HARNACK INEQUALITIES FOR EXOTIC BROWNIAN MOTIONS

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Abstract. Exotic Brownian motions are diffusion processes given by Dirichlet forms $\mathcal{E}$ on $L^2(\mathbb{S}, \mu)$, where the state space $\mathbb{S}$ is the support of a Radon measure μ in $\mathbb{R}^d$ and the energy form $\mathcal{E}$ is given by the integral of the canonical square field on $\mathbb{R}^d$ with respect to μ . In most cases we take μ in such a way that μ is singular to the d -dimensional Lebesgue measure and, in particular, μ does not satisfy the doubling condition. The purpose of this paper is to prove the parabolic Harnack inequalities for these diffusions. We will do this with a refinement such that the dependence of μ , Poincaré and Sobolev constants is clarified. Because of the singularity of μ , diffusions are expected to behave in rather an unusual fashion. These Harnack inequalities will be used to show exotic properties of our diffusion processes in a forthcoming paper.

1. Introduction

Let μ be a Radon measure on $\mathbb{R}^d$ and let $\mathbb{S}$ be the topological support of the measure μ , that is, $\mathbb{S}$ is the minimal closed set whose complement has μ -measure zero. Exotic Brownian motions are diffusion processes associated with Dirichlet forms $\mathcal{E}$ on $L^2(\mathbb{S}, \mu)$, where $\mathcal{E}$ is the closure of the form $\mathcal{E}$ (denoted by the same symbol $\mathcal{E}$):

$$\mathcal{E}(f, g) = \int_{\mathbb{S}} \mathbb{D}[f, g] d\mu \quad (1.1)$$

with suitably chosen domain $\mathcal{D}_0$. Here $\mathbb{D}$ is the canonical square field on $\mathbb{R}^d$

$$\mathbb{D}[f, g] = \frac{1}{2} \sum_{i=1}^d \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}.$$

The state space of the diffusion is $\mathbb{S}$ by construction.

We will take μ in such a way that it is a countable sum of weighted d_n -dimensional volume:

$$\mu = \sum_n w_n \lambda_n. \quad (1.2)$$

Here λ_n is the d_n -dimensional volume on a d_n -dimensional smooth submanifold S_n . We assume $1 \leq d_n \leq d$. We remark by construction that

$$\mathbb{S} = \overline{\cup_n S_n}.$$

Note that dimensions of submanifolds d_n may depend on n . In the most interesting cases, we assume $d_n = d - 1$ for all n . In such a case μ is singular to the d -dimensional Lebesgue measure; in particular, μ does not satisfy the doubling condition. When μ has a well-behaved density with respect to the d -dimensional Lebesgue measure, then $(\mathcal{E}, \mathcal{D})$ is called distorted Brownian motion [2, 3]. We can thus regard exotic Brownian motions as a generalization of distorted Brownian motions.

Under quite mild assumptions it is known that $(\mathcal{E}, \mathcal{D}_0)$ is closable on $L^2(\mathbb{S}, \mu)$ and its closure is a strongly local Dirichlet form (see [4]); henceforth the associated diffusion exists, which we call exotic Brownian motion. The purpose of this paper is to prove the parabolic Harnack inequality for (the transition density of) exotic Brownian motions. We will do this in such a way that the dependence on μ , Sobolev and Poincaré constants are clarified.

The diffusion is expected to behave in a rather unusual fashion because of the singularity of μ ; for example, in Example 2.1 below, in spite of $\mathbb{S} = \mathbb{R}^d$ ($d \geq 2$) and the diffusion being non-degenerate, the infinitesimal spectral dimension $d_s(x)$ at $x \in \mathbb{S}$ —the exponent such that $p(t, x, x) \sim t^{-d_s(x)/2}$ as $t \rightarrow 0$ —is different from d in a dense subset of $\mathbb{S} = \mathbb{R}^d$. The Harnack inequalities obtained here will be used to show exotic properties of our diffusion processes in a forthcoming paper [9].

2. Main results and examples

Let $a = (a_{ij})_{i,j=1,\dots,d}$ be a matrix valued function defined on $\mathbb{S}$ satisfying the following. There exists a positive constant c_1 such that for μ -a.e. x

$$c_1^{-1} |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j, \quad \left| \sum_{i,j=1}^d a_{ij}(x) \xi_i \eta_j \right| \leq c_1 |\xi| |\eta|$$

for all $\xi = (\xi_i), \eta = (\eta_i) \in \mathbb{R}^d$. (2.1)

We note that a is not necessary symmetric. Set

$$\mathcal{E}^a(f, g) = \int_{\mathbb{S}} \mathbb{D}^a[f, g] d\mu,$$

where $\mathbb{D}^a$ is the square field given by

$$\mathbb{D}^a[f, g] = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

When a is the d -dimensional unit matrix, we simply write $\mathbb{D}^a = \mathbb{D}$ and $\mathcal{E}^a = \mathcal{E}$, which coincides with the notation in Section 1. To simplify the notation we often write $\mathbb{D}^a[f, f] = \mathbb{D}^a[f]$ and $\mathcal{E}^a(f, f) = \mathcal{E}^a(f)$ and so on. Let $\mathcal{D}_0 = \{f|_{\mathbb{S}}; f \in \mathcal{D}_0(\mathbb{R}^d)\}$, where $\mathcal{D}_0(\mathbb{R}^d)$ is the set defined by

$$\mathcal{D}_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \rightarrow \mathbb{R}; f \text{ satisfies (2.2), (2.3), (2.4)}\};$$

$$f \text{ is a Lipschitz continuous function with compact support} \quad (2.2)$$

$$\|\mathbb{D}[f, f]\|_{L^\infty(\mu)} < \infty \quad (2.3)$$

$$\nabla_N f(x) = 0 \quad \text{for } \mu\text{-a.e. } x. \quad (2.4)$$

Here ∇_N is the normal derivative on S_k , where ∇_N is, by definition, a $(d - d_k)$ -dimensional vector perpendicular to S_k . Recall that each S_k is a submanifold, so (2.4) makes sense for $d_k \leq d - 1$. When $d_k = d$, we interpret (2.4) in such a way that (2.4) always holds on S_k .

We assume that $\mathcal{D}_0$ is dense in $C_0(\mathbb{S})$ with the uniform norm and also in $L^2(\mathbb{S}, \mu)$ and that $(\mathcal{E}^a, \mathcal{D}_0)$ is closable on $L^2(\mathbb{S}, \mu)$. Then it is clear that its closure $(\mathcal{E}^a, \mathcal{D})$ is a strongly local regular Dirichlet form; so by using the Dirichlet form theory, one sees there exists a diffusion process associated with $(\mathcal{E}^a, \mathcal{D}, L^2(\mathbb{S}, \mu))$ (see [4]).

A sufficient condition for the above assumption is as follows (see [8, 9]).

