EXTREMAL BINARY SELF-DUAL CODES OF LENGTHS 42 AND 44 WITH NEW WEIGHT ENUMERATORS

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Abstract. A paper of Conway and Sloane (J. H. Conway and N. J. A. Sloane. A new upper bound on the minimal distance of self-dual codes. IEEE Trans. Inform. Theory **36** (1990), 1319–1333) gives the putative weight enumerators of extremal self-dual binary codes for lengths $n \leq 64$ and $n = 72$. The construction of such codes with putative weight enumerators has been one of the fundamental problems in coding theory. In this paper, we construct extremal self-dual binary $[42, 21, 8]$ and extremal self-dual binary $[44, 22, 8]$ codes whose weight enumerators are not yet known to exist.

1. Introduction

We let V be a vector space over the finite field of two elements. If $C \subseteq V$, then C is called a binary linear code. An element of C is called a codeword.

Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in V^n$. The standard inner product on V^n is defined by

$$u \cdot v = \sum_{j=1}^n u_j v_j \pmod{2}.$$

We define the dual of C by

$$C^\perp = \{u \in V^n \mid u \cdot v = 0 \ \forall v \in C\}.$$

If $C \subset C^\perp$ (respectively $C = C^\perp$) then C is called *self-orthogonal* (respectively *self-dual*).

Two codes are said to be *equivalent* if one is obtained from the other by a permutation of coordinates. If Φ is a mapping that sends C into itself, then Φ is called an *automorphism* of C and the set of all automorphisms of C forms a group under composition called the *automorphism group* of C , which is denoted by $\text{Aut}(C)$ in this paper.

For $u \in V^n$, the *weight* of u (denoted by $\text{wt}(u)$) is the number of non-zero coordinates of u . The minimum weight of a code C is defined by

$$d(C) = \min\{\text{wt}(u) \mid u \in C, u \neq 0\}.$$

A fundamental problem in coding theory is to find a code $C \subset V^n$ with $\dim C = k$ such that $d(C)$ is as large as possible, for given positive integers n and k .

A code C is called *doubly-even* or Type II if for all $u \in C$,

$$\text{wt}(u) \equiv 0 \pmod{4}$$

holds. A code which is not doubly-even is called *singly-even* or Type I code.

THEOREM 1. (Rains [13]) *If C is a Type I code of length n , then*

$$d(C) \leq 4 \left\lceil \frac{n}{24} \right\rceil + 6 \quad \text{if } n \equiv 22 \pmod{24}$$

or

$$d(C) \leq 4 \left\lceil \frac{n}{24} \right\rceil + 4 \quad \text{if } n \not\equiv 22 \pmod{24}.$$

A self-dual code is called *extremal* if it has the highest minimum weight for that length.

Let C be a code and let A_i denote the number of codewords in C of weight i . The *weight enumerator* of C is a polynomial in y (where y is an indeterminate) expressed as

$$W_n(y) = \sum_{i=0}^n A_i y^i.$$

In [6], all the weight enumerators of extremal self-dual codes of lengths $n \leq 64$ and $n = 72$ were computed. In particular, the weight enumerator of code C of length 42 is either of the following:

$$W_{42,1}(y) = 1 + (84 + 8\beta)y^8 + (1449 - 24\beta)y^{10} + \dots,$$

or

$$W_{42,2}(y) = 1 + 164y^8 + 15088y^{10} + \dots.$$

For $W_{42,1}(y)$, β takes on the values $0 \leq \beta \leq 60$ and there exist self-dual codes with $\beta = 0, \dots, 16, 18, 20, 22, 24, 26, 32, 42$ (see [3] and the references therein).

While for a code C of length 44, the weight enumerator is either of the following:

$$W_{44,1}(y) = 1 + (44 + 4\beta)y^8 + (976 - 8\beta)y^{10} + \dots,$$

or

$$W_{44,2}(y) = 1 + (44 + 4\beta)y^8 + (1232 - 8\beta)y^{10} + \dots.$$

For $W_{44,1}(y)$, β takes on the values $10 \leq \beta \leq 122$ and there exist self-dual codes with $\beta = 10, \dots, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 70, 82, 90, 122$, while for $W_{44,2}(y)$, β takes on the values $0 \leq \beta \leq 154$ and there exist self-dual codes with $\beta = 0, \dots, 48, 50, 52, \dots, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154$ (see [3, 7–9]).

It is the purpose of this paper to fill in some of the gaps on the values of β in $W_{42,1}(y)$, $W_{44,1}(y)$ and $W_{44,2}(y)$.

2. Construction of the new codes

The vector in V^n whose entries are all 1 is called the all one vector (denoted by $\mathbf{1}$) and every self-dual code contains $\mathbf{1}$.

Self-dual codes C_1, C_2 of length n are called neighbors to each other if $C_1 \cap C_2$ has dimension $(n/2 - 1)$. Note that this intersection necessarily contains $\mathbf{1}$. Since C_1 and C_2 are self-dual, $\dim C_1 = \dim C_2$. It is known that neighbors are a way of constructing new self-dual codes from known self-dual codes and such results can be found in [1] and [6].

We solved our problem using a computer package called *Magma* and used a modified algorithm which was used in [11]. We will discuss what we did without writing down the *Magma* functions.

A self-dual code C of length n has $2^{n/2-1} - 1$ subcodes containing the all one vector $\mathbf{1}$. As a first step, we compute the automorphism group A of C and classify the neighbors of C up to the action of A . To this end, it suffices to classify subcodes of codimension 1 of C containing $\mathbf{1}$. Moreover, if we regard C as a vector space over V , then the subcodes of codimension 1 of C containing $\mathbf{1}$ are in one-to-one correspondence with the non-zero elements of the dual space of the quotient M of C by the one-dimensional space spanned by $\mathbf{1}$.

