z)|}{m(r, f_v)} - 1 \right| < \varepsilon \quad (|z| < r < R_\varepsilon) \quad (33)$$

in each set $E(\eta_v, \varepsilon)$ of r .

Thus we deduce

$$(1 - \varepsilon)m(r, f_v) < \log |f_v(z)| < (1 + \varepsilon)m(r, f_v) \quad (v = 1, 2, \dots, n) \quad (34)$$

by (33) in a set

$$E(\eta, \varepsilon) = \bigcap_v E(\eta_v, \varepsilon).$$

We have then from the right-hand side of (34), by taking the maximum for v ,

$$\max_v \log |f_v(z)| < (1 + \varepsilon) \max_v m(r, f_v)$$

and further, by taking the supremum at $|z| = r$ ($r \in E(\eta, \varepsilon)$) of both sides of the above inequality, we have by using Lemma 8,

$$\log M(r, f) - O(1) = \sup_{|z|=r} \max_v \log |f_v(z)| \leq (1 + \varepsilon) \max_v m(r, f_v).$$

Similarly we have from the left-hand side of (34), in view of Lemma 8,

$$\log L(r, f) + O(1) = \inf_{|z|=r} \max_v \log |f_v(z)| \geq (1 - \varepsilon) \max_v m(r, f_v).$$

Hence we deduce by using the above two inequalities

$$\frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{(1 + o(1)) \log L(r, f)}{(1 - o(1)) \log M(r, f)} \leq 1 \quad (35)$$

in a set $E(\eta, \varepsilon)$.

Therefore, it follows from (35) that

$$\begin{aligned} \frac{1 - \varepsilon}{1 + \varepsilon} \frac{\max_v m(r, f_v)}{T(r, f)} &\leq \frac{(1 + o(1)) \log L(r, f)}{T(r, f)} \\ &\leq \frac{(1 - o(1)) \log M(r, f)}{T(r, f)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \frac{\max_v m(r, f_v)}{T(r, f)} \end{aligned} \quad (36)$$

in $E(\eta, \varepsilon)$.

Since η and ε are arbitrary, we conclude that

$$\log L(r, f) \sim \log M(r, f) \quad (37)$$

from (35) too, which proves (5).

Also we deduce from the above inequality (36) that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f_v)} &\leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} \end{aligned} \quad (38)$$

in a set of r of upper density 1.

In view of (18), we now have

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} &\geq \liminf_{r \rightarrow \infty} \frac{\frac{1}{n} \sum_{v=1}^n m(r, f_v)}{T(r, f)} \geq \liminf_{r \rightarrow \infty} \frac{m(r, f) - O(1)}{T(r, f)} \\ &= \delta(\infty, f). \end{aligned} \quad (39)$$

Similarly we get, in view of (19),

$$\limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{nm(r, f) + O(1)}{T(r, f)} = n\Delta(\infty, f). \quad (40)$$

Thus we have, by combining (39) and (40) with (38),

$$\delta(\infty, f) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n\Delta(\infty, f).$$

This proves the inequality (6). Consequently, (7) is yielded obviously.

Next if $f_v(z)$ does not satisfy the asymptotic relation (29), we choose γ_v ($\neq 0$) such that

$$N\left(r, \frac{1}{f_v - \gamma_v}\right) \sim T(r, f_v) \quad \text{and} \quad f_v(0) \neq \gamma_v \quad (v = 1, 2, \dots, n).$$

Put $F_v(z) = c_v\{f_v(z) - \gamma_v\}$, where $F_v(0) = 1$. Then we have

$$T(r, f_v) \sim T(r, F_v) \sim N\left(r, \frac{1}{F_v}\right) \quad \text{and} \quad N(r, f_v) = N(r, F_v),$$

consequently $m(r, f_v) \sim m(r, F_v)$.

Thus the hypotheses in the theorem and the additional property hold with $f_v(z)$ replaced by $F_v(z)$. Hence the conclusion of the theorem holds with $f(z)$. This proves the general validity of the theorem. $\square$

5. Proof of Theorem 2

A discussion similar to that for Theorem 1 yields

$$\log L(r, f) \sim \log M(r, f)$$

in view of (35) since ε is arbitrary, in a set of r of upper density 1.