- (1) $S_m \cap S_n = \emptyset$ if $m \neq n$.
- (2) For each S_n its closure $\overline{S}_n$ is compact and there exists a smooth d_n -dimensional manifold $\hat{S}_n$ such that $\overline{S}_n \subset \hat{S}_n$.
- (3) $\mu(\partial S_n) = 0$ for all n .

Let I be a compact interval in $\mathbb{R}$. Let $\mathcal{H}^1(I \times \mathbb{S})$ denote the closure of $C^1(I) \otimes \mathcal{D}_0$ with respect to the norm

$$\sup_{t \in I} \|f(t, \cdot)\|_{L^2(\mathbb{S}, \mu)} + \left(\int_I \mathcal{E}^a(f(t, \cdot), f(t, \cdot)) dt \right)^{1/2}.$$

Note that $t \mapsto f(t, \cdot) \in L^2(\mathbb{S}, \mu)$ is continuous in $t \in I$.

Let $\mathcal{O}$ be an open set of $\mathbb{S}$. Here $\mathcal{O}$ is endowed with the relative topology from $\mathbb{R}^d$. It is clear that $\mu(\mathcal{O}) > 0$. Set

$$\begin{aligned}\mathcal{D}_0(\mathcal{O}) &= \{g = f|_{\mathcal{O}}; f \in \mathcal{D}_0, g \text{ has compact support}\}, \\ \mathcal{H}^1(I \times \mathcal{O}) &= \{f|_{I \times \mathcal{O}}; f \in \mathcal{H}^1(I \times \mathbb{S})\}.\end{aligned}$$

Set $\mathcal{E}_{\mathcal{O}}^a := \int_{\mathcal{O}} \mathbb{D}^a d\mu$ and let I° denote the open kernel of I . We say a measurable function $u : I \times \mathcal{O} \rightarrow \mathbb{R}$ is a solution in $I \times \mathcal{O}$ of the heat equation associated with the Dirichlet space $(\mathcal{E}^a, \mathcal{D}, L^2(\mathbb{S}, \mu))$ if:

$$u \in \mathcal{H}^1(I \times \mathcal{O}), \quad (2.5)$$

$$I^\circ \ni t \mapsto u(t, \cdot) \in L^2(\mathcal{O}, \mu) \text{ is differentiable}, \quad (2.6)$$

$$\int_{I \times \mathcal{O}} \partial_t u \varphi dt d\mu + \int_I \mathcal{E}_{\mathcal{O}}^a(u, \varphi) dt = 0, \quad (2.7)$$

for all $\varphi \in C^1(I) \otimes \mathcal{D}_0(\mathcal{O})$. For an open interval J we say a function u is a solution in $J \times \mathcal{O}$ if u is a solution in $I \times \mathcal{O}$ for each compact interval I such that $I \subset J$.

Throughout this paper we let $x_0 \in \mathbb{S}$ be fixed. Let ρ_0 be a metric on $\mathcal{O}$ that induces the same topology as the original and let $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function. We set $\rho(x) = h(\rho_0(x_0, x))$ and $\mathcal{O}(r) = \{x \in \mathbb{S}; \rho(x) < r\}$. Let

$$\mu_r = \mu(\mathcal{O}(r)). \quad (2.8)$$

Let r_0 be a positive constant such that $\mathcal{O}(r_0) \subset \mathcal{O}$ and let γ be a positive constant such that $2\gamma < r_0$. Assume there exists a positive constant c_2 such that

$$c_2^{-1} \leq \mathbb{D}[\rho(x)] \leq c_2 \quad \text{for } \mu\text{-a.e. } x \in \overline{\mathcal{O}(2\gamma)} \setminus \mathcal{O}(\gamma/2). \quad (2.9)$$

Let α be a positive constant such that $1 \leq \alpha < \infty$. For a given α we set

$$\kappa = \begin{cases} 1 + 2/\alpha & \text{if } \alpha > 2, \\ 1 + 2/(\alpha + 1) & \text{if } \alpha \leq 2. \end{cases} \quad (2.10)$$

We assume the following inequalities.

(1) (Sobolev inequality) There exists a positive constant c_3 such that

$$\|f\|_{L^{2\kappa}(\mathcal{O}(r), \mu)}^{2\kappa} \leq c_3 \|f\|_{L^2(\mathcal{O}(r), \mu)}^{2(\kappa-1)} \{ \mu_r^{1-\kappa} \|f\|_{L^2(\mathcal{O}(r), \mu)}^2 + \|\nabla f\|_{L^2(\mathcal{O}(r), \mu)}^2 \} \quad (2.11)$$

for all $f \in \mathcal{D}_0$, $\gamma/2 \leq r \leq \gamma$.

(2) (Weighted Poincaré inequality) There exists a positive constant c_4 such that

$$\|f - \bar{f}\|_{L^2(\mathcal{O}(2\gamma), \psi^2 \mu)}^2 \leq c_4 \gamma^2 \|\nabla f\|_{L^2(\mathcal{O}(2\gamma), \psi^2 \mu)}^2. \quad (2.12)$$

Here $\bar{f} = \{\int_{\mathcal{O}(2\gamma)} \psi^2 d\mu\}^{-1} \int_{\mathcal{O}(2\gamma)} f \psi^2 d\mu$ and $\psi(x) = \bar{\psi}(\rho(x))$, where $\bar{\psi} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the function defined by $\bar{\psi}(t) = 1$ for $0 \leq t \leq \gamma$, $\bar{\psi}(t) = -\gamma^{-1}t + 2$ for $\gamma \leq t \leq 2\gamma$ and $\bar{\psi}(t) = 0$ for $2\gamma \leq t < \infty$.

THEOREM 2.1. *Let $I_{\mp}$ be closed intervals such that $I_- = \gamma^2[-3/4, -1/4] + a$ and $I_+ = \gamma^2[1/4, 1] + a$ for some $a \in \mathbb{R}$. Let I be an open interval such that $\gamma^2[-1, 1] + a \subset I$. Let u be a non-negative solution in $I \times \mathcal{O}$. Let $\mathcal{O}(\gamma/2)$ be as above. Then there exists a version of $u(t, x)$ such that*

$$\sup_{I_- \times \mathcal{O}(\gamma/2)} u(t, x) \leq c_5 \inf_{I_+ \times \mathcal{O}(\gamma/2)} u(t, x), \quad (2.13)$$

where c_5 is a constant depending only on $c_1, c_2, c_3, c_4, \alpha$ and

$$\gamma^{-2/(\kappa-1)} \mu_{\gamma}. \quad (2.14)$$

Remark 2.1. (1) More precisely, the constant c_5 is given by $c_5 = \exp\{2c_6 c_7\}$, where c_6 and c_7 are constants given by

$$c_6 = \left(\frac{2}{\gamma}\right)^{\bar{\kappa}\tau(\sigma-1)} \cdot \left\{ \sum_{j=0}^{\infty} \left(\frac{1+\tau}{\sigma\tau}\right)^j ((1+j)(2+j))^{\bar{\kappa}\tau(\sigma-1)} \right\},$$

$$c_7 = \max \left\{ \sigma c_8 \left(\frac{\gamma}{2}\right)^{\bar{\kappa}\tau(\sigma-1)}, 2c_1 c_4 (2c_9 \gamma^{2-\bar{\kappa}} \mu_{\gamma})^{\tau(\sigma-1)} \right\}.$$

