UNIVERSAL COVERINGS OF SPACES OF HOLOMORPHIC MAPS

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1. Introduction

For each integer $d \geq 1$, let Hol_d denote the space consisting of all holomorphic maps f of degree d from the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ to itself. Similarly, let Hol_d^* denote the subspace of Hol_d consisting of all maps $f \in \text{Hol}_d$ which preserve the base points. These spaces are of interest both from a classical and a modern point of view (see [1, 4]). The spaces Hol_d and Hol_d^* can be easily identified with the following spaces of rational functions:

$$\begin{cases} \text{Hol}_d = \{p(z)/q(z) : p(z), q(z) \text{ satisfying the conditions (i), (ii)}\}, \\ \text{Hol}_d^* = \{p(z)/q(z) : p(z), q(z) \text{ satisfying the conditions (i), (iii)}\}, \end{cases}$$

where the conditions (i)–(iii) are given by the following:

- (i) $p(z), q(z) \in \mathbb{C}[z]$ are mutually coprime polynomials;
- (ii) $\max\{\deg(p(z)), \deg(q(z))\} = d$;
- (iii) $p(z), q(z) \in \mathbb{C}[z]$ are monic polynomials of degree d .

In particular, if $d = 1$, Hol_1 is the group of fractional linear transformations $\text{PSL}_2(\mathbb{C})$ and Hol_1^* may be identified with the affine transformation group of $\mathbb{C}$. It is an elementary and fundamental fact that Hol_d and Hol_d^* are connected spaces. More generally, the following result is known.

THEOREM 1.1. [5], [12] *Let $d \geq 1$ be an integer.*

- (i) $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$.
- (ii) $\pi_1(\text{Hol}_d^*) = \mathbb{Z}$.
- (iii) *If $i_d : \text{Hol}_d \rightarrow \text{Map}_d(S^2, S^2)$ and $\tilde{i}_d : \text{Hol}_d^* \rightarrow \Omega_d^2 S^2$ are inclusion maps, then i_d and $\tilde{i}_d$ are homotopy equivalences up to dimension d .*

Remark. The map $f : X \rightarrow Y$ is said to be a *homotopy equivalence up to dimension N* if the induced homomorphism $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ is bijective when $k < N$ and surjective when $k = N$.

Jones (cf. [12]) independently proved the assertion (ii) and Guest *et al.* [6] also obtained an easier proof of (ii). In contrast, it seems very difficult to understand Epshtein's original proof of (i). So in this paper, we also give another proof of (i) by using Segal's result. Since the spaces Hol_d and Hol_d^* are non-simply connected, we shall study the homotopy types of the universal coverings of Hol_d and Hol_d^* . We denote by $\tilde{\text{Hol}}_d$ and $\tilde{\text{Hol}}_d^*$ the universal coverings of Hol_d and Hol_d^* , respectively. Let $R : \text{Hol}_d^* \rightarrow \mathbb{C}^*$ be the resultant given by

$$R(p(z)/q(z)) = \prod_{1 \leq i, j \leq n} (\alpha_i - \beta_j) \quad \text{for } p(z)/q(z) \in \text{Hol}_d^*(S^2, S^2),$$

where

$$p(z) = \prod_{i=1}^d (z - \alpha_i), \quad q(z) = \prod_{j=1}^d (z - \beta_j) \quad (a, b \in \mathbb{C}^*).$$

Segal [12] has already studied the topology of $\tilde{\text{Hol}}_d^*$.

THEOREM 1.2. [12]

- (i) *There is a homotopy equivalence $R^{-1}(1) \simeq \tilde{\text{Hol}}_d^*$.*
- (ii) *There is a homeomorphism $\text{Hol}_d^* \cong R^{-1}(1) \times_{\mathbb{Z}/d^2} \mathbb{C}^*$, where the group $\mathbb{Z}/d^2$ act it by $\omega \cdot ((p(z), q(z)), a) = ((\omega \cdot p(z), \omega \cdot q(z)), \omega a)$. Here we identify $\mathbb{Z}/2d$ with the subgroup of $\mathbb{C}^*$ generated by the primitive d^2 th root of unity and the action of $\omega \in \mathbb{Z}/d^2$ on a monic polynomial $f(z)$ is given by $\omega \cdot f(z) = \prod_{j=1}^d (z - \omega \gamma_j)$ if $f(z) = \prod_{j=1}^d (z - \gamma_j)$.*

As a next step, we wish to study the space $\tilde{\text{Hol}}_d$. If $d = 1$ or 2 , the following result is known.

