A SCHWARZ MAP OF APPELL'S F_2 WITH PROJECTIVE MONODROMY GROUP OF ORDER 25 920

Mitsuo KATO

(Received 17 December 2001 and revised 28 February 2002)

1. Introduction

Appell's hypergeometric function

$$F_2(a; b, b'; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)(b, m)(b', n)}{(c, m)(c', n)(1, m)(1, n)} x^m y^n,$$

where $(a, n) = \Gamma(a+n)/\Gamma(a)$, satisfies a system $E_2(a; b, b'; c, c')$ of differential equations on the (x, y) -space $X (\simeq \mathbf{P}^2)$. The rank of E_2 is four, and its singular locus is given by

$$\text{Sing}(E_2) = \{(x, y) \mid xy(1-x)(1-y)(1-x-y) = 0\} \cup L_{\infty},$$

where L_{∞} is the line at infinity. In [Kt], we tabulated all the systems of parameters $(a; b, b'; c, c')$ such that each $E_2(a; b, b'; c, c')$ has a finite irreducible monodromy group. Among these finite groups, the biggest one G is of order $12 \times 25\,920$. In this paper, we treat

$$E := E_2\left(-\frac{1}{12}; \frac{1}{6}, \frac{1}{12}; \frac{1}{3}, \frac{1}{2}\right), \quad (1.1)$$

of which monodromy group is G . Note that there are infinitely many E_2 with the monodromy group G . The center $Z(G)$ of G is isomorphic to $\mathbf{Z}/12\mathbf{Z}$ and the projective monodromy group $P(G) := G/Z(G)$ is isomorphic to the quotient group of the reflection group G_{155520} (No. 32 in the list in [ST]) divided by its center.

A Schwarz map s_E of E defined by the ratio of four linearly independent solutions of E is a 25 920-valued map of $X - \text{Sing}(E)$ into $\mathbf{P}^3$. It is invariant under the involution

$$\iota : (x, y) \longrightarrow (-x/(1-x), y/(1-x)), \quad (1.2)$$

that is, the set $\{s_E(x, y)\}$ of distinct values of s_E at (x, y) is the same as that at $\iota(x, y)$. Let $\pi_{\iota} : X \longrightarrow Y (\simeq \mathbf{P}^2)$ be an ι -invariant 2:1 rational map. Then there exists a

FIGURE 1.

system E' of differential equations on Y such that a (multi-valued) function f on Y is a solution of E' if and only if $(1-x)^{1/24} \cdot (f \circ \pi_t)(x, y)$ is a solution of E . We define a Schwarz map $s_{E'}$ of E' so that the equality $s_E = s_{E'} \circ \pi_t$ holds.

The closure S of the image of s_E turns out to be an irreducible hypersurface of degree 90 on which $P(G)$ acts. The set $\{s_E(x, y)\}$ is an orbit of $P(G)$ for each $(x, y) \in X - \text{Sing}(E)$. As stated in [ST], there are basic invariant (homogeneous) polynomials $I_k(u) = I_k(u_1, u_2, u_3, u_4)$, $k = 12, 18, 24, 30$, of the reflection group G_{155520} , where $k = \deg(I_k)$. We make use of invariant polynomials F_k obtained by Maschke [Msk] to define our I_k .

The rational functions $\lambda := I_{24}/I_{12}^2$, $\mu := I_{18}^3/I_{12}^3$ and $v := I_{30}/I_{12}I_{18}$ are $P(G)$ -invariant and $\varpi_G(u) := [\lambda(u) : \mu(u) : v(u) : 1]$ defines a $P(G)$ -invariant rational map of S onto a quartic surface S_4 in $\mathbf{P}^3$ such that the generic fiber of ϖ_G is an orbit of $P(G)$, consisting of 25 920 points. There exists a birational map ψ of S_4 to Y such that $\psi \circ \varpi_G \circ s_E = \pi_t$. The rational map $\pi_G := \psi \circ \varpi_G$ is a single-valued inverse map of $s_{E'}$, that is, we have $\pi_G \circ s_{E'} = \text{id}_Y$.

We have the commutative diagram shown in Figure 1.

2. Maschke's invariants

We recall some notations from [Kt]. Let U be the $(4, 4)$ -matrix with expression (6.10) in [Kt] and $u_j(x, y)$, $1 \leq j \leq 4$, be the solutions of E defined by ${}^t(u_1, u_2, u_3, u_4) = U \cdot {}^t(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, where φ_j are defined by (2.5) in [Kt].

