EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

Aying WAN and Daqing JIANG

(Received 11 October 2000 and revised 21 March 2001)

1. Introduction

The purpose of this paper is to deal with the existence of positive periodic solutions for the general functional differential equations

\[ \dot{y}(t) = -a(t)y(t) + b(t)g(t, y(t - \tau(t))) \]  

where \( a(t) \in C(R, R), \int_0^\omega a(t) \, dt > 0, b(t) \in C(R, (0, \infty)), \tau(t) \in C(R, R), g \in C(R \times [0, \infty], [0, \infty]), \) and \( a(t), b(t), \tau(t), g(t, y) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant.

It is well known that the functional differential equation (1.1) includes many mathematical ecological equations.

For example, the Hematopoiesis model \([2, 4]\)

\[ \dot{y}(t) = -a(t)y(t) + b(t)e^{-\beta(t)(y(t - \tau(t)))}; \]  

more general the model of blood cell production \([2, 7, 9]\)

\[ \dot{y}(t) = -a(t)y(t) + b(t) \frac{1}{1 + y(t - \tau(t))^n}, \quad n > 0, \]  

\[ \dot{y}(t) = -a(t)y(t) + b(t) \frac{y(t - \tau(t))}{1 + y(t - \tau(t))^n}, \quad n > 0; \]  

and the more general Nicholson’s blowflies model \([2, 5, 6, 8]\)

\[ \dot{y}(t) = -a(t)y(t) + b(t)y(t - \tau(t))e^{-\beta(t)(y(t - \tau(t)))}. \]

The function \( a(t) \) is not necessarily positive. Since the environment fluctuates randomly, in bad conditions \( a(t) \) may be negative.

To the knowledge of the authors, there are very few works on the existence of positive periodic solutions for Equation (1.1), or even for (1.2)–(1.5). The systems
(1.2), (1.3) and (1.5) have been investigated in [4, 5, 7]. The estimates of solutions have been obtained, and they show that the solutions are uniformly bounded and uniformly-ultimately bounded. A group of conditions to guarantee the existence of positive $\omega$-periodic solutions for Equations (1.2), (1.3) and (1.5) have been obtained by applying the Yoshizawa theorem [10]. In all these papers, $a(t)$ is positive.

In this short paper, we apply a new method (the Krasnoselskii fixed point theorem [1, 3]) to establish a group of conditions to guarantee that (1.1) has to have positive periodic solutions. The conditions can be checked easily.

2. Existence of positive periodic solutions

Let $X$ be a real Banach space, and $K$ a closed, non-empty subset of $X$. $K$ is a cone provided: (i) $\alpha u + \beta v \in K$, for all $u, v \in K$ and all $\alpha, \beta \geq 0$; (ii) $u, -u \in K$ implies $u = 0$.

Suppose that:

$$(P_1) \quad a(t) \in C(R, R), \quad \int_0^\omega a(t) \, dt > 0, \quad b(t) \in C(R, (0, \infty)), \quad \tau(t) \in C(R, R),$$

$$g \in C(R \times [0, \infty), [0, \infty)), \text{ and } a(t), b(t), \tau(t), g(t, y) \text{ are all } \omega \text{-periodic functions.} \quad \omega > 0 \text{ is a constant.}$$

The main results of the present paper are as follows.

In view of the examples (1.2) and (1.3), we have the following.

**Theorem 2.1.** Assume that $(P_1)$ holds. Then Equation (1.1) has at least one $\omega$-periodic positive solution, provided the following condition holds:

$$(P_2) \quad \lim_{u \downarrow 0} \min_{t \in [0, \omega]} \frac{g(t, u)}{u} = \infty \quad \text{and} \quad \lim_{u \uparrow \infty} \max_{t \in [0, \omega]} \frac{g(t, u)}{u} = 0.$$

In view of the examples (1.4) and (1.5), we have

**Theorem 2.2.** Assume that $(P_1)$ holds and

$$(P_2) \quad \min_{t \in [0, \omega]} (b(t) - a(t)) > 0;$$

$$(P_3) \quad \text{there exists a } \varepsilon_0 > 0 \text{ such that } g(t, u) \text{ is increasing in } 0 \leq u \leq \varepsilon_0.$$ 

Then Equation (1.1) has at least one $\omega$-periodic positive solution, provided the following condition holds:

$$(P_4) \quad \lim_{u \downarrow 0} \min_{t \in [0, \omega]} \frac{g(t, u)}{u} = 1 \quad \text{and} \quad \lim_{u \uparrow \infty} \max_{t \in [0, \omega]} \frac{g(t, u)}{u} = 0.$$
The proofs of Theorems 2.1 and 2.2 will be based on an application of the following fixed point theorem due to Krasnoselskii.