THEOREM 1.3. [6]

- (i) *If $d = 1$, $\tilde{\text{Hol}}_1^*$ is contractible and there exists a homotopy equivalence $\tilde{\text{Hol}}_1 \simeq S^3$.*
- (ii) *If $d = 2$, there are homotopy equivalences*

$$\begin{cases} \tilde{\text{Hol}}_2^* \simeq S^2, \\ \tilde{\text{Hol}}_2 \simeq S^2 \times S^3. \end{cases}$$

The main purpose of this paper is to prove the following result.

THEOREM 1.4. (Theorem 4.1) *Let $d \geq 1$ be an integer. Then there is a homotopy equivalence $\tilde{\text{Hol}}_d \simeq \tilde{\text{Hol}}_d^* \times S^3$.*

COROLLARY 1.5. (Corollary 4.2) *Let $d \geq 1$ be an integer. Then there is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_k(\text{Hol}_d^*) \oplus \pi_k(S^3)$ for any $k \geq 2$. In particular, if $d > k \geq 2$, then there is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.*

Cohen *et al* [2] determined the stable homotopy type of Hol_d^* and computed its homology. So the space Hol_d^* is now largely understood. In general, the Serre spectral sequence of the evaluation fibration

$$\text{Hol}_d^* \xrightarrow{j_d} \text{Hol}_d \xrightarrow{ev} S^2 \quad (1.1)$$

does not collapse at the E^2 -term (cf. [10]) and it does not seem so easy to compute the (co)homology of Hol_d directly using (1.1). However, it follows from Theorem 1.4 that we can assert that the fibration (1.1) is trivial on the level of ‘universal coverings’ (if we replace S^2 by S^3).

The plan of this paper is as follows. In Section 2, we shall give an easier alternative proof of (i) of Theorem 1.1. In Section 3, we shall study the evaluation fibration and obtain a certain important fibration (see (3.1)). Finally, in Section 4 we shall prove the main theorem.

2. Fundamental groups

Let $\text{Map}(S^n, X)$ (respectively $\text{Map}^*(S^n, X)$) denote the space consisting of all (respectively basepoint preserving) continuous maps $f : S^n \rightarrow X$ with compact-open topology. For $f \in \text{Map}(S^n, X)$, let $\text{Map}_f(S^n, X)$ and $\text{Map}_f^*(S^n, X)$ be the path components of $\text{Map}(S^n, X)$ and $\text{Map}^*(S^n, X)$ containing the element f , respectively. Let $ev : \text{Map}(S^n, X) \rightarrow X$ be the evaluation map given by $ev(g) = g(s_0)$ for $g \in \text{Map}(S^n, X)$, where $s_0 \in S^n$ is a fixed base point. There is an evaluation fibration sequence $\text{Map}_f^*(S^n, X) \rightarrow \text{Map}_f(S^n, X) \xrightarrow{ev} X$.

For each integer $d \in \mathbb{Z}$, we denote by $\text{Map}_d(S^2, S^2)$ (respectively $\Omega_d^2 S^2$) the space consisting of all (respectively base point preserving) continuous maps $f : S^2 \rightarrow S^2$ of degree d . We recall the following well-known result due to Whitehead.

LEMMA 2.1. [14] *Let $\partial : \pi_k(X) \rightarrow \pi_{k-1}(\text{Map}_f^*(S^n, X))$ be the boundary operator of the above evaluation fibration.*

If we identify $\pi_{k-1}(\text{Map}_f^*(S^n, X)) \cong \pi_{k-1+n}(X)$, then ∂ is identified with the operator $\partial' : \pi_k(X) \rightarrow \pi_{k-1+n}(X)$, which is defined by the Whitehead product $\partial'(\alpha) = [\alpha, f]$ for $\alpha \in \pi_k(X)$.