Let $W = X - \text{Sing}(E)$ and $P_0 = (p_0, p_0)$, where p_0 is a sufficiently small positive number. Let $\rho = \rho_u$ be the group homomorphism of $\pi_1(W, P_0)$ into $GL(4, \mathbf{C})$ defined

by ${}^t(\gamma_*u_1, \gamma_*u_2, \gamma_*u_3, \gamma_*u_4) = \rho(\gamma) {}^t(u_1, u_2, u_3, u_4)$, where γ_*u_j is the analytic continuation of $u_j(x, y)$ along the loop γ . We call ρ the monodromy representation with respect to $u_j(x, y)$, $1 \leq j \leq 4$, and $G := \rho(\pi_1(W, P_0))$ the *monodromy group* of E . $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 are loops surrounding once the irreducible components $\{x = 0\}$, $\{y = 0\}$, $\{x + y = 1\}$, $\{x = 1\}$ and $\{y = 1\}$, respectively. Then G is generated by $\rho(\gamma_j)$, $1 \leq j \leq 5$. Let N_r be the smallest normal subgroup of G containing $\rho(\gamma_3)$, $\rho(\gamma_4)$ and $\rho(\gamma_5)$, which are reflections. Then N_r is isomorphic to the reflection group G_{155520} of No. 32 in the list in [ST].

The characteristic exponents of $E_2(a; b, b'; c, c')$ are known to be $\{0, 0, \lambda_1, \lambda_1\}$, $\lambda_1 = 1 - c$ along $\{x = 0\}$; $\{0, 0, \lambda_2, \lambda_2\}$, $\lambda_2 = 1 - c'$ along $\{y = 0\}$; $\{0, 0, 0, \lambda_3\}$, $\lambda_3 = c + c' - a - b - b'$ along $\{x + y = 1\}$; $\{0, 0, 0, \lambda_4\}$, $\lambda_4 = b' + c - a - b$ along $\{x = 1\}$; $\{0, 0, 0, \lambda_5\}$, $\lambda_5 = b + c' - a - b'$ along $\{y = 1\}$; $\{a, a, a, b + b'\}$ along L_∞ . So, in our case, the characteristic exponents of E are given as

$$\begin{aligned} &\{0, 0, 2/3, 2/3\}, \quad \text{along } \{x = 0\}, \\ &\{0, 0, 1/2, 1/2\}, \quad \text{along } \{y = 0\}, \\ &\{0, 0, 0, 2/3\}, \quad \text{along } \{x + y = 1\} \text{ and } \{y = 1\}, \\ &\{0, 0, 0, 1/3\}, \quad \text{along } \{x = 1\}, \\ &\{-1/12, -1/12, -1/12, 1/4\} \quad \text{along } L_\infty. \end{aligned} \tag{2.1}$$

LEMMA 2.1. *We have*

$$Z(N_r) = \{e^{2\pi i k/6} \cdot E_4 \mid k \in \mathbf{Z}\} \quad \text{and} \quad Z(G) = \{e^{2\pi i k/12} \cdot E_4 \mid k \in \mathbf{Z}\}, \tag{2.2}$$

where E_4 denotes the 4×4 identity matrix.

Proof. First we note that both $Z(G)$ and $Z(N_r)$ consist of scalar matrices because G and N_r are irreducible. Since $Z(N_r)$ has order six ([ST]), the first equality of (2.2) holds. Let γ_∞ be a loop surrounding once the line L_∞ at infinity. Since the characteristic exponents along L_∞ are $-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}$ and $\frac{1}{4}$, we have $\rho(\gamma_\infty)^3 = e^{-2\pi i/4} \cdot E_4$. Hence $e^{2\pi i/12} \cdot E_4$ is in $Z(G)$. The order of $Z(G)$ cannot be greater than 12 because we have the inclusion $N_r/Z(N_r) \longrightarrow G/Z(G)$. This proves the latter equality of (2.2). $\square$

COROLLARY 2.2. *The inclusion $N_r \longrightarrow G$ induces the isomorphism $N_r/Z(N_r) \simeq G/Z(G)$.*