Here the constant $\bar{\kappa}$ is defined by

$$\bar{\kappa} = \frac{\kappa}{\kappa-1} = \begin{cases} \frac{\alpha+2}{2} & (2 < \alpha), \\ \frac{\alpha+3}{2} & (1 \leq \alpha \leq 2), \end{cases}$$

and $c_8 = \max\{x \in A\}$ if $A \neq \emptyset$ and $c_8 = 0$ if otherwise, where

$$A = \left\{ x \in \mathbb{R}^+; \frac{1}{2c_1} = \frac{1}{x(\sigma-1)} \log \left(\frac{x}{2c_1 c_4} \right) \right\}.$$

σ and τ are any numbers satisfying $1 < \sigma, \tau < \infty$, $(1+\tau)/(\sigma\tau) < 1$ and c_9 is a positive constant defined by

$$c_9 = c_{10} \cdot (c_3 c_1)^{1/(\kappa-1)} \cdot (\max\{1, \gamma^2 \mu_{\gamma}^{1-\kappa}/c_1\})^{1/(\kappa-1)} \\ = c_{10} \cdot (c_3 \max\{c_1, \gamma^2 \mu_{\gamma}^{1-\kappa}\})^{1/(\kappa-1)}, \quad (2.15)$$

where c_{10} is a positive constant depending only on (κ, c_1, c_2) (see (6.16)).

(2) A simple calculation shows that, for given c_1, c_2, c_3, c_4 and α , we can take a non-negative and non-decreasing function $C : (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{s \rightarrow \infty} C(s) = \infty$, $\lim_{s \rightarrow 0} C(s) > 0$ and that

$$c_5 = C(\gamma^{-2/(\kappa-1)} \mu_\gamma). \quad (2.16)$$

(3) Let x_0 denote the center of domains. Suppose $\alpha > d_H > d - 1$, where d_H is the Hausdorff dimension of $\mathbb{S}$, and (2.11) and (2.12) hold for $\gamma_k = C^{-k}$ ($C > 1$, $k = 1, 2, \dots$) with constants uniform in k . Suppose $d_n = d - 1$ for all n and the doubling condition is not satisfied. This is the situation in Examples 2.1 and 2.2. If $x_0 \in \cup_n S_n$, then $\mu_{\gamma_k} \sim \gamma_k^{-(d-1)}$ as $\gamma_k \rightarrow 0$; so $\sup_k c_5(\gamma_k) = \infty$. We remark that, in the present situation, one cannot expect the *global* Harnack inequality, which means (2.13) holds with some $c_5(\gamma_k) > 0$ such that

$$\sup_{k \in \mathbb{N}, x_0 \in \mathbb{S}} c_5(\gamma_k) < \infty.$$

This is because $\sup_{k \in \mathbb{N}, x_0 \in \mathbb{S}} c_5(\gamma_k) < \infty$, together with (2.11) and (2.12) for $\gamma(k) = 3^{-k}$, implies the doubling condition of μ by the argument of Grigor'yan [5]. The doubling condition is commonly assumed in [5, 10, 11]. We emphasize that, even if μ does not satisfy the doubling condition, one can prove (non-uniform) Harnack inequalities in some cases. Such a Harnack inequality is weaker than those in [5, 10, 11]. It is still useful to obtain the locally uniform lower bound of transition densities of associated diffusion processes, as this yields the non-degeneracy of diffusion processes.

We now show examples.

Example 2.1. Exotic Brownian motions over $\mathbb{R}^d$ ($d \geq 2$). Let $d \geq 2$ and let N be a natural number such that $N \geq 2$. Let $\mathbf{B} = \cup_{i=1}^d \{x = (x_1, \dots, x_d) \in \mathbb{R}^d; x_i \in \mathbb{Z}, x_j \notin \mathbb{Z} \text{ for all } j \neq i\}$ and $\mathbf{B}^n = (1/N)^n \mathbf{B}$. We set

$$S_n = \mathbf{B}^{n+1} \setminus \mathbf{B}^n.$$

For given $\alpha > d$ we take the weight w_n in (1.2) such that

$$w_n = N^{-(\alpha-d+1)n}.$$

Then $\mu = \mu^{[\alpha]}$ is a Radon measure on $\mathbb{R}^d$ whose support equals $\mathbb{R}^d$. The measure μ satisfies the self-similarity in the sense that

$$\mu(NA) = N^\alpha \cdot \mu(A) \quad \text{for all } A \in \mathfrak{B}(\mathbb{R}^d) \quad (2.17)$$

and the cell translation invariance in the sense of [8]. By using the method developed in [8] we see that p satisfies a global Gaussian upper bound; there exists a positive constant c_{11} (and a version p) such that

$$p(t, x, y) \leq c_{11} t^{-\alpha/2} e^{-|x-y|^2/c_{11}t} \quad \text{for all } 0 < t < \infty, x, y \in \mathbb{R}^d$$

and that geometric inequalities (2.11) and (2.12) hold for sufficiently many domains. From this and Theorem 2.1, one can prove the transition densities $p = p(t, x, y)$ are locally uniformly positive in the sense of (2.19) below—henceforth, diffusions are non-degenerate. In addition, there exists a dense subset R in $\mathbb{R}^d$ satisfying the following. The transition densities are continuous in $(0, \infty) \times R \times R$ and

$$c_{12}^{-1} t^{-\alpha/2} \leq p(t, x, x) \leq c_{12} t^{-\alpha/2} \quad \text{for all } 0 < t < 1, x \in R.$$

Here $c_{12} = c_{12}(x)$ are positive constants depending on $x \in R$. The last inequality means that the short time asymptotic behavior of the transition density is different from the standard Brownian motion in the dense subset R , in spite of the diffusion begin non-degenerate and moving around over $\mathbb{R}^d$.

Example 2.2. Exotic Brownian motions on Sierpinski carpets [8]. Let F denote the d -dimensional Sierpinski carpet and d_H the Hausdorff dimension of F . If we replace N by 3, $\mathbb{R}^d$ by F and d by d_H in Example 2.1, then the same conclusion as Example 2.1 holds.