Proof of (i) of Theorem 1.1. Since $\text{Hol}_1 \cong \text{PSL}_2(\mathbb{C})$, there is a homotopy equivalence $\text{Hol}_1 \simeq \mathbb{RP}^3$. Hence $\pi_1(\text{Hol}_1) \cong \pi_1(\mathbb{RP}^3) = \mathbb{Z}/2$. So the case $d = 1$ is trivial.

From now on, we assume $d \geq 2$ and consider the following commutative diagram (up to sign)

$$\begin{array}{ccccccc}
 \pi_2(S^2) & \xrightarrow{\partial} & \pi_1(\text{Hol}_d^*) & \longrightarrow & \pi_1(\text{Hol}_d) & \longrightarrow & 0 \\
 \downarrow = & & \downarrow (\tilde{i}_d)_* & & \downarrow (i_d)_* \cong & & \\
 \pi_2(S^2) & \xrightarrow{\partial_2} & \pi_1(\Omega_d^2 S^2) & \xrightarrow{j_*} & \pi_1(\text{Map}_d(S^2, S^2)) & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow = & & \\
 \pi_2(S^2) & \xrightarrow{\partial'_2} & \pi_3(S^2) & \longrightarrow & \pi_1(\text{Map}_d(S^2, S^2)) & \longrightarrow & 0
 \end{array}$$

where horizontal sequences are exact.

We note that $\pi_2(S^2) = \mathbb{Z} \cdot \iota_2$ and $\pi_3(S^2) = \mathbb{Z} \cdot \eta_2$, where η_2 denotes the Hopf map. Since $[\iota_2, \iota_2] = 2\eta_2$ [13], using Lemma 2.1, $\partial'_2(\iota_2) = [\iota_2, d\iota_2] = 2d\eta_2$; hence $\pi_1(\text{Map}_d(S^2, S^2)) = \mathbb{Z}/2d$. Since $d \geq 2$, by (iii) of Theorem 1.1 $(i_d)_*$ is bijective. So $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$ and this completes the proof. $\square$

From the above proof, it is also easy to see the following result.

COROLLARY 2.2. *Let $j_d : \text{Hol}_d^* \xrightarrow{\subset} \text{Hol}_d$ denote the inclusion map. Then there is a commutative diagram*

$$\begin{array}{ccc}
 \pi_1(\text{Hol}_d^*) & \xrightarrow{(j_d)_*} & \pi_1(\text{Hol}_d) \\
 \cong \downarrow & & \cong \downarrow \\
 \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/2d
 \end{array}$$

where $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/2d$ denotes the natural projection homomorphism.

3. Evaluation fibrations and group actions

In this section we consider the evaluation fibration $ev : \text{Hol}_d \rightarrow S^2$ which is defined by $ev(f) = f(\infty)$ for $f \in \text{Hol}_d$. Let $\iota : \mathbb{Z}/2d \rightarrow S^1$ and $\iota_1 : S^2 = \mathbb{CP}^1 \rightarrow \mathbb{CP}^\infty = BS^1$ be inclusion maps, where we identify $\mathbb{Z}/2d$ with the subgroup $\{\xi \in \mathbb{C}^* : \xi^{2d} = 1\} \subset S^1$. Then, if we regard the resultant R as the map $R : \text{Hol}_d^* \rightarrow \mathbb{C}^* \simeq S^1 \simeq B\mathbb{Z}$, it follows from Corollary 2.2 that there is a homotopy

commutative diagram

$$\begin{array}{ccccc}
 \tilde{\text{Hol}}_d^* & \xrightarrow{\tilde{j}_d} & \tilde{\text{Hol}}_d & & \\
 r_d \downarrow & & r'_d \downarrow & & \\
 \text{Hol}_d^* & \xrightarrow{j_d} & \text{Hol}_d & \xrightarrow{ev} & S^2 \\
 R \downarrow & & R_1 \downarrow & & \\
 B\mathbb{Z} & \xrightarrow{B\pi} & B(\mathbb{Z}/2d) & \xrightarrow{B\iota} & BS^1 = K(\mathbb{Z}, 2)
 \end{array}$$

where all horizontal and vertical sequences are fibrations. Then we obtain the following result.