LEMMA 2.3. *We have*

$$\rho(\gamma_1), \rho(\gamma_3), \rho(\gamma_4), \rho(\gamma_5) \in N_r, \quad (2.3)$$

$$\rho(\gamma_2) = e^{2\pi i/12} \rho(\gamma) \quad \text{for some } \rho(\gamma) \in N_r. \quad (2.4)$$

Proof. We have proved (2.3) and $\rho(\gamma_2) \notin N_r$ in [Kt]. From (2.2) and the above corollary, we have (2.4). $\square$

Maschke [Msk] found basic invariant polynomials $F_k(z_0, z_1, z_2, z_3)$, $k = 12, 18, 24, 30$, of G_{155520} , where $k = \deg(F_k)$. Since his embedding of G_{155520} into $GL(4, \mathbb{C})$ is a little bit different from our ρ (see [Kt, p. 297] and [Cxt, Sections 12.5, 13.9]), we slightly modify them and define our basic invariant polynomials I_k of N_r as

$$I_k(u_1, u_2, u_3, u_4) = c_k F_k(u_4, -u_3, -u_1, -u_2), \quad 1 \leq j \leq 4, \quad (2.5)$$

where the constants c_k are determined later. Put

$$f_k(x, y) = I_k(u(x, y)) = I_k(u_1(x, y), u_2(x, y), u_3(x, y), u_4(x, y)). \quad (2.6)$$

Then we have the following lemma.

LEMMA 2.4. $f_{12}(x, y)$, $f_{18}(x, y)/\sqrt{y}$, $f_{24}(x, y)$ and $f_{30}(x, y)/\sqrt{y}$ are polynomials in x, y . The degrees of the first two are at most one and those of the latter two are at most two.

Proof. Since $I_k(u)$ are absolute invariants of N_r , we have, from Lemma 2.3, that $\gamma_{j*} f_k(x, y) = f_k(x, y)$ for $j = 1, 3, 4, 5$ and that $\gamma_{2*} f_k(x, y) = e^{2\pi i k/12} f_k(x, y)$. Consequently, the four functions in this lemma are single-valued in $\mathbb{C}^2$.

Since the characteristic exponents of E are given by (2.1), we know that $f_{12}(x, y)$, $f_{18}(x, y)/\sqrt{y}$, $f_{24}(x, y)$, $f_{30}(x, y)/\sqrt{y}$ are polynomials and that the degree of $f_k(x, y)$ (respectively $f_k(x, y)/\sqrt{y}$) is at most $k/12$ (respectively $k/12 - 1/2$). This proves the lemma. $\square$

The lemma above enables us to find the explicit forms of $f_k(x, y)$. If we choose suitable constants c_k in (2.5), we have

$$\begin{cases} f_{12}(x, y) = 1 - x/2 + y/5, \\ f_{18}(x, y) = \sqrt{y}(1 - x/2 - y/5), \\ f_{24}(x, y) = 1 - x + y/2 + 3x^2/32 - xy/4, \\ f_{30}(x, y) = \sqrt{y}(1 - x - y/13 + x^2/104 + xy/26). \end{cases} \quad (2.7)$$

3. Schwarz maps and several rational maps

Definition 3.1. (Schwarz map) Let $P_0 = (p_0, p_0) \in X - \text{Sing}(E)$ be as in the beginning of Section 2 and V a sufficiently small open neighborhood of P_0 . Then we have a holomorphic map

$$u(x, y) := [u_1(x, y) : u_2(x, y) : u_3(x, y) : u_4(x, y)]$$

of V to $\mathbf{P}^3$. By analytic continuations of u along curves in $X - \text{Sing}(E)$, we have a multi-valued map $s_E : X - \text{Sing}(E) \longrightarrow \mathbf{P}^3$, which we call a *Schwarz map* of E .

We denote by S the closure of the image of the Schwarz map s_E . Since the monodromy group G is finite, S is an irreducible algebraic hypersurface of $\mathbf{P}^3$.

Let $\{s_E(x, y)\}$ denote the set of distinct values of s_E at (x, y) . Of course we have $\sharp\{s_E(x, y)\} \leq \sharp P(G) = 25\,920$.