Further, (36) in Theorem 1 shows that

$$\begin{aligned} \frac{1 - \varepsilon}{1 + \varepsilon} \frac{\max_v m(r, f_v)}{T(r, f)} &\leq \frac{(1 + o(1)) \log L(r, f)}{T(r, f)} \\ &\leq \frac{(1 - o(1)) \log M(r, f)}{T(r, f)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \frac{\max_v m(r, f_v)}{T(r, f)} \end{aligned} \quad (41)$$

in $E(\eta, \varepsilon)$.

The left-hand side of (41) is

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} &\geq \liminf_{r \rightarrow \infty} \frac{\frac{1}{n} \max_v m(r, f_v)}{T(r, f)} \\ &\geq \liminf_{r \rightarrow \infty} \frac{\frac{1}{n} \sum_{v=1}^n m(r, f_v)}{T(r, f)} \\ &\geq \liminf_{r \rightarrow \infty} \frac{m(r, f) - O(1)}{T(r, f)} \\ &= \delta(\infty, f) \\ &\geq \max_v \delta(\infty, f_v) \end{aligned}$$

by using (22) in Lemma 5.

The right-hand side of (41) is

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} &\leq \liminf_{r \rightarrow \infty} \frac{nm(r, f) + O(1)}{T(r, f)} \\ &= n \liminf_{r \rightarrow \infty} \frac{m(r, f)}{T(r, f)} \\ &= n\delta(\infty, f) \leq n^2 \max_v \delta(\infty, f_v) \end{aligned}$$

in view of (19) and (24).

Thus we obtain the desired result, which proves Theorem 2. $\square$

6. Proof of Theorem 3

Similarly by using (20), the left-hand side is

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} &\geq \limsup_{r \rightarrow \infty} \frac{\frac{1}{n} n \max_v m(r, f_v)}{T(r, f)} \\ &\geq \limsup_{r \rightarrow \infty} \frac{\frac{1}{n} \sum_{v=1}^n m(r, f_v)}{T(r, f)} \\ &\geq \limsup_{r \rightarrow \infty} \frac{m(r, f) - O(1)}{T(r, f)} = \Delta(\infty, f) \\ &\geq \Delta(\infty, f_v) \quad (v = 1, 2, \dots, n) \end{aligned}$$

in view of (23).

Thus

$$\limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} \geq \max_v \limsup_{r \rightarrow \infty} \frac{m(r, f_v)}{T(r, f_v)} = \max_v \Delta(\infty, f_v).$$

In view of (21), the right-hand side is

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f)} &\leq \limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{\frac{1}{n} T(r, f_v) - O(1)} \quad (v = 1, 2, \dots, n) \\ &= n \limsup_{r \rightarrow \infty} \frac{\max_v m(r, f_v)}{T(r, f_v)} \\ &= n \max_v \limsup_{r \rightarrow \infty} \frac{m(r, f_v)}{T(r, f_v)} \\ &= n \max_v \Delta(\infty, f_v). \end{aligned}$$

Thus we deduce from (41)

$$\max_v \Delta(\infty, f_v) \leq \limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \max_v \Delta(\infty, f_v).$$

This proves Theorem 3. $\square$

7. Proofs of Corollaries 1 and 2

Proof of Corollary 1. It follows from (22) that

$$\sum_{v=1}^n \delta(\infty, f_v) \leq \sum_{v=1}^n \delta(\infty, f) = n\delta(\infty, f) \quad (42)$$

then we have by combining the right-hand side of (8) and the left-hand side of (6)

$$\delta(\infty, f) \leq \liminf_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n\delta(\infty, f).$$

This proves (10). $\square$

Proof of Corollary 2. It follows from (26) that

$$\Delta(\infty, f) \leq n \max_v \Delta(\infty, f_v).$$

Then we have by combining the left-hand side of (9) and the right-hand side of (6)

$$\frac{1}{n} \Delta(\infty, f) \leq \limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{T(r, f)} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{T(r, f)} \leq n \Delta(\infty, f).$$

This proves (11). $\square$

8. Example

If the hypothesis $\delta(\infty, f) > 0$ is omitted, the conclusion of Theorem 1 is no longer true. For this reason $\delta(\infty, f) > 0$ is the essential condition for Theorem 1.