PROPOSITION 3.1. *There is a fibration sequence (up to homotopy)*

$$\tilde{\text{Hol}}_d^* \xrightarrow{\tilde{j}_d} \tilde{\text{Hol}}_d \xrightarrow{p_d} S^3. \quad (3.1)$$

Proof. If we recall Theorem 1.3, the assertion clearly holds when $1 \leq d \leq 2$. So from now on, we assume $d \geq 3$. Since $S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{\iota_1} K(\mathbb{Z}, 2)$ is a fibration sequence, it suffices to show that the following diagram is homotopy commutative:

$$\begin{array}{ccc}
 \text{Hol}_d & \xrightarrow{ev} & S^2 \\
 R_1 \downarrow & & \downarrow \iota_1 \\
 B(\mathbb{Z}/2d) & \xrightarrow{B\iota} & K(\mathbb{Z}, 2)
 \end{array} \quad (3.2)$$

In fact, if this is true, it follows from [3, (2.1)] that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 \tilde{\text{Hol}}_d^* & \xrightarrow{\tilde{j}_d} & \tilde{\text{Hol}}_d & \xrightarrow{p_d} & S^3 \\
 r_d \downarrow & & r'_d \downarrow & & \downarrow \eta_2 \\
 \text{Hol}_d^* & \xrightarrow{j_d} & \text{Hol}_d & \xrightarrow{ev} & S^2 \\
 R \downarrow & & R_1 \downarrow & & \downarrow \iota_1 \\
 B\mathbb{Z} & \xrightarrow{B\pi} & B(\mathbb{Z}/2d) & \xrightarrow{B\iota} & BS^1 = K(\mathbb{Z}, 2)
 \end{array}$$

where all horizontal and vertical sequences are fibrations. Hence we obtain the fibration (3.1). So it remains to show that (3.2) is homotopy commutative. For proving

this, it is sufficient to show that two homomorphisms

$$\begin{cases} \phi_1 : \mathbb{Z} = H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \xrightarrow{\iota_1^*} H^2(S^2, \mathbb{Z}) \xrightarrow{ev^*} H^2(\text{Hol}_d, \mathbb{Z}), \\ \phi_2 : \mathbb{Z} = H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \xrightarrow{B\iota_1^*} H^2(B(\mathbb{Z}/2d), \mathbb{Z}) \xrightarrow{R_1^*} H^2(\text{Hol}_d, \mathbb{Z}), \end{cases}$$

coincide (up to units). We remark that ι_1^* is an isomorphism and that $B\iota_1^*$ can be identified with the natural projection homomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/2d$. Consider the Serre spectral sequence of the evaluation fibration: $E_2^{p,q} = H^p(S^2, \mathbb{Z}) \otimes H^q(\text{Hol}_d^*, \mathbb{Z}) \Rightarrow H^{p+q}(\text{Hol}_d, \mathbb{Z})$.

Since $H^1(\text{Hol}_1^*, \mathbb{Z}) = \mathbb{Z}$, $H^2(\text{Hol}_d^*, \mathbb{Z})$ is a torsion group and $\pi_1(\text{Hol}_d) = \mathbb{Z}/2d$, the differential $d_2 : \mathbb{Z} = E_2^{0,1} \rightarrow E_2^{2,0} = \mathbb{Z}$ is identified with the $2d$ -times multiplication homomorphism ρ_d . Hence $E_3^{2,0} = E_\infty^{2,0} = H^2(\text{Hol}_d, \mathbb{Z}) = \mathbb{Z}/2d$ and we can identify $ev^* = \pi : \mathbb{Z} = H^2(S^2, \mathbb{Z}) \rightarrow H^2(\text{Hol}_d, \mathbb{Z}) = \mathbb{Z}/2d$, so that $\phi_1 = \pi : \mathbb{Z} \rightarrow \mathbb{Z}/2d$.