**Theorem A.** (Krasnoselskii fixed point theorem [1, 3]) Let $X$ be a Banach space, and let $K \subset X$ be a cone in $X$. Assume $\Omega_1, \Omega_2$ are open subsets of $X$ with $0 \in \Omega_1, \Omega_1 \subset \Omega_2$, and let

$$
\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K
$$

be a completely continuous operator such that

$$
\|\Phi y\| \geq \|y\| \forall y \in K \cap \partial \Omega_1 \quad \text{and} \quad \|\Phi y\| \leq \|y\| \forall y \in K \cap \partial \Omega_2.
$$

Then $\Phi$ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

### 2.1. Proof of Theorem 2.1

First of all, we point out that to find a $\omega$-periodic solution of Equation (1.1) is equivalent to finding a $\omega$-periodic solution of the integral equation

$$
y(t) = \int_t^{t+\omega} G(t, s)b(s)g(s, y(s - \tau(s))) ds,
$$

where

$$
G(t, s) := \frac{\exp \left( \int_s^t a(\xi) d\xi \right)}{\exp \left( \int_0^\omega a(\xi) d\xi \right) - 1}.
$$

Let

$$
X = \{ y(t) : y(t) \in C(\mathbb{R}, \mathbb{R}), y(t + \omega) = y(t) \}
$$

and define

$$
\|y\| = \sup_{t \in [0, \omega]} \{|y(t)| : y \in X\}.
$$

Then $X$ with the norm $\| \cdot \|$ is a Banach space.

Define a operator on $X$ as follows,

$$
y = \Phi y
$$

where $\Phi$ is defined by

$$
(\Phi y)(t) = \int_t^{t+\omega} G(t, s)b(s)g(s, y(s - \tau(s))) ds,
$$

for $y \in X$. Clearly, $\Phi$ is a completely continuous operator on $X$. 
Let
\[ K = \{ y \in X : y(t) \geq 0 \text{ and } y(t) \geq \sigma \|y\| \} \]
where \(0 < \sigma = A/B < 1\), and
\[ A := \min\{G(t, s) : 0 \leq t, s \leq \omega\} > 0, \quad B := \max\{G(t, s) : 0 \leq t, s \leq \omega\} > 0. \]

(2.5)

It is not difficult to verify that \( K \) is a cone in \( X \).

**LEMMA 2.3.** Assume that \((P_1)\) holds. Then \( \Phi(K) \subseteq K \).

**Proof.** For any \( y \in K \), we have
\[ \|\Phi y\| \leq B \int_0^\omega b(s)g(s, y(s - \tau(s))) \, ds, \]
and
\[ (\Phi y)(t) \geq A \int_0^t b(s)g(s, y(s - \tau(s))) \, ds. \]
So we have
\[ (\Phi y)(t) \geq \frac{A}{B} \|\Phi y\| = \sigma \|\Phi y\|. \]
i.e. \( \Phi y \in K \). This completes the proof of Lemma 2.3. \( \square \)

We prove the conclusion under the assumptions \((P_1)\) and \((P_2)\).

Since \( \lim_{u \downarrow 0} \min_{t \in [0,\omega]} \frac{g(t, u)}{u} = \infty \), for \( M > 0 \) with \( \sigma AM \int_0^\omega b(s) \, ds > 1 \), there exists a positive constant \( R_1 > 0 \) such that
\[ g(t, u) \geq Mu, \quad \text{whenever } 0 \leq u \leq R_1. \]

(2.6)