For instance, consider the functions

$$A_0(z) = \prod \left(1 - \frac{z}{e^n}\right), \quad A_1(z) = \prod \left(1 + \frac{z}{e^n}\right), \quad (43)$$

and put

$$\varphi(z) = \frac{A_1(z)}{A_0(z)}. \quad (44)$$

Then $\varphi(z)$ is of order zero and obviously

$$\log L(r, \varphi) = -\log M(r, \varphi).$$

Let $f(z)$ be an algebroid function defined by

$$F(z, f) \equiv A_0(z)f^n - A_1(z)f^{n-1} + A_1(z) - A_0e^{i\theta} = 0 \quad (\theta \text{ is irrational}) \quad (45)$$

which is irreducible.

The irreducibility of this equation (45) can be showed by the following.

Suppose that

$$A_0f^n - A_1f^{n-1} + A_1 - A_0e^{i\theta} \equiv A(z, f)B(z, f). \quad (46)$$

When $n = 2$, the irreducibility of (46) is obvious. When $n = 3$, suppose that

$$A_0f^3 - A_1f^2 + A_1 - A_0e^{i\theta} = (\alpha_2f^2 + \alpha_1f + \alpha_0)(\beta_1f + \beta_0),$$

then we have

$$\alpha_0\beta_0 = A_1 - A_0e^{i\theta}, \quad \alpha_1\beta_0 + \alpha_0\beta_1 = 0, \quad \alpha_2\beta_0 + \alpha_1\beta_1 = -A_1.$$

Hence there is some suitable function k which satisfies

$$\alpha_1 = k\alpha_0, \quad \beta_1 = -k\beta_0$$

from the second equation, where α_0, β_0 are prime for each other. Then we deduce the identity

$$\alpha_2\beta_0 - k^2\alpha_0\beta_0 = -A_1.$$

From the third equation which shows that A_1 is a multiple of β_0 . But the zeros of β_0 are the zeros of $A_1 - A_0e^{i\theta}$ from the first equation, which is a contradiction.

Thus equation (46) is irreducible.

For general $n \in \mathbb{N}$, take m, l such that $m \geq l$ and $m + l = n$, and

$$\begin{aligned} A_0f^n - A_1f^{n-1} + A_1 - A_0e^{i\theta} &= (\alpha_m f^m + \alpha_{m-1} f^{m-1} + \cdots + \alpha_1 f + \alpha_0) \\ &\quad \times (\beta_l f^l + \beta_{l-1} f^{l-1} + \cdots + \beta_1 f + \beta_0). \end{aligned}$$

Then we have $n + 1$ identities as follows:

$$\alpha_0\beta_0 = A_1 - A_0e^{i\theta}, \quad (47)$$

$$\alpha_1\beta_0 + \alpha_0\beta_1 = 0, \quad (48)$$

$$\alpha_2\beta_0 + \alpha_1\beta_1 + \alpha_0\beta_2 = 0, \quad (49)$$

$$\alpha_3\beta_0 + \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_0\beta_3 = 0, \quad (50)$$

...

$$\alpha_m\beta_0 + \alpha_{m-1}\beta_1 + \alpha_{m-2}\beta_2 + \cdots + \alpha_{m-l}\beta_l = 0, \quad (51)$$

$$\alpha_m\beta_1 + \alpha_{m-2}\beta_2 + \cdots + \alpha_{m-l+1}\beta_l = 0,$$

...

$$\alpha_m\beta_{l-2} + \alpha_{m-1}\beta_{l-2} + \alpha_{m-2}\beta_l = 0,$$

$$\alpha_m\beta_{l-1} + \alpha_{m-1}\beta_l = -A_1, \quad (52)$$

$$\alpha_m\beta_l = A_0. \quad (53)$$

Hence similarly we get

$$\alpha_1 = k\alpha_0, \quad \beta_1 = -k\beta_0 \quad (54)$$

from (48), where α_0, β_0 are prime for each other, and k is some suitable function. Then we deduce the identity

$$\alpha_2\beta_0 - k^2\alpha_0\beta_0 + \alpha_0\beta_2 = 0$$

from (49), and we get

$$\alpha_2 = K_2\alpha_0, \quad \beta_2 = K'_2\beta_0 \quad (55)$$

where K_2, K'_2 are some suitable functions.