Example 2.3. Exotic Brownian motions on bubbles. Let $d \geq 2$ and let B_n denote a closed ball with radius r_n . Let S_n denote the boundary of B_n . Assume

$$\begin{aligned} B_n \cap B_m &= \emptyset \quad \text{for } n \neq m \\ \overline{\bigcup_{n \in \mathbb{N}} B_n} &= \mathbb{R}^d. \end{aligned}$$

Let us take the weight w_n in such a way that $w_n = r_n$, then it is known that a global Gaussian upper bound holds; there exists a positive constant c_{13} such that

$$p(t, x, y) \leq c_{13} t^{-d/2} e^{-|x-y|^2/c_{13}t} \quad \text{for a.e. } 0 < t < \infty, x, y \in \mathbb{S}. \quad (2.18)$$

Here and below a.e. is with respect to $dt \times \mu \times \mu$. In addition, for sufficiently many domains (2.11) and (2.12), henceforth (2.13) by Theorem 2.1, are satisfied with $\alpha = d$. From this and (2.18) above we see that p is locally uniform positive in the sense that for any compact set $K \subset (0, \infty) \times \mathbb{S} \times \mathbb{S}$ there exists a positive constant $c_{14} = c_{14}(K)$ such that

$$0 < c_{14} \leq p(t, x, y) \quad \text{for a.e. } (t, x, y) \in K. \quad (2.19)$$

There is a celebrated work by Moser [7], which together with [6] proves parabolic Harnack inequalities for uniformly elliptic operators. Since then, many extensions have been made (see, for example, [10, 11] and references therein). In these works the doubling condition is a pivotal assumption. One of our contributions is that we carry out Moser's argument without the doubling condition. This is imperative, because the examples cited above do not satisfy the doubling condition. The diffusions in these examples are somehow exotic, but still well-behaved, diffusions that have various invariance. To apply Moser's theory to these diffusions we relax the assumptions. Our second contribution is to clarify the dependence of c_5 on the constants such as c_1 , c_3 , etc. This refinement will also be used in [9] to obtain fine estimates of diffusions on $\mathbb{R}^d$.

As for work on Moser's iterations on fractal sets, we refer to the work of Barlow and Bass [1]. In their paper the reference measure essentially satisfies the doubling condition.

The organization of the rest of this paper is as follows. In Section 3 we prepare three lemmas, Lemmas 3.1–3.3. In Section 4 we prove the main theorem by using the lemmas in Section 3. In Section 5 we do some calculations on Dirichlet form. In Sections 6–8, we prove Lemmas 3.1–3.3, respectively.

3. Preliminary lemmas

In this section we prepare three lemmas which we will prove in Sections 6–8. In the next section we will prove the main theorem by using these lemmas.

Let $J_-(r) = [-r^2, 0]$, $J_+(r) = [0, r^2]$ and $J(r) = J_-(r) \cup J_+(r)$. Set $R_\pm(r) = J_\pm(r) \times \mathcal{O}(r)$ and $R(r) = J(r) \times \mathcal{O}(r)$. Let $\bar{\mu}$ denote the product measure $\bar{\mu} = dt \times \mu$ and let κ and $\bar{\kappa}$ be as in Section 2. We recall

$$\bar{\kappa} = \begin{cases} \frac{\alpha + 2}{2} & (2 < \alpha), \\ \frac{\alpha + 3}{2} & (1 \leq \alpha \leq 2). \end{cases}$$

LEMMA 3.1. *Let u be a uniformly positive solution in a neighborhood of $R(\gamma)$; then for all $\gamma/2 < r_1 < r_2 \leq \gamma$ and $0 < p < 1/(2c_1)$ we have*

$$\sup_{R(r_1)} u \leq \left\{ \frac{c_9}{(r_2 - r_1)^{2\bar{\kappa}}} \right\}^{1/p} \|u\|_{L^p(R(r_2), \bar{\mu})}, \quad (3.1)$$

$$\sup_{R_-(r_1)} \frac{1}{u} \leq \left\{ \frac{c_9}{(r_2 - r_1)^{2\bar{\kappa}}} \right\}^{1/p} \left\| \frac{1}{u} \right\|_{L^p(R_-(r_2), \bar{\mu})}. \quad (3.2)$$

LEMMA 3.2. *Let u be a uniformly positive solution in a neighborhood of $R(2\gamma)$. Let c_{15} be the constant defined by*

$$c_{15} = 2c_1^3 c_2 \frac{\mu_{2\gamma}}{\mu_\gamma} - \int_{O(\gamma)} \log u(0, x) d\mu. \quad (3.3)$$

Then for all $s > 0$ the following holds:

$$\bar{\mu}(R_-(\gamma) \cap \{u > e^{s-c_{15}}\}) \leq \frac{2c_1 c_4 \gamma^2 \mu_\gamma}{s}; \quad (3.4)$$

$$\bar{\mu}(R_+(\gamma) \cap \{u < e^{-s-c_{15}}\}) \leq \frac{2c_1 c_4 \gamma^2 \mu_\gamma}{s}. \quad (3.5)$$

Let $\{Q(r)\}$ ($\gamma/2 \leq r \leq \gamma$) denote any family of measurable subsets in $\mathbb{R}^{d+1}$ such that $Q(r_1) \subset Q(r_2)$ for $r_1 \leq r_2$.

LEMMA 3.3. *Let β and θ be constants such that $0 < \beta < \infty$ and $1/2 \leq \theta < 1$. Let f be a positive measurable function satisfying the following. For positive constants c_{16} and c_{17}*

$$\sup_{Q(r_1)} f \leq \left\{ \frac{c_{16}}{(r_2 - r_1)^\beta \bar{\mu}(Q(\gamma))} \right\}^{1/p} \|f\|_{L^p(Q(r_2), \bar{\mu})}$$

for all $\theta\gamma \leq r_1 < r_2 \leq \gamma$, $0 < p < \frac{1}{2c_1}$ (3.6)

$$\bar{\mu}(f \geq e^s) \leq \frac{2c_1 c_{17}}{s} \cdot \bar{\mu}(Q(\gamma)), \quad (3.7)$$

then

$$\sup_{Q(\theta\gamma)} f \leq c_{18} \quad (3.8)$$

where $c_{18} = c_{18}(\gamma, \theta, \beta, c_{16}, c_{17})$ is given by

$$c_{18} = \exp\{c_{19} c_{20}\}. \quad (3.9)$$

Here $c_{19} = c_{19}(\gamma, \theta, \beta, \sigma, \tau)$ and $c_{20} = c_{20}(\gamma, \theta, \beta, c_{16}, c_{17}, \sigma, \tau)$ are constants given by

$$c_{19} = \left(\frac{1}{\gamma(1-\theta)} \right)^{\beta\tau(\sigma-1)} \sum_{j=0}^{\infty} \left(\frac{1+\tau}{\sigma\tau} \right)^j ((1+j)(2+j))^{\beta\tau(\sigma-1)}, \quad (3.10)$$

$$c_{20} = \max\{\sigma c_{21}(\gamma(1-\theta))^{\beta\tau(\sigma-1)}, 2c_1 c_{17} (2c_{16})^{\tau(\sigma-1)}\} \quad (3.11)$$

where c_{21} is defined similarly to c_8 with the replacement of c_4 by c_{17} and σ, τ are positive numbers such that $((1+\tau)/(\sigma\tau)) < 1$.

