CAUCHY–SCHWARZ TYPE INEQUALITIES FOR CATEGORIES

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Abstract. We study a Cauchy–Schwarz type inequality for categories with respect to a certain ‘inner product’ for objects. This turns out to be equivalent to showing a positivity of certain matrices called ‘Laplacians’ constructed from categories.

0. Introduction

The purpose of this paper is to study a positivity of certain matrices constructed from categories stressing the case of size 2 where the positivity is equivalent to an inequality of a Cauchy–Schwarz type. These matrices are considered as Laplacians of categories. When there exist associated zeta functions having the determinant expressions via such Laplacians, the spectra (eigenvalues) explain the zeros and poles of zeta functions. A typical example of such is the determinant expression of the Selberg zeta function in terms of the hyperbolic Laplacian of a Riemann surface (see [K1], [KW] and their references). We studied examples of the spectra of Laplacians of categories in a previous paper [KST] from the view point of zeta functions. The positivity of the Laplacian has, therefore, a definitive importance to investigating the nature of locations and the multiplicity of zeros and poles of the zeta function in question if it has a determinant expressions via the Laplacian.

Recall the Cauchy–Schwarz inequality for the Euclidean space $\mathbb{R}^n$:

$$\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \geq \langle \mathbf{x}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{x} \rangle.$$

Here $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product. Note that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ and hence the right-hand side is equal to $\langle \mathbf{x}, \mathbf{y} \rangle^2$. As is well known, this inequality is the most basic one and there are many generalizations (see [HLP]). In the present paper, we introduce a different situation as follows.

Let $\mathcal{C}$ be a category. For objects X, Y of $\mathcal{C}$, we write

$$\langle X, Y \rangle = \# \operatorname{Hom}_{\mathcal{C}}(X, Y).$$

(We use the notation of $[\mathbf{M}]$ for categories.) Obviously, $\langle X, Y \rangle$ is not symmetric in general. We study the following problem:

Does the Cauchy–Schwarz type inequality $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle \langle Y, X \rangle$ hold whenever every quantity appearing in the inequality is finite?

The purpose of this paper is to show that this inequality is valid for the familiar categories $\mathbf{Ab}(= \text{Mod}(\mathbb{Z}))$, $\text{Mod}(\mathbb{F}_q)$, $\mathbf{Set}(= \text{Mod}(\mathbb{F}_1))$ and $\mathbf{PB}$.

1. The category $\mathbf{Ab}$

We now study the category $\mathbf{Ab}$ of abelian groups and prove the following theorem.

THEOREM 1. *For finite abelian groups X and Y , we have the Cauchy–Schwarz type inequality $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle \langle Y, X \rangle$.*

Proof. Let

$$X = \oplus_i \mathbb{Z}/m_i \mathbb{Z}, \quad Y = \oplus_j \mathbb{Z}/n_j \mathbb{Z}$$

be decompositions into cyclic modules. Then it is easy to see that

$$\begin{aligned} \langle X, X \rangle &= \prod_{i_1, i_2} (m_{i_1}, m_{i_2}), \\ \langle Y, Y \rangle &= \prod_{j_1, j_2} (n_{j_1}, n_{j_2}), \\ \langle X, Y \rangle &= \langle Y, X \rangle = \prod_{i, j} (m_i, n_j), \end{aligned}$$

where (m, n) denotes the greatest common divisor of the integers m and n . Hence the inequality to be proved is

$$\prod_{i_1, i_2} (m_{i_1}, m_{i_2}) \prod_{j_1, j_2} (n_{j_1}, n_{j_2}) \geq \prod_{i, j} (m_i, n_j)^2.$$

To prove this inequality, we look at the p -part of it for each prime p , that is, the greatest divisor among the powers of p of (m_i, n_j) , etc. In fact, if we denote by p^{a_i} (respectively p^{b_j}) the p -factor of the integer m_i (respectively n_j), it is hence sufficient to show that the inequality

$$\sum_{i_1, i_2} \min\{a_{i_1}, a_{i_2}\} + \sum_{j_1, j_2} \min\{b_{j_1}, b_{j_2}\} \geq 2 \sum_{i, j} \min\{a_i, b_j\} \quad (*)$$

holds for non-negative integers $a_1, \dots, a_k$ and $b_1, \dots, b_\ell$. It is clear that we may assume $k \geq \ell$ and

$$a_1 \geq a_2 \geq \dots \geq a_k, \quad b_1 \geq b_2 \geq \dots \geq b_\ell.$$

Then, setting $b_j = 0$ for $\ell < j \leq k$, we may assume further that $k = \ell$. Now, the left-hand side of (*) is calculated as

$$\sum_{i_1, i_2} \min\{a_{i_1}, a_{i_2}\} = a_1 + 3a_2 + \dots + (2k - 1)a_k$$

and

$$\sum_{j_1, j_2} \min\{b_{j_1}, b_{j_2}\} = b_1 + 3b_2 + \dots + (2k - 1)b_k.$$

Thus it suffices to show the following inequalities:

$$\begin{aligned} a_1 + b_1 &\geq 2 \min\{a_1, b_1\}, \\ 3(a_2 + b_2) &\geq 2\{\min\{a_2, b_1\} + \min\{a_2, b_2\} + \min\{a_1, b_2\}\}, \\ &\dots\dots\dots \\ (2i - 1)(a_i + b_i) &\geq 2\{\min\{a_i, b_1\} + \dots + \min\{a_i, b_i\} + \dots + \min\{a_1, b_i\}\} \\ &\dots\dots\dots \\ (2k - 1)(a_k + b_k) &\geq 2\{\min\{a_k, b_1\} + \dots + \min\{a_k, b_k\} + \dots + \min\{a_1, b_k\}\}. \end{aligned}$$

These are easily seen since

$$\begin{aligned} 2a_i &\geq 2 \min\{a_i, b_1\}, \\ &\dots\dots\dots \\ 2a_i &\geq 2 \min\{a_i, b_{i-1}\}, \\ a_i + b_i &\geq 2 \min\{a_i, b_i\}, \\ 2b_i &\geq 2 \min\{a_{i-1}, b_i\}, \\ &\dots\dots\dots \\ 2b_i &\geq 2 \min\{a_1, b_i\}. \end{aligned} \quad \square$$

Remark 1. From the above proof it is easy to extend our result for categories $\text{Mod}(A)$ of A -modules for some ring A . For example, the case of the ring of integers A of an algebraic number field is a quite similar one to $\mathbb{Z}$ in the aforementioned theorem in view of the Dedekindness. Actually we have the following theorem.

THEOREM 1'. Let $\mathcal{C} = \text{Mod}(A)$ for $A = \mathcal{O}_F$ where $\mathcal{O}_F$ is the ring of integers of an algebraic number field F . Then we have $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle \langle Y, X \rangle$ for finite A -modules X, Y , where $\langle X, Y \rangle = \# \text{Hom}_{A\text{-modules}}(X, Y)$.

Remark 2. A similar result to the above inequality was studied by Isbell [I] from a different point of view.

Remark 3. If we define the norm $N(X)$ of an object X of a category $\mathcal{C}$ by $N(X) = \langle X, X \rangle$, the cardinality of endomorphisms of X , the zeta function of $\mathcal{C}$ can be defined by the Euler product (see [K2]):

$$\zeta(s, \mathcal{C}) = \prod_P (1 - N(P)^{-s})^{-1},$$

where P runs over the isomorphism classes of all finite simple objects of $\mathcal{C}$. Then, for instance, if $\mathcal{C} = \mathbf{Ab}$ we conclude that $\zeta(s, \mathbf{Ab}) = \zeta(s)$, the Riemann zeta function.

2. Set and $\text{Mod}(\mathbb{F}_q)$

For the category **Set** of sets and the category $\text{Mod}(\mathbb{F}_q)$ of $\mathbb{F}_q$ -modules, we have the following theorem.

THEOREM 2. For **Set** and $\text{Mod}(\mathbb{F}_q)$, the inequality $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle \langle Y, X \rangle$ holds.

Proof. The proofs are almost trivial and are given respectively as follows.

- (1) **Set:** Put $\#X = m$ and $\#Y = n$. Then the inequality is transformed in to the following easy one:

$$m^m n^n \geq n^m m^n.$$

- (2) $\text{Mod}(\mathbb{F}_q)$: Let $X = \mathbb{F}_q^m$ and $Y = \mathbb{F}_q^n$. Then the inequality is

$$q^{m^2} q^{n^2} \geq q^{2mn},$$

and this is obviously true. $\square$

Remark 4. For **Set** = $\text{Mod}(\mathbb{F}_1)$ (see [KW] or [KOW]), the inequality $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle^2$ is not valid in general. Indeed, let $\#X = 5$ and $\#Y = 4$ for instance. Then

$$\langle X, X \rangle \langle Y, Y \rangle = 5^5 4^4 = 800000,$$

while

$$\langle X, Y \rangle^2 = 4^{10} = 1048576.$$

3. **PB** associated to the symmetric group

Let $\mathfrak{S}_n$ be the group of permutations of a set of n elements. Let $\mathfrak{S}_\infty$ be the group of all permutations of the series of positive integers. We study the category **PB** attached to this infinite symmetric group $\mathfrak{S}_\infty$ (see [N] for details).