Similarly, if we compute the Serre spectral sequence of the fibration sequence $\tilde{\text{Hol}}_d \xrightarrow{r'_d} \text{Hol}_d \xrightarrow{R_1} B(\mathbb{Z}/2d)$, we can easily see that R_1^* is an isomorphism. Hence ϕ_2 can be also identified with π . So $\phi_1 = \phi_2$ and this completes the proof. $\square$

Now we identify $\text{Hol}_1 = \text{PSL}_2(\mathbb{C})$ and consider the left action of Hol_1 on Hol_d defined by post-composition:

$$A \cdot f(z) = A(f(z)) \quad \text{for } (A, f(z)) \in \text{Hol}_1 \times \text{Hol}_d.$$

We can define the left action of Hol_1^* on Hol_d^* by post-composition in a similar way. We recall the following two results, which we use in the next section.

LEMMA 3.2. [6] *Let $d \geq 1$ be an integer. Then the group Hol_1 acts on Hol_d freely by post-composition. Similarly, the group Hol_1^* also acts on Hol_d^* freely by post-composition.*

Proof. This follows from [6, (3.1)]. $\square$

LEMMA 3.3. [6] *Let $d \geq 1$ be an integer. Then the natural inclusion $j_d : \text{Hol}_d^* \rightarrow \text{Hol}_d$ induces a homeomorphism $\pi_d : \text{Hol}_1^* \setminus \text{Hol}_d^* \xrightarrow{\cong} \text{Hol}_1 \setminus \text{Hol}_d$ such that the diagram*

$$\begin{array}{ccccc} \text{Hol}_1^* & \longrightarrow & \text{Hol}_d^* & \xrightarrow{q_d} & \text{Hol}_1^* \setminus \text{Hol}_d^* \\ j_1 \downarrow & & j_d \downarrow & & \pi_d \downarrow \cong \\ \text{Hol}_1 & \longrightarrow & \text{Hol}_d & \xrightarrow{q'_d} & \text{Hol}_1 \setminus \text{Hol}_d \end{array}$$

is commutative, where horizontal sequences are principal fibration sequences.

Proof. This easily follows from [6, (3.2)]. $\square$

4. The relation between $\tilde{\mathbf{Hol}}_d$ and $\mathbf{Hol}_d^*$

Now we can state the main result of this paper.

THEOREM 4.1. *Let $d \geq 1$ be an integer. Then there is a homotopy equivalence $\tilde{\mathbf{Hol}}_d \simeq \mathbf{Hol}_d^* \times S^3$.*

Proof. Let $X_d = \mathbf{Hol}_1^* \setminus \mathbf{Hol}_d^*$ and consider the homotopy exact sequence of the principal fibration sequence $\mathbf{Hol}_1^* \xrightarrow{s_d} \mathbf{Hol}_d^* \xrightarrow{q_d} \mathbf{Hol}_1^* \setminus \mathbf{Hol}_d^* = X_d$. Since $\mathbf{Hol}_1^* \simeq S^1$, $(q_d)_* : \pi_k(\mathbf{Hol}_d^*) \xrightarrow{\cong} \pi_k(\mathbf{Hol}_1^* \setminus \mathbf{Hol}_d^*) = \pi_k(X_d)$ is bijective for any $k \geq 2$. If we denote the universal covering of X_d by $r_d'' : \tilde{X}_d \rightarrow X_d$, this induces a commutative diagram

$$(i) \quad \begin{array}{ccc} \tilde{\mathbf{Hol}}_d^* & \xrightarrow{\tilde{q}_d} & \tilde{X}_d \\ r_d \downarrow & & \downarrow r_d'' \\ \mathbf{Hol}_d^* & \xrightarrow{q_d} & X_d = \mathbf{Hol}_1^* \setminus \mathbf{Hol}_d^* \end{array}$$