4. Proof of the main theorem

In this section we deduce Theorem 2.1 from the lemmas in Section 3.

LEMMA 4.1. *Let u be a uniformly positive solution in a neighborhood of $R(2\gamma)$. Then*

$$\sup_{R_-(\gamma/2)} u \leq c_{18} e^{-c_{15}}. \quad (4.1)$$

Here c_{18} is given by (3.9) with the choice of $\theta = 1/2$, $\beta = \bar{\kappa}$, $c_{16} = c_9 \gamma^2 \mu_\gamma / \gamma^{\bar{\kappa}}$ and $c_{17} = c_4$.

Proof. Apply Lemma 3.3 to $f = u e^{c_{15}}$, $Q(r) = [-r^2 - \gamma^2/2, r^2 - \gamma^2/2] \times \mathcal{O}(r)$, θ, β, c_{16} and c_{17} as above. Since $Q(r)$ thus defined are translations of $R(r)$ in time, one can apply Lemma 3.1 to obtain (3.1) for $Q(r)$; so we obtain (3.6). Since $Q(\gamma) = R_-(\gamma)$, we obtain (3.7) by (3.4). Hence the assumptions of Lemma 3.3 are fulfilled, which yields (4.1). $\square$

LEMMA 4.2. *Let u be a uniformly positive solution in a neighborhood of $R(2\gamma)$. Let $R_*(r) = [\gamma^2 - r^2, \gamma^2] \times \mathcal{O}(r)$. Then*

$$\sup_{R_*(\gamma/2)} u^{-1} \leq c_{18} e^{c_{15}}. \quad (4.2)$$

Here c_{18} is given by (3.9) with the choice of $\theta = 1/2$, $\beta = \bar{\kappa}$, $c_{16} = c_9 \gamma^2 \mu_\gamma / \gamma^{\bar{\kappa}}$ and $c_{17} = c_4$.

Proof. Note that $R_*(r)$ is the translation of $R_-(r)$ in time. Apply Lemma 3.3 to $f = (u e^{c_{15}})^{-1}$, $Q(r) = R_*(r)$, $\theta = 1/2$, we obtain (4.2) from Lemma 3.1, (3.2), (3.5) and Lemma 3.3. $\square$

Proof of Theorem 2.1. Without loss of generality, one can assume $a = 0$. Then we see $I_- \times \mathcal{O}(\gamma/2) = R_-(\gamma/2)$ and $I_+ \times \mathcal{O}(\gamma/2) = R_*(\gamma/2)$.

Let $u_\epsilon = u + \epsilon$, where ϵ is an arbitrary positive constant, then u_ϵ is a uniformly positive solution. So we can apply Lemmas 4.1 and 4.2 to u_ϵ . Multiplying (4.1) and (4.2) we obtain (2.13) for u_ϵ . Since the constant c_5 is independent of ϵ , (2.13) is also valid for u . This completes the proof. $\square$

5. Calculation based on Dirichlet forms

In this section we prove an equality for solutions of (2.5)–(2.7) by using the Dirichlet form structure. We will use this equality in the subsequent sections to deduce basic inequalities.

Let u be a solution of (2.5)–(2.7) in $I \times \mathcal{O}$. We assume u is positive. In the subsequent sections we need inequalities for functions v of the form

$$v := g(u).$$

Here $g : (0, \infty) \rightarrow \mathbb{R}$ is a function. We will take $g(t) = t^{p/2}$ in Lemma 6.1 and $g(t) = -\log t$ in Lemma 7.1.

LEMMA 5.1. *Let $I = [a, b]$, $\phi \in C^1(I) \otimes \mathcal{D}_0(\mathcal{O})$ and let $f, g : (0, \infty) \rightarrow \mathbb{R}$ be smooth functions. Set $U = f(u(t, x))$, $V = g(u(t, x))$ and let $\eta = \pm 1$ be a constant. Suppose $f'' = \eta(g')^2$. Then*

$$\begin{aligned} \int_{\mathcal{O}} U(t, \cdot) \phi^2 d\mu|_a^b - 2 \int_{I \times \mathcal{O}} U \partial_t \phi \phi d\bar{\mu} \\ + \int_I \{ \eta \mathcal{E}_{\mathcal{O}}^a(V, V; \phi^2) + 2\mathcal{E}_{\mathcal{O}}^a(U, \phi; \phi) \} dt = 0. \end{aligned} \quad (5.1)$$

Here $\mathcal{E}_{\mathcal{O}}^a(\cdot, \cdot; h) = \int_{\mathcal{O}} \mathbb{D}^a[\cdot, \cdot] h d\mu$ and $\bar{\mu} = dt \times \mu$ as before.

Proof. Let $\varphi := f'(u)\phi^2$, then the direct calculation shows

$$\begin{aligned} \mathbb{D}^a[u, \varphi] &= \mathbb{D}^a[u, u] f''(u) \phi^2 + 2\mathbb{D}^a[u, \phi] f'(u) \phi \\ &= \eta \mathbb{D}^a[g(u), g(u)] \phi^2 + 2\mathbb{D}^a[f(u), \phi] \phi. \end{aligned}$$

Hence (5.1) follows immediately from this and (2.7) with integration by parts in t . $\square$

6. Proof of Lemma 3.1

Let $J^*(r)$ be one of $J(r)$, $J_{\pm}(r)$, which were defined at the beginning of Section 3. Let $R^*(r) = J^*(r) \times \mathcal{O}(r)$.

LEMMA 6.1. *Let $v = u^{p/2}$, $\epsilon(p) = \frac{1}{2}|1 - p^{-1}|$, let $\gamma/2 \leq r_1 < r_2 \leq \gamma$ and let $c_{22} = c_{22}(p)$ be the constant defined by*

$$c_{22}(p) = 2 + \frac{c_1^3 c_2}{\epsilon(p)}. \quad (6.1)$$

We consider inequalities for domains of the form $R^(r) = J^*(r) \times \mathcal{O}(r)$*

$$\sup_{t \in J^*(r_1)} \|v(t, \cdot)\|_{L^2(\mathcal{O}(r_1), \mu)}^2 + \frac{2\epsilon(p)}{c_1} \|\nabla v\|_{L^2(R^*(r_1), \bar{\mu})}^2 \leq \frac{c_{22}(p)}{(r_2 - r_1)^2} \|v\|_{L^2(R^*(r_2), \bar{\mu})}^2. \quad (6.2)$$

- (1) Let $0 < p < 1$; then (6.2) holds for $J^* = J_+$ and $R^* = R_+$.
 (2) Let $1 < p < \infty$; then (6.2) holds for $J^* = J_-$ and $R^* = R_-$.
 (3) Let $0 < p < \infty$ and $p \neq 1$; then (6.2) holds for $J^* = J$ and $R^* = R$.