Thus, if \( y \in K \) with \( \|y\| = R_1 \), then \( y(t) \geq \sigma R_1 \). It follows from (2.5) and (2.6) that we have
\[ (\Phi y)(t) \geq A \int_0^{t+\omega} b(s)g(s, y(s - \tau(s))) \, ds \]
\[ \geq AM \int_0^{t+\omega} b(s)y(s - \tau(s)) \, ds \]
\[ \geq AM \sigma R_1 \int_0^\omega b(s) \, ds \]
\[ > R_1 = \|y\|. \]

This implies that
\[ \|\Phi y\| > \|y\| \]
for \( y \in K \cap \partial \Omega_1 \), where \( \Omega_1 := \{ y \in X : \|y\| < R_1 \} \).
On the other hand, since \( \lim_{u \to \infty} \max_{t \in [0, \omega]} (g(t, u)/u) = 0 \), for any \( \varepsilon > 0 \) with \( B \varepsilon \int_0^\infty b(s) \, ds < 1/2 \), there exists a positive constant \( N > R_1 \) such that
\[
0 \leq g(t, u) \leq \varepsilon u
\] (2.7)
for \( u > N \).

Let \( R_2 > 2N + 2B \int_0^\infty b(s) \, ds \max_{0 \leq t \leq \omega} \max_{0 \leq u \leq N} |g(t, u)| \). (2.8)

Then for \( y \in K \) with \( \|y\| = R_2 \), we have
\[
(\Phi y)(t) \leq B \int_0^\infty b(s) g(s, y(s - \tau(s))) \, ds
= B \int_{y(s - \tau(s)) \leq N} b(s) g(s, y(s - \tau(s))) \, ds
+ B \int_{y(s - \tau(s)) > N} b(s) g(s, y(s - \tau(s))) \, ds
\leq B \int_0^\infty b(s) \, ds \max_{0 \leq t \leq \omega} \max_{0 \leq u \leq N} |g(t, u)| + B \varepsilon \int_0^\infty b(s) \, ds \|y\|
< \frac{R_2}{2} + \frac{\|y\|}{2} = \|y\|.
\]

This implies that
\[
\|\Phi y\| < \|y\|
\]
for \( y \in K \cap \partial \Omega_2 \), where \( \Omega_2 := \{ y \in X : \|y\| < R_2 \} \).

Therefore, by Theorem A, it follows that \( \Phi \) has a fixed point \( y \in K \cap (\tilde{\Omega}_2 \setminus \Omega_1) \).
Furthermore, \( R_1 \leq \|y\| \leq R_2 \) and \( y(t) \geq \sigma R_1 > 0 \), which means that \( y(t) \) is an \( \omega \)-periodic positive solution of (1.1). This completes the proof Theorem 2.1.

2.2. Proof of Theorem 2.2

Put
\[
\eta = \min_{t \in [0, \omega]} \int_t^{t+\omega} G(t, s) b(s) \, ds.
\] (2.9)

It follows from \( (P_2) \) that
\[
\eta > \min_{t \in [0, \omega]} \int_t^{t+\omega} G(t, s) a(s) \, ds = 1.
\] (2.10)
Since \( \lim_{u \to 0} \min_{t \in [0, \omega]} \frac{g(t, u)}{u} = 1 \), there exists a positive constant \( 0 < \theta < \sigma \varepsilon_0 \) such that
\[
g(t, u) \geq \frac{u}{\eta}, \quad \text{whenever } 0 \leq u \leq \theta.
\] (2.11)

Here \( G(t, s) \) and \( \sigma \) are defined in Section 2.1.

We consider the modified equation of (1.1):
\[
\dot{y}(t) = -a(t)y(t) + b(t)g^*(t, y(t - \tau(t)))
\] (2.12)

where
\[
g^*(t, u) = \begin{cases} 
  g(t, \theta), & u \leq \theta, \\
  g(t, u), & u > \theta.
\end{cases}
\]

Obviously, \( g^*(t, u) \) is non-decreasing in \([0, \varepsilon_0]\).

