A REMARK ON THE FRAMED NULL-COBORDANTNESS OF THE EXCEPTIONAL LIE GROUP $E_6$

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Dedicated to the memory of Professor Katsuo Kawakubo

1. Introduction and statement of a result

The null-cobordantness of a framed compact connected Lie group $G$ has been discussed in [7] and many other papers. As is well known, the left (or right) invariant tangent fields over $G$ yield a trivialization of its tangent bundle and so induce a framing of the stable normal bundle of $G$ in a canonical way. This defines an element, denoted simply by $[G]$, of the stable homotopy group $\pi_\ast^S$ of spheres via the Pontrjagin–Thom construction. The vanishing of $[G]$ means, of course, that such a framed Lie group $G$ is null-cobordant.

Ossa [7] has shown the following result of a general nature:

$$8 \cdot 3^2 [G] = 0.$$ 

Furthermore, for a few special groups we have a stronger estimation of the order of $[G]$ such that $[2, 3, 4, 6]$

$$[SO(2n)] = 0 \ (n \geq 2), \quad [SU(n)]_{(3)} = 0 \ (n \geq 3) \quad \text{and} \quad [F_4]_{(3)} = 0,$$

where $[G]_{(3)}$ denotes the 3-primary component of $[G]$. In this paper we will prove the following.

**Theorem.** The odd-component of $[E_6]$ is trivial.

2. Preliminaries

Before proving the theorem we recall two facts. From now on let $G$ and $K$ denote $E_6$ and $F_4$ respectively and let $p$ be an odd prime. Let $\sigma$ be the involutive automorphism of $G$ having $K$ as a fixed point set. According to [1] (also cf. [5]) we then have

$$0 \rightarrow \pi_i(K) \stackrel{i_\ast}{\rightarrow} \pi_i(G) \stackrel{\pi_\ast}{\rightarrow} \pi_i(G/K) \rightarrow 0 \quad (i \geq 1) \quad (2.1)$$
are exact and split modulo the class of 2-primary abelian groups where \( i : K \rightarrow G \) denotes the inclusion and \( \pi : G \rightarrow G/K \) denotes the projection. The splitting is given by the map \( q : G/K \rightarrow G \) defined by \( q(gK) = g\sigma(g)^{-1} \) for \( g \in G \).

Consider the map
\[
\varphi : K \times G/K \rightarrow G
\]
given by \( \varphi(k, gK) = g\sigma(g)^{-1}k \) for \( k \in K, g \in G \). From (2.1) and the fact that the multiplication of the loop space \( \Omega G \) induced by the group multiplication of \( G \) and the standard multiplication of the loop space are homotopic, it follows that for all \( i \geq 1 \),

\[
\varphi_* : \pi_i(K \times G/K)(p) \rightarrow \pi_i(G)(p)
\]
is an isomorphism, where \( M(p) \) denotes the \( p \)-localization of a module \( M \). This implies that \( \varphi \) is a \( p \)-equivalence for all \( p \). Hence

\[
\varphi_* : \pi_0(K \times G/K)(p) \rightarrow \pi_0(G)(p)
\]  \hspace{1cm} (2.2)
is an isomorphism, where \( X^+ \) denotes the disjoint union of \( X \) and a single point. Further, we see that

\[
(1 \times \pi)^* : \pi_0(K \times G/K)(p) \rightarrow \pi_0(K \times G^+)(p)
\]  \hspace{1cm} (2.3)
is a split monomorphism. Actually, the splitting is given by the map \( 1 \times q \).

The construction of \([G]\) mentioned below, which is done by making use of the fixed point index of a fiber-preserving map, is due to Becker and Schultz [2]. Let \( \zeta : G \rightarrow \mathbb{R}^\ell \) be an embedding with a normal bundle \( \nu \). Then the Pontrjagin–Thom construction yields a map
\[
\zeta^\sharp : S^\ell \rightarrow G\nu,
\]
where \( G\nu \) denotes the Thom space of \( \nu \). Define maps
\[
\eta : G^+ \wedge G^\nu \rightarrow G^+ \wedge G^\nu \quad \text{and} \quad d : G^\nu \rightarrow G^+ \wedge G^\nu
\]
by \( \eta(g', v) = (g'g, v) \) for \( g', g \in G \) and \( d(v) = (g, v) \) for \( v \in v_g \), where \( v_g \) denotes the normal space at \( g \in G \). Let \( \tau \) be the tangent bundle of \( G \). Then we have a sequence of maps
\[
G^+ \wedge S^\ell \xrightarrow{1 \times \zeta^\sharp} G^+ \wedge G^\nu \xrightarrow{\eta} G^+ \wedge G^\nu \xrightarrow{d^\sharp} G^\tau \oplus \psi \xrightarrow{\psi} G^+ \wedge S^\ell \xrightarrow{p} S^\ell. \]  \hspace{1cm} (2.4)
Here \( d^\sharp \) is the Pontrjagin–Thom map similar to \( \zeta^\sharp \), \( \psi \) denotes the homeomorphism induced from the trivialization \( \tau \oplus \psi \cong G \times \mathbb{R}^\ell \) associated with the embedding \( \zeta \) and \( p \) denotes the evident projection.
Null-cobordantness of the exceptional Lie group $E_6$