Proof. We prove (1), so assume $0 < p < 1$. Let us take $f(t) = 1/4t^p$, $g(t) = \sqrt{2\epsilon(p)}t^{p/2}$ and $\eta = -1$ in Lemma 5.1; then $f'' = \eta(g')^2$ is satisfied. So we can apply Lemma 5.1. Let ϕ be such that $\phi(r_2, x) = 0$. Let $I_r = [r, r_2]$ for $r_1 \leq r < r_2$ and $I = [r_1, r_2]$. Then by (5.1) and the relation

$$V^2 = 8\epsilon(p)U,$$

which follows from $g^2 = 8\epsilon(p)f$, $V = g(u)$ and $U = f(u)$, yields

$$\begin{aligned} \frac{1}{8\epsilon(p)} \int_{\mathcal{O}} V^2(r_1, \cdot) \phi^2 d\bar{\mu} + \int_{I_r} \mathcal{E}_{\mathcal{O}}^a(V, V; \phi^2) dt \\ = \frac{-1}{4\epsilon(p)} \int_{I_r \times \mathcal{O}} V^2 \partial_t \phi \phi d\bar{\mu} + \frac{1}{2\epsilon(p)} \int_{I_r} \mathcal{E}_{\mathcal{O}}^a(V, \phi; V\phi) dt. \end{aligned}$$

Since

$$\frac{1}{2\epsilon(p)} \mathcal{E}_{\mathcal{O}}^a(V, \phi; V\phi) \leq \frac{1}{2} \mathcal{E}_{\mathcal{O}}^a(V, V; \phi^2) + \frac{c_1^2}{8\epsilon(p)^2} \mathcal{E}_{\mathcal{O}}^a(\phi, \phi; V^2),$$

we obtain

$$\begin{aligned} \frac{1}{8\epsilon(p)} \int_{\mathcal{O}} V^2(r_1, \cdot) \phi^2 d\bar{\mu} + \frac{1}{2} \int_{I_r} \mathcal{E}_{\mathcal{O}}^a(V, V; \phi^2) dt \\ \leq \frac{-1}{4\epsilon(p)} \int_{I_r \times \mathcal{O}} V^2 \partial_t \phi \phi d\bar{\mu} + \frac{c_1^2}{8\epsilon(p)^2} \int_{I_r} \mathcal{E}_{\mathcal{O}}^a(\phi, \phi; V^2) dt. \end{aligned}$$

Taking ϕ in such a way that $\phi(t, x) = 1$ for $(t, x) \in R_+(r_1)$, $\phi(t, x) = 0$ for $(t, x) \notin R_+(r_2)$, $|\mathbb{D}^a[\phi, \phi]| \leq c_1 c_2 / (r_2 - r_1)^2$ and $|\partial_t \phi| \leq 1/(r_2 - r_1)^2$ yields (1). The proof of (2) is similar and (3) follows immediately from (1) and (2). $\square$

LEMMA 6.2. Let $c_{23}(p)$ and $c_{24}(p)$ be positive constants defined by

$$c_{23}(p) = 2c_1 \left(\frac{\gamma^2 \mu_\gamma^{1-\kappa}}{c_1} + \frac{1}{4\epsilon(p)} \right), \quad (6.3)$$

$$c_{24}(p) = \{c_3 c_{23}(p)\}^{1/\kappa p} c_{22}(p)^{1/p}. \quad (6.4)$$

Then for each $\gamma/2 \leq r_1 < r_2 \leq \gamma$ and $0 < p < \infty$, $p \neq 1$ we have

$$\|u\|_{L^{\kappa p}(R(r_1), \bar{\mu})} \leq \frac{c_{24}(p)}{(r_2 - r_1)^{2/p}} \|u\|_{L^p(R(r_2), \bar{\mu})}. \quad (6.5)$$

Proof. Applying (2.11) to $v = u^{p/2}$ and integrating over $[-r_1^2, r_1^2]$ in t we obtain

$$\|v\|_{L^{2\kappa}(R(r_1), \bar{\mu})}^{2\kappa} \leq c_3 \left\{ \sup_t \|v(t, \cdot)\|_{L^2(\mathcal{O}(r_1), \mu)}^{2(\kappa-1)} \cdot \{\mu_\gamma^{1-\kappa} \|v\|_{L^2(R(r_1), \bar{\mu})}^2 + \|\nabla v\|_{L^2(R(r_1), \bar{\mu})}^2\} \right\} \quad (6.6)$$

By (3) of Lemma 6.1 we have

$$\sup_t \|v(t, \cdot)\|_{L^2(\mathcal{O}(r_1), \mu)}^{2(\kappa-1)} \leq \left(\frac{c_{22}(p)}{(r_2 - r_1)^2} \right)^{\kappa-1} \|v\|_{L^2(R(r_2), \bar{\mu})}^{2(\kappa-1)}. \quad (6.7)$$

By (3) of Lemma 6.1 and $|J(r_1)| \leq 2\gamma^2$ we have

$$\frac{1}{2\gamma^2} \|v\|_{L^2(R(r_1), \bar{\mu})}^2 + \frac{2\epsilon(p)}{c_1} \|\nabla v\|_{L^2(R(r_1), \bar{\mu})}^2 \leq \frac{c_{22}(p)}{(r_2 - r_1)^2} \|v\|_{L^2(R(r_2), \bar{\mu})}^2,$$

which immediately implies

$$\mu_\gamma^{1-\kappa} \|v\|_{L^2(R(r_1), \bar{\mu})}^2 + \|\nabla v\|_{L^2(R(r_1), \bar{\mu})}^2 \leq c_{23}(p) \left(\frac{c_{22}(p)}{(r_2 - r_1)^2} \right) \|v\|_{L^2(R(r_2), \bar{\mu})}^2.$$

Putting this and (6.7) into (6.6) yields

$$\|v\|_{L^{2\kappa}(R(r_1), \bar{\mu})}^{2\kappa} \leq c_3 c_{23}(p) \cdot \left(\frac{c_{22}(p)}{(r_2 - r_1)^2} \right)^\kappa \|v\|_{L^2(R(r_2), \bar{\mu})}^{2\kappa}.$$

Recalling $v = u^{p/2}$ and $R(r_1) \subset R(r_2)$ we obtain (6.5). $\square$

Proof of Lemma 3.1. For $m \in \mathbb{N} = \{0, 1, 2, \dots\}$ let

$$\begin{aligned} r_{1,m} &= r_1 + (r_2 - r_1)2^{-(m+1)} \\ r_{2,m} &= r_1 + (r_2 - r_1)2^{-m} \\ p_m &= p\kappa^m, \end{aligned}$$

where $p = \frac{1}{2}\kappa^n(\kappa + 1)$ for some $n \in \mathbb{Z}$. We choose n in such a way that

$$|p_m - 1| \geq 1 - \kappa^{-1} \quad \text{for all } m \in \mathbb{N}, \quad (6.8)$$

$$\frac{1}{\epsilon(p_m)} \leq \frac{\kappa^m}{1 - \kappa^{-1}} \quad \text{for all } m \in \mathbb{N}. \quad (6.9)$$