Similar to as in Section 2.1, we point out that to find a \( \omega \)-periodic solution of Equation (2.12) is equivalent to finding a \( \omega \)-periodic solution of the integral equation
\[
y(t) = \int_{t}^{t+\omega} G(t, s)b(s)g^*(s, y(s - \tau(s)))ds.
\] (2.13)

Let \( y(t) \) be a positive \( \omega \)-periodic solution of Equation (2.12), then
\[
\|y\| \leq B \int_{0}^{\omega} b(s)g^*(s, y(s - \tau(s)))ds
\]
and
\[
y(t) \geq A \int_{0}^{\omega} b(s)g^*(s, y(s - \tau(s)))ds.
\]

From these we have
\[
y(t) \geq \frac{A}{B} \|y\| = \sigma \|y\|.
\] (2.14)

**Lemma 2.4.** Assume that \((P_1)\)–\((P_4)\) hold. Let \( y(t) \) be a positive \( \omega \)-periodic solution of Equation (2.12), then \( y(t) \geq \theta \).

This means that each positive \( \omega \)-periodic solution of Equation (2.12) is a positive \( \omega \)-periodic solution of Equation (1.1).

**Proof.** Assume that there is a point \( t_0 \in [0, \omega] \) such that \( y(t_0) < \theta \). It follows from (2.14) that we have
\[
\|y\| \leq \frac{1}{\sigma} y(t_0) < \theta/\sigma < \varepsilon_0.
\]
Then by \((P_3)\) and \((2.9)\)–\((2.11)\), we obtain

\[
y(t_0) = \int_{t_0}^{t_0 + \omega} G(t_0, s)b(s)g^*(s, y(s - \tau(s))) \, ds
\]

\[
\geq \int_{t_0}^{t_0 + \omega} G(t_0, s)b(s)g^*(s, 0) \, ds
\]

\[
= \int_{t_0}^{t_0 + \omega} G(t_0, s)b(s)g(s, \theta) \, ds
\]

\[
\geq \int_{t_0}^{t_0 + \omega} G(t_0, s)b(s) \frac{\theta}{\eta} \, ds
\]

\[
\geq \theta,
\]

which is a contradiction. This completes the proof. \(\square\)

Now, we will seek the positive \(\omega\)-periodic solution of Equation \((2.12)\).

Let \(X, K\) be the same as those in Section 2.1. Define an operator on \(X\) as follows:

\[
y = \Phi y
\]

where \(\Phi\) is defined by

\[
(\Phi y)(t) = \int_{t}^{t + \omega} G(t, s)b(s)g^*(s, y(s - \tau(s))) \, ds,
\]

for \(y \in X\). Clearly, \(\Phi\) is a completely continuous operator on \(X\).

In the same way as in Section 2.1, we have \(\Phi(K) \subset K\).

Let \(y \in K\) with \(\|y\| = \theta\). It follows from \((2.9)\)–\((2.12)\) that we have

\[
(\Phi y)(t) = \int_{t}^{t + \omega} G(t, s)b(s)g^*(s, y(s - \tau(s))) \, ds
\]

\[
\geq \int_{t}^{t + \omega} G(t, s)b(s)g(s, \theta) \, ds
\]

\[
\geq \int_{t}^{t + \omega} G(t, s)b(s) \frac{\theta}{\eta} \, ds
\]

\[
\geq \theta = \|y\|.
\]

This implies that

\[
\|\Phi y\| \geq \|y\|
\]

for \(y \in K \cap \partial \Omega_1\), where \(\Omega_1 := \{y \in X : \|y\| < \theta\}\).
On the other hand, since \( \lim_{u \uparrow \infty} \max_{t \in [0, \omega]} (g^*(t, u)/u) = 0 \), similar to as in Section 2.1, there exists a \( R_2 > \varepsilon_0 \) such that
\[
\|\Phi y\| < \|y\|
\]
for \( y \in K \cap \partial \Omega_2 \), where \( \Omega_2 := \{ y \in X : \|y\| < R_2 \} \).

Therefore, by Theorem A, it follows that \( \Phi \) has a fixed point \( y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1) \).

Furthermore, \( \theta \leq \|y\| \leq R_2 \) and \( y(t) \geq \sigma \theta > 0 \), which means that \( y(t) \) is a \( \omega \)-periodic positive solution of (2.12). Then by Lemma 2.4, this means that \( y(t) \) is also a positive \( \omega \)-periodic solution of Equation (1.1).