By using (54) and (55) with (50), we can obtain

$$\alpha_3 = K_3\alpha_0, \quad \beta_3 = K'_3\beta_0$$

where K_3, K'_3 are some suitable functions.

Now we repeat the same procedure to the identity (51), we can deduce at last

$$\alpha_j = K_j\alpha_0 \quad (j = 1, 2, \dots, m), \quad \beta_j = K'_j\beta_0 \quad (j = 1, 2, \dots, l) \quad (56)$$

where K_j, K'_j are suitable functions.

Thus we have the identity

$$K_m\alpha_0K'_{l-1}\beta_0 + K_{m-1}\alpha_0K'_l\beta_0 = -A_1,$$

from (52) which shows that A_1 is a multiple of $\alpha_0\beta_0$.

Hence it follows also that A_0 is a multiple of $\alpha_0\beta_0$ from (47) or (53), which is a contradiction.

Next we consider the case that β_0 may be zero-free, for example a constant. When $n = 3$, suppose that

$$A_0f^3 - A_1f^2 + A_1 - A_0e^{i\theta} = (\alpha_2f^2 + \alpha_1f + \alpha_0)(\beta_1f + \beta_0),$$

then we have

$$\alpha_2\beta_1 = A_0, \quad \alpha_2\beta_0 + \alpha_1\beta_1 = -A_1, \quad \alpha_1\beta_0 + \alpha_0\beta_1 = 0, \quad \alpha_0\beta_0 = A_1 - A_0e^{i\theta}.$$

Suppose that $\beta_0 = c$, so we have

$$\alpha_1 = -\frac{\beta_1}{c}\alpha_0 \tag{57}$$

from the third equation.

From the first equation we have

$$c\alpha_0 = A_1 - A_0e^{i\theta} = A_1 - \alpha_2\beta_1e^{i\theta}$$

by using the fourth equation.

We have from the second equation that $\alpha_2 = -(A_1 + \alpha_1\beta_1)/c$, so then we have

$$c\alpha_0 = A_1 + \frac{A_1 + \alpha_1\beta_1}{c}\beta_1e^{i\theta}.$$

Hence we get

$$c\alpha_0 + \frac{\alpha_0\beta_0^3}{c}e^{i\theta} = \left(1 + \frac{\beta_1}{c}e^{i\theta}\right)A_1$$

in view of (57), and consequently

$$\frac{\alpha_0}{c}(c^3 + \beta_1^3e^{i\theta}) = (c + \beta_1e^{i\theta})A_1,$$

where $c^3 + \beta_1^3e^{i\theta}$, $c + \beta_1e^{i\theta}$ are prime for each other. This shows that A_1 is a multiple of α_0 . Thus we have a contradiction from the fourth equation.

For the general case, we use $n + 1$ identities (47)–(53) similarly. Firstly we have

$$\alpha_1 = -\frac{\beta_1}{c}\alpha_0$$

from (48), and we then get

$$\alpha_1 = K_1\alpha_0$$

where K_1 is some suitable function. Then we have from (49)

$$\alpha_2 = \left(\frac{\beta_1^2}{c^2} - \beta_2 \right) \alpha_0 \equiv K_2 \alpha_0.$$

Similarly we have from (50)

$$\alpha_3 = \left(\frac{\beta_1}{c^2} - \frac{\beta_1^2}{c^3} + \frac{\beta_2}{c} - \frac{\beta_3}{c} \right) \alpha_0 \equiv K_3 \alpha_0$$

where K_2, K_3 are some suitable functions.