LEMMA 3.1. *The set of points $(x, y) \in X - \text{Sing}(E)$ such that $\sharp\{s_E(x, y)\} = 25\,920$ is an open dense subset of $X - \text{Sing}(E)$.*

Proof. Let X' denote the set in question. It is obvious that X' is open. Assume that X' is not dense. Then there exists an open set $X'' \subset X - \text{Sing}(E) - X'$, a non-scalar matrix M in G and an eigenvalue c of M such that $M \cdot {}^t(u_1, u_2, u_3, u_4) = c \cdot {}^t(u_1, u_2, u_3, u_4)$ in X'' . Hence u_j , $1 \leq j \leq 4$, are linearly dependent in X'' . This contradiction proves the lemma. $\square$

We define a $P(G)$ -invariant rational map ϖ_G of S into $\mathbf{P}^3$ by

$$\varpi_G([u_1 : u_2 : u_3 : u_4]) = [\lambda(u) : \mu(u) : \nu(u) : 1], \quad (3.1)$$

$$\lambda(u) = \frac{I_{24}(u)}{I_{12}(u)^2}, \quad \mu(u) = \frac{I_{18}(u)^2}{I_{12}(u)^3}, \quad \nu(u) = \frac{I_{30}(u)}{I_{12}(u)I_{18}(u)}. \quad (3.2)$$

The single-valued map

$$(\varpi_G \circ s_E)(x, y) = \varpi_G(u(x, y)) = [\lambda(x, y) : \mu(x, y) : \nu(x, y) : 1]$$

of $X - \text{Sing}(E) - \{(\frac{8}{9}, \frac{25}{9}), (-8, 25)\}$ into $\mathbf{P}^3$ can be expressed by

$$\begin{cases} \lambda(x, y) = \frac{1 - x + y/2 + 3x^2/32 - xy/4}{(1 - x/2 + y/5)^2}, \\ \mu(x, y) = \frac{y(1 - x/2 - y/5)^2}{(1 - x/2 + y/5)^3}, \\ \nu(x, y) = \frac{1 - x - y/13 + x^2/104 + xy/26}{(1 - x/2 + y/5)(1 - x/2 - y/5)}, \end{cases} \quad (3.3)$$

from (2.7). Eliminating x and y from (3.3), we obtain the defining function $s_4(\lambda, \mu, \nu)$ of $\varpi_G(S)$ as

$$\begin{aligned} s_4(\lambda, \mu, \nu) = & \mu\nu^3 - \frac{3}{26}\mu\nu^2 + \frac{1}{13^2}\left(-1000\lambda^2 - 720\lambda\mu + 1550\lambda + \frac{2043}{4}\mu - \frac{4805}{8}\right)\nu \\ & + \frac{1}{13^3}\left(20\,000\lambda^3 - 38\,000\lambda^2 + 3240\lambda\mu + \frac{45\,725}{2}\lambda + \frac{3456}{5}\mu^2\right. \\ & \left. - \frac{20\,921}{8}\mu - \frac{33\,635}{8}\right). \end{aligned} \quad (3.4)$$

Put

$$S_4 = \varpi_G(S);$$

this is an irreducible quartic surface in $\mathbf{P}^3$. The defining function $s_4(\lambda, \mu, \nu)$ of S_4 will be obtained in Lemma 3.4 more simply. Let us define a polynomial $I_{90}(u)$ in u_j , $1 \leq j \leq 4$, of degree 90 by

$$I_{90}(u) = s_4\left(\frac{I_{24}(u)}{I_{12}(u)^2}, \frac{I_{18}(u)^2}{I_{12}(u)^3}, \frac{I_{30}(u)}{I_{12}(u)I_{18}(u)}\right)I_{12}(u)^6I_{18}(u), \quad (3.5)$$

where $I_k(u) = I_k(u_1, u_2, u_3, u_4)$. By the definition of I_{90} , we know that

$$S \subset \{u \in \mathbf{P}^3 \mid I_{90}(u) = 0\}. \quad (3.6)$$

We will show that $I_{90}(u)$ is in fact a defining function of S in Corollary 3.6.