This completes the proof of Theorem 2.2. \( \square \)

Remark 1. In a similar way as in this paper, we can deal with the existence of positive periodic solutions for the general Volterra integro-differential equations
\[
\dot{y}(t) = -a(t)y(t) + b(t) \int_{-\infty}^{0} K(r)g(t, y(t + r)) \, dr
\]
where \( a(t) \in C(R, R) \), \( \int_0^\infty a(t) \, dt > 0 \), \( b(t) \in C(R, (0, \infty)) \), \( g \in C(R \times [0, \infty], [0, \infty)) \), and \( a(t), b(t), g(t, y) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant. Moreover, \( K(r) \in C((-\infty, 0], [0, \infty)) \) and \( \int_{-\infty}^{0} K(r) \, dr = 1 \).

3. Examples

In this section, we apply the main results obtained in the previous section to study some examples which have some biological background.

From Theorem 2.1, we have the following.

**Corollary 3.1.** Assume that:
\( (H_1) \) \( a(t) \in C(R, R) \), \( \int_0^\infty a(t) \, dt > 0 \), \( b(t) \in C(R, (0, \infty)) \), \( \beta(t) \in C(R, (0, \infty)) \), \( \tau(t) \in C(R, R) \), \( \alpha(t), \beta(t), \tau(t) \) and \( \beta(t) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant.

Then Equation (1.2) has at least one \( \omega \)-periodic positive solutions.

**Corollary 3.2.** Assume that:
\( (H_1) \) \( a(t) \in C(R, R) \), \( \int_0^\infty a(t) \, dt > 0 \), \( b(t) \in C(R, (0, \infty)) \), \( \tau(t) \in C(R, R) \), \( a(t), b(t) \) and \( \tau(t) \) are all \( \omega \)-periodic functions. \( \omega > 0 \) is a constant.

Then Equation (1.3) has at least one \( \omega \)-periodic positive solution.

From Theorem 2.2, we have the following.
Corollary 3.3. Assume that:

(H1) \[ a(t) \in C(R, R), \int_0^\infty a(t) \, dt > 0, b(t) \in C(R, (0, \infty)), \tau(t) \in C(R, R), a(t), \]
\[ b(t) \text{ and } \tau(t) \text{ are all } \omega \text{-periodic functions. } \omega > 0 \text{ is a constant;} \]
(H2) \[ \min_{t \in [0, \omega]} \{b(t) - a(t)\} > 0. \]

Then Equation (1.4) has at least one \( \omega \)-periodic positive solution.

Corollary 3.4. Assume that:

(H1) \[ a(t) \in C(R, R), \int_0^\infty a(t) \, dt > 0, b(t) \in C(R, (0, \infty)), \beta(t) \in C(R, (0, \infty)), \tau(t) \in C(R, R), a(t), b(t), \tau(t) \text{ and } \beta(t) \text{ are all } \omega \text{-periodic functions. } \omega > 0 \text{ is a constant;} \]
(H2) \[ \min_{t \in [0, \omega]} \{b(t) - a(t)\} > 0. \]

Then Equation (1.5) has at least one \( \omega \)-periodic positive solution.

Corollaries 3.1–3.3 can be checked easily.

For Corollary 3.4, in fact, \( g(t, u) = u e^{-\beta(t)u} \) is increasing for \( 0 \leq u \leq 1/\beta(t) \). Let \( \varepsilon_0 = 1/(\max_{t \in [0, \omega]} \beta(t))^{-1} \). Hence, under the conditions (H1) and (H2), all the assumptions in Theorem 2.2 are satisfied, and the conclusion follows.

Remark 2. The systems (1.2), (1.3) and (1.5) have been investigated by [4, 5, 7]. In all these papers \( a(t) > 0 \). They have obtained a group of conditions to guarantee the existence of positive \( \omega \)-periodic solution for Equations (1.2), (1.3) and (1.5). For the system (1.5), in paper [5], (H2) in Corollary 3.4 is \( b_0/a_0 > 1 \), where
\[ b_0 = \min_{t \in [0, \omega]} b(t), \quad a_0 = \max_{t \in [0, \omega]} a(t). \]

We improve their results.

Acknowledgement. This work was supported by NNSF of China.

References


Aying Wan  
Department of Mathematics  
Hulunber College  
Hailar 021008  
People’s Republic of China

Daqing Jiang  
Department of Mathematics  
Northeast Normal University  
Changchun 130024  
People’s Republic of China