Because r_d and r_d'' are also universal covering projections,

$$\begin{cases} (r_d)_* : \pi_k(\tilde{\mathbf{Hol}}_d^*) \xrightarrow{\cong} \pi_k(\mathbf{Hol}_d^*), \\ (r_d'')_* : \pi_k(\tilde{X}_d) \xrightarrow{\cong} \pi_k(X_d), \end{cases}$$

are isomorphisms for any $k \geq 2$. Hence $\tilde{q}_{d*} : \pi_k(\tilde{\mathbf{Hol}}_d^*) \xrightarrow{\cong} \pi_k(\tilde{X}_d)$ is also an isomorphism for any $k \geq 2$. However, since $\tilde{\mathbf{Hol}}_d^*$ and $\tilde{X}_d$ are simply connected, the map $\tilde{q}_d : \tilde{\mathbf{Hol}}_d^* \xrightarrow{\cong} \tilde{X}_d$ is in fact a homotopy equivalence. Consider the map of composite $\pi_d^{-1} \circ q_d' \circ r_d' : \tilde{\mathbf{Hol}}_d \rightarrow X_d$:

(ii)

$$\begin{array}{ccccc} & & & & \tilde{X}_d \\ & & & & \downarrow r_d'' \\ \tilde{\mathbf{Hol}}_d^* & \xrightarrow{r_d} & \mathbf{Hol}_d^* & \xrightarrow{q_d} & X_d = \mathbf{Hol}_1^* \setminus \mathbf{Hol}_d^* \\ \tilde{j}_d \downarrow & & j_d \downarrow & & \downarrow \pi_d \cong \\ \tilde{\mathbf{Hol}}_d & \xrightarrow{r_d'} & \mathbf{Hol}_d & \xrightarrow{q_d'} & \mathbf{Hol}_1 \setminus \mathbf{Hol}_d \end{array}$$

Since $\tilde{\text{Hol}}_d$ is simply connected and r_d'' is a universal covering projection, there is a lifting $\theta_d : \tilde{\text{Hol}}_d \rightarrow \tilde{X}_d$ such that $r_d'' \circ \theta_d = \pi_d^{-1} \circ q_d' \circ r_d'$. Hence, using two commutative diagrams (i) and (ii), we have the relation $\theta_d \circ \tilde{j}_d = \tilde{q}_d$:

$$\begin{array}{ccc} \tilde{\text{Hol}}_d^* & \xrightarrow[\simeq]{\tilde{q}_d} & \tilde{X}_d \\ \tilde{j}_d \downarrow & & \downarrow = \\ \tilde{\text{Hol}}_d & \xrightarrow{\theta_d} & \tilde{X}_d \end{array}$$

Let $\tilde{q}_d^{-1} : \tilde{X}_d \xrightarrow{\simeq} \tilde{\text{Hol}}_d^*$ denote a homotopy inverse of $\tilde{q}_d$ and define the map $s_d : \tilde{\text{Hol}}_d \rightarrow \tilde{\text{Hol}}_d^*$ by the formula $s_d = \tilde{q}_d^{-1} \circ \theta_d$. Clearly $s_d \circ \tilde{j}_d \simeq \text{id} : \tilde{\text{Hol}}_d^* \rightarrow \tilde{\text{Hol}}_d^*$ (homotopic). Hence it follows from the homotopy exact sequence of the fibration (3.1) that the map $\phi_d = (s_d, p_d) : \tilde{\text{Hol}}_d \rightarrow \tilde{\text{Hol}}_d^* \times S^3$ induces an isomorphism on π_k for any k . Hence $\phi_d : \tilde{\text{Hol}}_d \xrightarrow{\simeq} \tilde{\text{Hol}}_d^* \times S^3$ is a homotopy equivalence. $\square$

COROLLARY 4.2. *Let $d \geq 1$ be an integer. Then there is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_k(\text{Hol}_d^*) \oplus \pi_k(S^3)$ for any $k \geq 2$. In particular, if $d > k \geq 2$, then there is an isomorphism $\pi_k(\text{Hol}_d) \cong \pi_{k+2}(S^2) \oplus \pi_k(S^3)$.*

Proof. The first assertion is trivial. It follows from Segal's result (Theorem 1.1(iii)) that $\pi_k(\text{Hol}_d^*) \cong \pi_k(\Omega_d^2 S^2) \cong \pi_{k+2}(S^2)$ if $k < d$. Hence the second assertion also follows. $\square$

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