The composition $f$ of the maps of (2.4) defines an element of $\pi^0_3(G^+)$, denoted by $I_G(G)$ using the same notation as in [2]. Clearly this map is constant on the outside of some coordinate neighbourhood $U$ of the identity element $e \in G$. So $f$ factors as

$$f : G^+ \wedge S^d \xrightarrow{c \times 1} \frac{G^+}{G - U} \wedge S^d \to S^d,$$

where $c$ denotes the collapse map. Identify $S^d = \frac{G^+}{G - U}$, where $d = \dim G$, then it is seen that the second unnamed map defines just $[G] \in \pi^3_d$. Now the map $c^* : \pi^3_d(S^d) \to \pi^0_3(S^d(G^+))$ is a split monomorphism; in fact, the splitting is given by taking the Kronecker product with the homotopy fundamental class of $G$. So if we identify $\pi^3_d$ with its image by $c^*$, then $I_G(G)$ coincides with $[G]$.

## 3. Proof of the theorem

To prove the theorem it suffices to show by (2.2) that the composition

$$(K \times G/K)^+ \wedge S^d \xrightarrow{\psi \wedge 1} G^+ \wedge S^d \xrightarrow{f'} S^d$$

is null-homotopic. It is well known that $G$ has a faithful complex representation of dimension 27, for which we write $\rho : G \to U(27)$, and $\rho$ satisfies the equality $\rho(\sigma(g)) = \overline{\rho(g)}$, the complex conjugate of $\rho(g)$, for $g \in G$. So we have $\rho(K) \subset SO(2 \cdot 27)$.

Let $M(n, \mathbb{R})$ and $M(n, \mathbb{C})$ be the spaces of $n \times n$ real and complex matrices and let $M(n, \mathbb{C})$ be embedded in $M(2n, \mathbb{R})$ in the usual way. Then by identifying $M(2n, \mathbb{R})$ with $\mathbb{R}^{4n^2}$ we have an embedding

$$\zeta : G \to \mathbb{R}^\ell \quad (\ell = 4 \cdot 27^2)$$

via the above representation $\rho$.

Take this one as an embedding $\zeta$ of Section 2 and identify $G$ with the image $\zeta(G)$. We now also use all the other notation of Section 2 without references. Then it follows that $v_g = R_g v_e$ and $\tau_g = R_g \tau_e$ for all $g \in G$. Here $\tau_g$ is the tangent space at $g$, $v_g$ is as in Section 2 and $R_g : \mathbb{R}^\ell \to \mathbb{R}^\ell$ denotes right multiplication by $g$ induced from the product of matrices under the identification $\mathbb{R}^\ell = M(2 \cdot 27, \mathbb{R})$.

Now we consider the composition

$$\tilde{f} : (K \times G)^+ \wedge S^d \xrightarrow{(1 \wedge \tilde{\pi}) \wedge 1} (K \times G/K)^+ \wedge S^d \xrightarrow{\psi \wedge 1} G^+ \wedge S^d \xrightarrow{f'} G^{\tau \oplus v} ,$$

where $f' = d^\ell \circ \eta \circ (1 \wedge \zeta^2)$. To simplify the argument we now take $\rho \oplus 1$ instead of $\rho$, where 1 denotes one-dimensional complex representation and view $g' \in G$ as
an element of $SO(2 \cdot 28)$. Then we can represent $g'$, for example, in the form of $g' = \left( \begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix} \right)$, where $\alpha$ and $\beta$ are real square matrices of degree 28 and so the equality $g' = \left( \begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix} \right) g' \left( \begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix} \right)$ holds, where $I$ denotes the identity matrix. There is an obvious path $p(t)$ in $SO(2 \cdot 28)$ with $\left( \begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix} \right)$ as the initial point and $e = \left( \begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix} \right)$ as the terminal point. Because $\sigma(g') = \bar{g}'$ in $U(28)$, using this path we can define a homotopy

$$F_t : (K \times G)^+ \land S^\ell \to G^{\ell+\nu}$$

which is given by

$$F_t(k, g', R_g v) = R_g(u_t, v_t + v) \quad \text{for } k \in K, \quad g', g \in G \text{ and } v \in \mathcal{V}_e.$$ 

Here $u_t$ and $v_t$ denote the tangential and normal components over $G$ of the vector in $\mathbb{R}^\ell$ with $e$ as the initial point and $g'(p(t)) g'(p(t))^{-1} k$ as the terminal point. In fact, these vectors are uniquely determined because $\tau_e \oplus \mathcal{V}_e = \mathbb{R}^\ell$. Then it is easily seen that $F_0 = \bar{f}$ and $F_1$ is null-homotopic. Hence we also see that

$$f \circ (\varphi \land 1) \circ ((1 \times \pi) \land 1) : (K \times G)^+ \land S^\ell \to S^\ell$$

is null-homotopic. We thus obtain

$$(1 \times \pi)^* (\varphi^*(I_G(G))) = 0 \quad \text{in } \pi^0_\ell(K \times G^+)(p).$$

So by (2.3) we have $\varphi^*(I_G(G)) = 0$ in $\pi^0_\ell(K \times G/K^+)(p)$. (We note here that the equality $\varphi^*(I_G(G)) = I_K(G) \times 1$ holds using the same notation as in [2].) From (2.2) it therefore follows that $I_G(G) = 0$, so that $[G] = 0$. This completes the proof of the theorem.

REFERENCES

Null-cobordantness of the exceptional Lie group $E_6$

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