We note by $p_m = p\kappa^m$,

$$\sum_{m=0}^{\infty} \frac{1}{p_m} = \frac{\kappa}{p(\kappa - 1)} = \frac{\bar{\kappa}}{p}, \quad (6.10)$$

$$\sum_{m=0}^{\infty} \frac{m+1}{p_m} = \frac{\kappa^2}{p(\kappa - 1)^2} = \frac{\bar{\kappa}^2}{p}. \quad (6.11)$$

By (6.5), $r_{1,\infty} = r_{2,\infty} = r_1$ and $r_{2,0} = r_2$ we have

$$\begin{aligned} \sup_{R(r_1)} u &= \lim_{m \rightarrow \infty} \|u\|_{L^{p_m}(R(r_1(m)), \bar{\mu})} \\ &\leq \left\{ \prod_{m \in \mathbb{N}} c_{24}(p_m) \frac{2^{(m+1)/p_m}}{(r_2 - r_1)^{2/p_m}} \right\} \|u\|_{L^p(R(r_2), \bar{\mu})} \\ &= \left\{ \prod_{m \in \mathbb{N}} c_{24}(p_m) \right\} \cdot \left(\frac{2^{\bar{\kappa}^2}}{(r_2 - r_1)^{2\bar{\kappa}}} \right)^{1/p} \|u\|_{L^p(R(r_2), \bar{\mu})}. \end{aligned} \quad (6.12)$$

Here we used (6.10) and (6.11) for the last line. Let $c_{25} = \prod_{m \in \mathbb{N}} \{c_3 c_{23}(p_m)\}^{1/\kappa p_m}$ and $c_{26} = \prod_{m \in \mathbb{N}} c_{22}(p_m)^{1/p_m}$, then by definition we have

$$\prod_{m \in \mathbb{N}} c_{24}(p_m) = c_{25} \cdot c_{26}. \quad (6.13)$$

By (6.9) and (6.11) and the definition of c_{23} and c_{22} , we easily see there exist positive constants c_{27} and c_{28} depending only on κ and (κ, c_1, c_2) , respectively, satisfying the following:

$$\begin{aligned} c_{25} &\leq \prod_{m \in \mathbb{N}} \left\{ 2c_3 c_1 \left(\frac{\gamma^2 \mu_\gamma^{1-\kappa}}{c_1} + \frac{\kappa^{m+1}}{\kappa - 1} \right) \right\}^{1/\kappa p_m} \\ &\leq \left[c_{27} (c_3 c_1)^{1/(\kappa-1)} \left\{ \max \left\{ \frac{\gamma^2 \mu_\gamma^{1-\kappa}}{c_1}, 1 \right\} \right\}^{1/(\kappa-1)} \right]^{1/p}, \end{aligned} \quad (6.14)$$

$$c_{26} \leq \prod_{m \in \mathbb{N}} \left\{ 2 + c_1^3 c_2 \left(\frac{\kappa^{m+1}}{\kappa - 1} \right) \right\}^{1/p_m} = c_{28}^{1/p}. \quad (6.15)$$

Putting (6.13), (6.14) and (6.15) into (6.12) and letting

$$c_{10} = c_{27} c_{28} 2^{\bar{\kappa}^2} \quad (6.16)$$

proves (3.1). The inequality (3.2) is derived similarly to (2) of Lemma 6.1. $\square$

7. Proof of Lemma 3.2

LEMMA 7.1. *Let $v = -\log u$ and $\psi \in \mathcal{D}_0(\mathcal{O})$, then*

$$\int v \psi^2 d\mu|_{t_1}^{t_2} + \frac{1}{2} \int_{t_1}^{t_2} \mathcal{E}_{\mathcal{O}}^a(v, v; \psi^2) dt \leq 2c_1^2 \mathcal{E}_{\mathcal{O}}^a(\psi, \psi)(t_2 - t_1). \quad (7.1)$$

Proof. Let us take in Lemma 5.1 as $f(t) = -\log t$, $g(t) = \log t$, $I = [t_1, t_2]$, $\phi = 1 \otimes \psi$ and $\eta = 1$. Then $U = -V = v$. Combining this with (5.1) we see

$$\begin{aligned} \int_{\mathcal{O}} v(t, \cdot) \psi^2 d\mu|_{t_1}^{t_2} + \int_I \mathcal{E}_{\mathcal{O}}^a(v, v; \psi^2) dt &= -2 \int_I \mathcal{E}_{\mathcal{O}}^a(v, \psi; \psi) \\ &\leq \frac{1}{2} \int_I \mathcal{E}_{\mathcal{O}}^a(v, v; \psi^2) dt + 2c_1^2 \int_I \mathcal{E}_{\mathcal{O}}^a(\psi, \psi; 1) dt. \end{aligned}$$

This implies (7.1). $\square$

LEMMA 7.2. Let $V(t) = \int v(t, \cdot) \psi^2 d\mu / \int \psi^2 d\mu$, where ψ is the function defined after (2.12). Set $c_{29} = 2c_1^2 c_2 \mu_{2\gamma} / \gamma^2$ and $c_{30} = 2c_1 c_4 \gamma^2$, then for a.e. t

$$\frac{dV(t)}{dt} + \frac{1}{c_{30}\mu_{\gamma}} \int_{\mathcal{O}(\gamma)} |v(t, x) - V(t)|^2 d\mu \leq \frac{c_{29}}{\mu_{\gamma}}. \quad (7.2)$$

Proof. By (2.9) and the definition of ψ , we see $\mathbb{D}^a[\psi, \psi] \leq c_1 c_2 \gamma^{-2}$. So by (2.1) and Lemma 7.1, we have

$$\int v \psi^2 d\mu|_{t_1}^{t_2} + \frac{1}{2c_1} \int_{t_1}^{t_2} dt \int \mathbb{D}[v, v] \psi^2 d\mu \leq c_{29}(t_2 - t_1).$$

Hence, by using the weighted Poincaré inequality (2.12), we obtain

$$\frac{(V(t_2) - V(t_1)) \int \psi^2 d\mu}{t_2 - t_1} + \frac{1}{c_{30}(t_2 - t_1)} \int_{t_1}^{t_2} dt \int |v(t, x) - V(t)|^2 \psi^2 d\mu \leq c_{29}.$$

Since $\psi(x) = 1$ on $\mathcal{O}(\gamma)$ and $|\psi| \leq 1$, this yields

$$\frac{(V(t_2) - V(t_1))\mu_{\gamma}}{(t_2 - t_1)} + \frac{1}{c_{30}(t_2 - t_1)} \int_{t_1}^{t_2} dt \int_{\mathcal{O}(\gamma)} |v(t, x) - V(t)|^2 d\mu \leq c_{29}.$$

Then dividing both sides by μ_{γ} and taking $t_2 \rightarrow t_1$, we obtain (7.2). $\square$