The category of finite partial bijections **PB** is defined as follows. The objects of the category **PB** are the sets $\emptyset, \{1\}, \{1, 2\}, \dots, \{1, 2, 3, \dots, n\}, \dots$ (of finite length). We denote these sets by $[0], [1], [2], \dots, [n], \dots$, respectively. A morphism $\varphi : [n] \rightarrow [m]$ of this category is a partial bijection, that is, a bijection φ of a (possibly empty) subset $\text{Dom}(\varphi) (\subset [n])$ onto a subset $\text{Im}(\varphi) (\subset [m])$. Denote by $\text{Hom}_{\mathbf{PB}}([n], [m])$ the set of morphisms between indicated objects. It is clear that $\text{Aut}_{\mathbf{PB}}([n], [n]) \simeq \mathfrak{S}_n$ ($\text{Aut}_{\mathbf{PB}}(\mathbb{N}) \simeq \mathfrak{S}_\infty$). Multiplication of morphisms is a composition of partially defined maps described naturally. Namely, for two given morphisms $\varphi : [n] \rightarrow [m]$ and $\rho : [m] \rightarrow [\ell]$, the composition $\rho \circ \varphi : [n] \rightarrow [\ell]$ of the partial bijections is defined to be that we have $\text{Dom}(\rho \circ \varphi) = \text{Dom}(\varphi)$ if $\text{Im}(\varphi) \subset \text{Dom}(\rho)$, otherwise we have $\text{Dom}(\rho \circ \varphi) = \varphi^{-1}(\text{Dom}(\rho))$.

We then have the following Cauchy–Schwarz type inequality for the category $\mathcal{C} = \mathbf{PB}$.

THEOREM 3. *The inequality $\langle X, X \rangle \langle Y, Y \rangle \geq \langle X, Y \rangle \langle Y, X \rangle$ holds for all objects X and Y of **PB**.*

Proof. Put

$$d_{ij} = \langle [i], [j] \rangle = \# \text{Hom}_{\mathbf{PB}}([i], [j]).$$

It is easy to see that

$$d_{ij} = \sum_{k=0}^{\min\{i,j\}} \binom{i}{k} \binom{j}{k} k!.$$

It is enough to show that $d_{ii}d_{jj} \geq d_{ij}d_{ji}$. Obviously d_{ij} is symmetric. We may assume $i \geq j$. We define $\binom{j}{k} = 0$ for $j < k$. Now put $a_k^{(i)} = \binom{i}{k} \sqrt{k!}$. Then, since

$$d_{ii} = \sum_{k=0}^i (a_k^{(i)})^2, \quad d_{jj} = \sum_{k=0}^j (a_k^{(j)})^2,$$

the ordinary Cauchy–Schwarz inequality asserts that

$$d_{ii}d_{jj} \geq \left(\sum_{k=0}^i a_k^{(i)} a_k^{(j)} \right)^2 = \left(\sum_{k=0}^j a_k^{(i)} a_k^{(j)} \right)^2 = d_{ij}^2.$$

This proves the theorem. $\square$

4. Spectra

For objects X, Y of a category $\mathcal{C}$, form the matrix

$$\Delta_{\mathcal{C}}(X, Y) = \begin{pmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle Y, X \rangle & \langle Y, Y \rangle \end{pmatrix}.$$

The Cauchy–Schwarz type inequality is obviously equivalent to $\det \Delta_{\mathcal{C}}(X, Y) \geq 0$.

Since the characteristic polynomial of $\Delta_{\mathcal{C}}(X, Y)$ is

$$t^2 - \{\langle X, X \rangle + \langle Y, Y \rangle\}t + \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle \langle Y, X \rangle,$$

the discriminant is $\{\langle X, X \rangle - \langle Y, Y \rangle\}^2 + 4\langle X, Y \rangle \langle Y, X \rangle$. This is non-negative since $\langle X, Y \rangle$ is always non-negative. It follows immediately that the eigenvalues of $\Delta_{\mathcal{C}}(X, Y)$ are non-negative real if the Cauchy–Schwarz type inequality holds. Hence the theorems we have obtained in the preceding sections show that the eigenvalues of $\Delta_{\mathcal{C}}(X, Y)$ are non-negative real numbers for (finite) objects X, Y of a category $\mathcal{C} = \mathbf{Ab}, \mathbf{Mod}(\mathbb{F}_q), \mathbf{Set}$ and $\mathbf{PB}$.

More generally, the same problem should be studied for

$$\Delta_{\mathcal{C}}(X_{\lambda}(\lambda \in \Lambda)) \stackrel{\text{def}}{=} (\langle X_{\lambda}, X_{\mu} \rangle)_{\lambda, \mu \in \Lambda},$$

where $X_{\lambda}(\lambda \in \Lambda)$ is a family of objects of the category $\mathcal{C}$. This matrix is considered as a—not necessarily symmetric—Laplacian (of the higher order). Nevertheless one can conjecture that it is in general positive definite. It is also interesting to see what one can say about relations between the positivity of the matrices above and the analytic properties of the zeta functions for categories. (See [K1, K2, KST].) We hope to treat these problems on another occasion.

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