LEMMA 3.2. *The Schwarz map s_E is invariant under the involution ι at (1.2), that is, we have $\{s_E(\iota(x, y))\} = \{s_E(x, y)\}$.*

Proof. For small positive numbers x and y , choose a curve γ starting at (x, y) and ending at $\iota(x, y)$ such that $\arg(-x/(1-x)) = 3\pi$ and $\arg(y/(1-x)) = 0$. Then, from the well-known formula (see, for example, [Bly, p. 78])

$$F_2(a; b, b'; c, c'; \iota(x, y)) = (1-x)^a F_2(a; c-b, b'; c, c'; x, y), \quad (3.7)$$

we have $(\gamma_*\varphi_j)(\iota(x, y)) = (1-x)^a \varphi_j(x, y)$, $1 \leq j \leq 4$, where φ_j are the solutions of E appeared at the beginning of Section 2. This proves the lemma. $\square$

For any single-valued function $f(x, y)$, $f(x, y) + f(\iota(x, y))$ is invariant under ι . For example,

$$\begin{cases} \xi_1(x, y) := x - x/(1-x) = x^2/(x-1), \\ \eta(x, y) := y + y/(1-x) = (x-2)y/(x-1), \\ \xi(x, y) := (\xi_1 - 4)/(5\xi_1 - 2\eta - 20) = (x-2)/(5x-2y-10) \end{cases} \quad (3.8)$$

are invariant under ι . Let π_ι be an ι -invariant rational map of X to $Y (\simeq \mathbf{P}^2)$ defined by

$$\pi_\iota([x : y : 1]) = [\xi(x, y) : \eta(x, y) : 1]. \quad (3.9)$$

It is easily verified that $X^* \xrightarrow{\pi_\iota} \pi_\iota(X^*)$ is an unbranched double cover, where

$$X^* = X - \text{Sing}(E) - \{(x, y) \mid (x-2)(5x-2y-10) = 0\}. \quad (3.10)$$

LEMMA 3.3. *Assume $c = 2b$. Then there exists a system E'_2 of differential equations on Y of rank four such that a multi-valued function f on Y is a solution of E'_2 if and only if $(1-x)^{-a/2} \cdot (f \circ \pi_\iota)(x, y)$ is a solution of $E_2(a; b, b'; c, c')$.*

Proof. Let $P_0 = (p_0, p_0)$ and V be as in Definition 3.1. Put $V' = V \cup \iota(V)$. By (3.7), the holomorphic function $F(x, y) := (1-x)^{a/2} F_2(a; b, b'; c, c'; x, y)$ in V' is invariant under ι . Hence it defines a holomorphic function $F'(\xi, \eta)$ in $\pi_\iota(V') = \pi_\iota(V)$ such that $F(x, y) = (F' \circ \pi_\iota)(x, y)$ in V' . Let E^* be the system of differential equations on X satisfied by $F(x, y)$. It is of rank four and defined by two differential equations of the form

$$P_j(x, y, \partial_x, \partial_y)v = \left(\sum_{m+n \leq 2} a_{jmn}(x, y) \partial_x^m \partial_y^n \right) v = 0, \quad j = 1, 2, \quad (3.11)$$

where $a_{jmn}(x, y)$ are polynomials in x, y . If the unknown $v(x, y)$ is replaced by $v'(\xi(x, y), \eta(x, y))$, then the equations in (3.11) change into those of the form

$$P'_j(x, y, \partial_\xi, \partial_\eta)v'(\xi(x, y), \eta(x, y)) = 0. \quad (3.12)$$

Taking the sum and difference

$$P'_j(x, y, \partial_\xi, \partial_\eta) \pm P'_j(\iota(x, y), \partial_\xi, \partial_\eta),$$

we have four differential equations of the form

$$\left(\sum b_{jmn}^+(\xi, \eta) \partial_\xi^m \partial_\eta^n \right) v'(\xi, \eta) = 0 \quad \text{and} \quad \left(\sum b_{jmn}^-(\xi, \eta) \partial_\xi^m \partial_\eta^n \right) v'(\xi, \eta) = 0$$

satisfied by $F'(\xi, \eta)$, where $b_{jmn}^\pm(\xi, \eta)$ are polynomials in ξ, η . Let E'_2 be the system of these four differential equations on Y . Then we know that a holomorphic function $f(\xi, \eta)$ in $\iota(V)$ is a solution of E'_2 if and only if $(f \circ \pi_\iota)(x, y)$ is a solution of E^* in V (and $\iota(V)$). This proves the lemma. $\square$