Proof of Lemma 3.2. We only prove (3.5) since the proof of (3.4) is similar. Let $c_{31} = \int_{\mathcal{O}(\gamma)} v(0, x) d\mu$, $c_{32} = c_{29}/\mu_{\gamma}$ and

$$w(t, x) = v(t, x) - c_{32}t - c_{31}, \quad W(t) = V(t) - c_{32}t - c_{31}.$$

Then by (7.2) we have

$$\frac{dW(t)}{dt} + \frac{1}{c_{30}\mu_{\gamma}} \int_{\mathcal{O}(\gamma)} |w(t, x) - W(t)|^2 d\mu \leq 0. \quad (7.3)$$

For $0 < t \leq \gamma^2$ let $\mathcal{O}_{s,t} = \{x \in \mathcal{O}(\gamma); s < w(t, x)\}$. Then for $s > 0$ and $x \in \mathcal{O}_{s,t}$ we have

$$w(t, x) - W(t) > s - W(t) > 0$$

since $W(t) \leq W(0) = 0$ in $0 < t \leq \gamma^2$, which follows from (7.3). By restricting the integral in (7.3) to $\mathcal{O}_{s,t}$ we obtain

$$\frac{dW(t)}{dt} + \frac{\mu(\mathcal{O}_{s,t})}{c_{30}\mu_\gamma} |s - W(t)|^2 \leq 0, \quad (7.4)$$

which immediately implies

$$\frac{\mu(\mathcal{O}_{s,t})}{\mu_\gamma} \leq \frac{c_{30}}{(s - W(t))^2} \cdot \frac{d(s - W(t))}{dt}.$$

Integrating from $t = 0$ to $t = \gamma^2$ and using $W(0) = 0$ we see

$$\frac{1}{\mu_\gamma} \bar{\mu}(\{(t, x); 0 < t < \gamma^2, s < w(t, x)\}) \leq \frac{c_{30}}{s - W(0)} = \frac{c_{30}}{s}.$$

This combined with

$$\begin{aligned} & \bar{\mu}(\{(t, x); 0 < t < \gamma^2, s < w(t, x)\}) \\ &= \bar{\mu}(\{(t, x); 0 < t < \gamma^2, s < v(t, x) - c_{32}t - c_{31}\}) \\ &\geq \bar{\mu}(\{(t, x); 0 < t < \gamma^2, s < v(t, x) - c_{32}\gamma^2 - c_{31}\}) \\ &= \bar{\mu}(\{(t, x); 0 < t < \gamma^2, u(t, x) < \exp(-s - (2c_1^3 c_2 \mu_{2\gamma} / \mu_\gamma) - c_{31})\}) \end{aligned}$$

completes the proof of (3.5). $\square$

8. Proof of Lemma 3.3

Let σ and τ be constants such that $1 < \sigma, \tau < \infty$. Let $c_{33} = 2c_1 c_{17}$ and let c_{34} be the constant defined by

$$c_{34} = \max\{\sigma c_{21}(r_2 - r_1)^{\beta\tau(\sigma-1)}, c_{33}(2c_{16})^{\tau(\sigma-1)}\}.$$

LEMMA 8.1. *Let $\varphi(r) = \sup_{Q(r)} \log f$. Then*

$$\varphi(r_1) < \frac{1 + \tau}{\sigma\tau} \varphi(r_2) + \frac{c_{34}}{(r_2 - r_1)^{\beta\tau(\sigma-1)}} \quad \text{for all } \theta\gamma \leq r_1 < r_2 \leq \gamma. \quad (8.1)$$

Proof. Let $\bar{\varphi} = \varphi(r_2)/\sigma$. Decompose $Q(r_2)$ as $Q(r_2) = Q_+ + Q_-$, where $Q_+ = \{x \in Q(r_2); \log f > \bar{\varphi}\}$ and $Q_- = Q(r_2) \setminus Q_+$, then by (3.7) we have

$$\int_{Q(r_2)} f^p d\bar{\mu} \leq \left(e^{p\varphi(r_2)} \frac{c_{33}}{\bar{\varphi}} + e^{p\bar{\varphi}} \right) \bar{\mu}(Q(\gamma)). \quad (8.2)$$

We choose p in such a way that the two terms in the brackets on the right-hand side are equal,

$$p = \frac{1}{\bar{\varphi}(\sigma - 1)} \log \left(\frac{\bar{\varphi}}{c_{33}} \right) \quad (8.3)$$

provided this quantity is less than $1/2c_1$ so that $0 < p < 1/2c_1$ holds. This requires that

$$\bar{\varphi} > c_{21}. \quad (8.4)$$

Assume (8.3) holds, then we have

$$\int_{Q(r_2)} f^p d\bar{\mu} \leq 2 e^{p\bar{\varphi}} \bar{\mu}(Q(\gamma)).$$

Combining this with (3.6) yields

$$\begin{aligned} \varphi(r_1) &\leq \frac{1}{p} \log \left(\frac{c_{16}}{(r_2 - r_1)^\beta} 2 e^{p\bar{\varphi}} \right) \\ &= \frac{1}{p} \left(\log \frac{2c_{16}}{(r_2 - r_1)^\beta} \right) + \bar{\varphi} \\ &= \bar{\varphi} \left\{ (\sigma - 1) \frac{\log((2c_{16})/(r_2 - r_1)^\beta)}{\log(\bar{\varphi}/c_{33})} + 1 \right\} \end{aligned}$$

by (8.3). Let $1 < \tau < \infty$. If the first term in the parentheses is less than $1/\tau$, that is:

$$\bar{\varphi} > c_{33} \left(\frac{2c_{16}}{(r_2 - r_1)^\beta} \right)^{\tau(\sigma-1)}, \quad (8.5)$$

then we have

$$\varphi(r_1) < \frac{1 + \tau}{\sigma \tau} \varphi(r_2). \quad (8.6)$$

Next assume one of provisions (8.4) and (8.5) is violated, then

$$\varphi(r_2) \leq \frac{c_{34}}{(r_2 - r_1)^{\beta\tau(\sigma-1)}}.$$

Since $\varphi(r_1) \leq \varphi(r_2)$, we obtain (8.1). $\square$

Proof of Lemma 3.3. Let $r_j = \gamma(1 - (1 - \theta)/(1 + j))$, then

$$r_{j+1} - r_j = \frac{\gamma(1 - \theta)}{(1 + j)(2 + j)}.$$

By (8.1)

$$\varphi(\gamma\theta) \leq c_{20} \left(\frac{1}{\gamma(1 - \theta)} \right)^{\beta\tau(\sigma-1)} \sum_{j=0}^{\infty} \left(\frac{1 + \tau}{\sigma\tau} \right)^j ((1 + j)(2 + j))^{\beta\tau(\sigma-1)}.$$

Here $c_{20} = \max\{\sigma c_{21}(\gamma(1 - \theta))^{\beta\tau(\sigma-1)}, c_{33}(2c_{16})^{\tau(\sigma-1)}\}$ as before. Equation (3.8) immediately follows from this. $\square$

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