By using (51), similarly we can get

$$\alpha_j = K_j \alpha_0 \quad (j = 1, 2, \dots, n), \quad (58)$$

where K_j is some suitable function.

Hence we have from (52) by using (58)

$$K_m \alpha_0 \beta_{l-1} + K_{m-1} \alpha_0 \beta_l = -A_1,$$

i.e.

$$\alpha_0 (K_m \beta_{l-1} + K_{m-1} \beta_l) = -A_1,$$

which shows that A_1 is a multiple of α_0 .

On the other hand, from (53) we have

$$K_m \alpha_0 \beta_l = A_0,$$

which shows A_0 is a multiple of α_0 also, which is a contradiction.

Consequently, this contradiction of (46) gives us that equation (45) is irreducible. $\square$

Now we consider

$$F(z, f) = A_0 f^n - A_1 f^{n-1} + A_1 - A_0 e^{i\theta} = 0$$

and

$$a_j = \exp \left[\frac{2\pi j}{n} i + \frac{\theta}{n} i \right] \quad (j = 1, 2, \dots, n).$$

Then we have

$$F(z, a_j) = \left(1 - \exp \left(\frac{n-1}{n} 2\pi j i - \frac{n-1}{n} \theta i \right) \right) A_1 \equiv \lambda_j A_1 \quad (\lambda_j \neq 0).$$

Hence $F(z, a_j) = 0$ yields $A_1 = 0$ ($j = 1, 2, \dots, n$). Then since

$$g_j(z) = F(z, a_j) = \lambda_j A_1,$$

we have, in view of (44),

$$f_j(z) = \frac{g_j(z)}{A_0(z)} = \frac{\lambda_j A_1(z)}{A_0(z)} = \lambda_j \varphi \quad (j = 1, 2, \dots, n). \quad (59)$$

Then we also have

$$\begin{aligned} N(r, a_\nu, f) &= \frac{1}{n} N(r, 0, g_\nu) = \frac{1}{n} N(r, 0, A_1), \\ N(r, \infty, f) &= \frac{1}{n} N(r, 0, A_0) = \frac{1}{n} N(r, \varphi). \end{aligned}$$

We note that

$$\begin{aligned} T(r, f) + O(1) &= \mu(r, A) = \frac{1}{2\pi n} \int_0^{2\pi} \log \max\{1, |A_0|, |A_1|\} d\theta \\ &\sim \frac{1}{n} T\left(r, \frac{A_1}{A_0}\right) = \frac{1}{n} T(r, \varphi). \end{aligned}$$

This shows at once that $f(z)$ is of order zero.

Now we estimate $m(r, \varphi)$ of function

$$\varphi(z) = \frac{A_1(z)}{A_0(z)} = \prod_{n=1}^{\infty} \frac{1 + (z/e^n)}{1 - (z/e^n)}.$$

Since $\varphi(z) = A_1(z)/A_0(z)$ satisfies a q -difference equation

$$(1 - qz)\varphi(qz) = (1 + qz)\varphi(z) \quad \left(q = \frac{1}{e}\right)$$

hence [1]

$$m(r, f) = o(\log r)^2.$$

On the other hand, since

$$n(r, \varphi) \sim \log r,$$

we have

$$N(r, \varphi) = \int_1^r \frac{\log t}{t} dt = \frac{1}{2}(\log r)^2(1 + o(1)).$$

Hence we deduce

$$T(r, \varphi) = \frac{1}{2}(\log r)^2(1 + o(1)).$$

Thus this yields

$$\delta(\infty, f) = \Delta(\infty, f) = 0.$$

Also, in view of (59),

$$\begin{aligned} \log M(r, f) &= \sup_{|z|=r} \max_v \{\log |f_1|, \log |f_2|, \dots, \log |f_n|\} \\ &= \sup_{|z|=r} (\log |\varphi| + O(1)) \\ &= \sup_{|z|=r} \log |\varphi| + O(1) \\ &= \log M(r, \varphi) = -\log L(r, \varphi) = -\inf_{|z|=r} \log |\varphi| + O(1) \\ &= -\log L(r, f) \end{aligned}$$

which means consequently that (5) does not hold.

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