We define two rational maps

$$\phi([\xi : \eta : 1]) = [\lambda(\xi, \eta) : \mu(\xi, \eta) : \nu(\xi, \eta) : 1] : Y \longrightarrow S_4 \quad (3.13)$$

and

$$\psi([\lambda : \mu : \nu : 1]) = [\xi(\lambda, \mu, \nu) : \eta(\lambda, \mu, \nu) : 1] : S_4 \longrightarrow Y \quad (3.14)$$

by the following equations.

$$\lambda(\xi, \eta) = \frac{25\xi(-17\xi\eta - 50\xi + 4\eta + 10)}{8\eta}, \quad (3.15)$$

$$\mu(\xi, \eta) = -5(10\xi - 1)^2(5\xi - 1), \quad (3.16)$$

$$\nu(\xi, \eta) = \frac{25\xi(11\xi\eta - 250\xi - 2\eta + 50)}{26\eta(10\xi - 1)}, \quad (3.17)$$

$$\xi(\lambda, \mu, \nu) = \frac{-2600\lambda\nu + 4100\lambda + 576\mu - 1690\nu^2 + 2665\nu - 2880}{25(960\lambda - 676\nu^2 + 260\nu - 625)}, \quad (3.18)$$

$$\eta(\lambda, \mu, \nu) = \frac{250\xi(\lambda, \mu, \nu)(1 - 5\xi(\lambda, \mu, \nu))}{425\xi(\lambda, \mu, \nu)^2 - 100\xi(\lambda, \mu, \nu) + 8\lambda}. \quad (3.19)$$

LEMMA 3.4. ψ defines a birational map of S_4 to Y satisfying $\psi^{-1} = \phi$ and

$$\psi \circ \varpi_G \circ s_E = \pi_t. \quad (3.20)$$

The defining function $s_4(\lambda, \mu, \nu)$ of S_4 can also be obtained by eliminating ξ and η from (3.15)–(3.17).

Proof. We eliminate x and y from (3.3) and (3.8). Substituting $y = (x - 1)/(x - 2)\eta$ and $x^2 = (x - 1)\xi_1$ to the right-hand sides of (3.3), we obtain (3.15)–(3.17). This implies that ϕ defines a rational map of Y onto S_4 satisfying

$$\phi \circ \pi_t = \varpi_G \circ s_E. \quad (3.21)$$

We next eliminate ξ and η from (3.15)–(3.17) to get (3.18), (3.19) and the defining function $s_4(\lambda, \mu, \nu)$ of S_4 . From (3.15), we have

$$\eta = 250\xi(1 - 5\xi)/(425\xi^2 - 100\xi + 8\lambda). \quad (3.22)$$

Substituting this into (3.17), we have

$$\nu = 5(240\xi^2 - 55\xi + 4\lambda)/13(10\xi - 1). \quad (3.23)$$

Eliminating ξ^2 and ξ^3 from (3.16) and (3.22), we obtain (3.18). Equation (3.19) follows from (3.23). This implies that ϕ is birational and that $\phi = \psi^{-1}$. The equality (3.20) then follows from (3.21).

Substituting (3.18) with (3.23), we obtain $s_4(\lambda, \mu, \nu)$ at (3.4). This completes the proof. $\square$

We put

$$\pi_G = \psi \circ \varpi_G, \quad (3.24)$$

which is a G -invariant rational map of S to Y .

Now we are ready to state the main theorem.

THEOREM 3.5. *Let s_E be the Schwarz map of $E = E_2(-\frac{1}{12}; \frac{1}{6}, \frac{1}{12}; \frac{1}{3}, \frac{1}{2})$ given by Definition 3.1 and S be the closure of its image. Let $Y (\simeq \mathbf{P}^2)$ be the image of π_ι defined by (3.9) and π_G be the G -invariant rational map of S to Y defined above. Then the following hold.*

- (1) *There exists a system E' of differential equations on Y of rank four such that a Schwarz map $s_{E'}$ of E' defined by the ratio of suitable solutions of E' satisfies*

$$s_E = s_{E'} \circ \pi_\iota \quad (3.25)$$

on an open dense subset of X .

- (2) *The rational map π_G is a 25 920:1 map satisfying*

$$\pi_G \circ s_{E'} = \text{id}_Y \quad (3.26)$$

on $Y - \text{Sing}(E')$. Thus we have the commutative diagram (Figure 1).

- (3) *The degree of S is 90.*

Proof. (1) Let P_0 and V be as in Definition 3.1. From Lemma 3.3, there exists a holomorphic solution $u'_j(\xi, \eta)$ of E' in $\pi_\iota(V)$ such that $u_j(x, y) = (1-x)^{1/24}(u'_j \circ \pi_\iota)(x, y)$ holds in V for each $j = 1, 2, 3, 4$. The Schwarz map $s_{E'}$ of E' defined by the ratio $[u'_1 : u'_2 : u'_3 : u'_4]$ then satisfies the equality $s_E = s_{E'} \circ \pi_\iota$ on the open dense subset (3.10) of X , thanks to the invariance of s_E under ι (see Lemma 3.2).

(2) The equalities (3.20), (3.24) and (3.25) imply (3.26).

Next, we prove that $\sharp(\pi_G)^{-1}(y) = 25\,920$ for generic $y \in Y$. We know, from Lemma 3.1 and the assertion (1) proved above, that there is an open dense subset Y' of $Y - \text{Sing}(E')$ such that for each $y \in Y'$, the set $\{s_{E'}(y)\}$ of distinct values of $s_{E'}$ at y contains 25 920 points. Note that

$$s_{E'}(Y') \xrightarrow{\pi_G} Y' \quad (3.27)$$

is a 25 920-sheeted covering map. Assume that π_G is defined (and hence continuous) at $Q_0 \in S$ and that $y_0 := \pi_G(Q_0)$ is in Y' . Since S is a closure of $s_{E'}(Y')$, there exist sequences $\{y_n\} \subset Y'$ and $\{Q_n\} \subset S$ such that $Q_n \in \{s_{E'}(y_n)\}$ and $Q_n \rightarrow Q_0$ as $n \rightarrow \infty$. Then we have $y_n \rightarrow y_0$ by the continuity of π_G at Q_0 . Hence, Q_0 must be in $\{s_{E'}(y_0)\}$ because (3.27) is an unbranched cover. This implies $(\pi_G)^{-1}(y_0) = \{s_{E'}(y_0)\}$, which proves $\sharp(\pi_G)^{-1}(y_0) = 25\,920$.

(3) Assume $\lambda_0\mu_0 \neq 0$. Because I_k , $1 \leq k \leq 4$, are algebraically independent, they have no common zero in $\mathbf{P}^3$. Consequently, $I_{12}(u)I_{18}(u)$ does not vanish on the set

$$S(\lambda_0, \mu_0) := \{u \mid I_{24}(u) - \lambda_0 I_{12}(u)^2 = 0, I_{18}(u)^2 - \mu_0 I_{12}(u)^3 = 0\} \cap S,$$

whence ϖ_G is defined there. Let v_j , $j = 0, 1, 2$, be the solution of $s_4(\lambda_0, \mu_0, v) = 0$. Then we have $S(\lambda_0, \mu_0) = \bigcup_{j=0,1,2} (\varpi_G)^{-1}([\lambda_0 : \mu_0 : v_j : 1])$. From Bézout's theorem, we have $24 \times 36 \times d \geq 3 \times 25\,920$, which implies $d \geq 90$. Since $S \subset \{u \mid I_{90}(u) = 0\}$, we have $d = 90$ and (3) is proved. $\square$

COROLLARY 3.6. *The polynomial $I_{90}(u_1, u_2, u_3, u_4)$ given by (3.5) is irreducible and is a defining function of S .*

Acknowledgement. The author would like to thank the referee for his valuable advice.

REFERENCES

- [Bly] W. N. Bailey. Generalized Hypergeometric Series (Cambridge Tracts in Mathematics and Mathematical No. 32). Cambridge University Press, Cambridge, 1935.
- [Cxt] H. S. M. Coxeter. Regular Complex Polytopes, 2nd edn. Cambridge University Press, Cambridge, 1991.
- [Kt] M. Kato. Appell's hypergeometric systems F_2 with finite irreducible monodromy groups. Kyushu J. Math. **54** (2000), 279–305.
- [Msk] H. Maschke. Aufstellung des vollen Formensystems einer quaternären Gruppe von 51 840 linearen Substitutionen. Math. Ann. **33** (1888), 317–344.
- [ST] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. Canad. J. Math. **6** (1954), 274–304.

Mitsuo Kato
Department of Mathematics
College of Education
University of the Ryukyus
Nishihara-cho, Okinawa 903-0213
Japan
(E-mail: mkato@edu.u-ryukyu.ac.jp)