This then leads us to construct the action of A on M . To do this, we define a permutation module P for A over V and construct a submodule N of P generated by

the vectors in C . Now we define $M = N/\langle \mathbf{1} \rangle$. As N is defined to be an A -module, M is also an A -module.

Next, we construct the matrix representation G of A in the dual space of M and define X as the set of orbit representatives of G on the set of non-zero vectors of $V^{n/2-1}$.

Let x be an element of X . As we wish to regard x as an element of the dual space of M , we should be able to construct a subcode of codimension 1 of M . We will show how this can be done. First, we define N_0 as a subspace of the vector space of P generated by the vectors in C . We let $M_0 = N_0/\langle \mathbf{1} \rangle$. We also let K_1 be a vector space of degree 1 over V . Now we define the dual of M_0 to be the vector space of homomorphisms of M_0 and K_1 . The subcode M_h of codimension 1 of M can be defined by the kernel of x in the dual of M_0 .

Now M_h is a subcode of codimension 1 of M . We can construct a subcode of codimension 1 (say H) of C by taking the preimage of the natural mapping from C to its quotient by $\mathbf{1}$.

Finally, we can construct the neighbor of C sharing H . We let H_0 be the vector space defined by H and let H_d be vector space defined by the dual of H . We define T to be the sequence of transversals for H_d in H_0 such that the weight of each transversal is equal to 8. We then take the first element of T (say t) and define a subspace C' of H_d generated by the elements of H and t . Now, C' is the neighbor of C sharing H . By doing this procedure to all the elements of X , we can construct all the neighbors of C .

We give some remarks on the algorithm mentioned above. The code C has any of the following parameters: $[42, 21, 6]$, $[42, 21, 8]$, $[44, 22, 6]$ or $[44, 22, 8]$. It is possible to find all codes with putative weight enumerators by iterating this algorithm but it is difficult to find an appropriate code C . We can choose C to have $d(C)$ equal to two or four. However, the probability of constructing the neighbors of these codes that can attain a minimum distance eight is lower than if we choose C to have $d(C)$ equal to six or eight. We mainly chose C to have $d(C) = 6$, since work in this area has never existed before.

3. Results

We ran the algorithm on the $[42, 21, 8]$ and $[44, 22, 8]$ codes found in [14]. We remark that even the results were recursively used thus making our work an exhaustive search. The generator matrix for the $[42, 21, 8]$ code found in [14] is

defined as

$$\begin{pmatrix} 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 0 \dots 0 & 1 \dots 1 \\ S & S & S & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ T & T & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ T & 0 \dots 0 & T & 0 \dots 0 & 0 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & 0 \dots 0 & S & S & S \\ 0 \dots 0 & 0 \dots 0 & 0 \dots 0 & T & T & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & 0 \dots 0 & T & 0 \dots 0 & T \end{pmatrix},$$

where S, T are (3×7) circulant matrices with first rows 1110100, 1001011, respectively, and the generator matrix for the $[44, 22, 8]$ code found in [14] is defined as

$$\begin{pmatrix} 1 \dots 1 & 0 \dots 0 & 1 \dots 1 & 0 \dots 0 \\ 0 \dots 0 & 1 \dots 1 & 0 \dots 0 & 1 \dots 1 \\ M & N & 0 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & 0 \dots 0 & M & N \end{pmatrix},$$

where M, N are (10×11) circulant matrices with first rows $01 \dots 1, 11011100010$, respectively.

$C_{42,1}$ is a $[42, 21, 6]$ code and is a neighbor of the $[42, 21, 8]$ code found in [14]. $C_{42,2}$ is a neighbor of $C_{42,1}$ and is a $[42, 21, 8]$ code. The automorphism groups of these codes are permutation groups acting on a set of cardinality 42. Their orders and generators are presented in Table 1.

$C_{44,j}$ ($j = 1, \dots, 9$) are $[44, 21, 6]$ codes and all of which are neighbors of the $[44, 22, 8]$ code found in [14]. The automorphism groups of these codes are permutation groups acting on a set of cardinality 44. Their orders and generators are presented in Table 2.

A neighbor of C can be constructed from one vector $v \in C$ with the subcode of C consisting of codewords orthogonal to v . Thus, giving one vector is enough since the subcode is automatically obtained from the vector. We present the vectors that define the generator matrices of the codes $C_{42,1}$, $C_{42,2}$ and $C_{44,j}$ ($j = 1, \dots, 9$) in hexadecimal notation (i.e. $\mathbf{0} = (0, 0, 0, 0)$, $\mathbf{1} = (0, 0, 0, 1)$, $\dots$, $\mathbf{F} = (1, 1, 1, 1)$) and note that they are right justified. The vectors are given in Tables 3 and 4.

By using the code $C_{42,1}$ and the defining vector $[2, 0, 0, 0, 0, 0, A, 4, 2, A, 1]$, we found the code C'_1 with the weight enumerator $W_{42,1}$ for a new value $\beta = 17$. Similarly, we found the codes C'_i ($i = 2, 3, 4$) for $\beta = 19, 21, 28$ from $C_{42,1}$, C'_1 ,

TABLE 1. The automorphism groups of $C_{42,1}$ and $C_{42,2}$.

$C_{42,i}$	Order	Generators
$C_{42,1}$	3870720	$(9, 42, 14, 28)(12, 25, 13, 39)(15, 29)(16, 30)(17, 31)(18, 32)(19, 33)(20, 34)(21, 35)(22, 23, 27, 38)$ $(24, 41, 37, 36)(26, 40), (1, 2)(3, 5)(16, 18)(17, 31)(19, 34)(20, 33)(21, 35)(30, 32),$ $(2, 3)(4, 6)(9, 42, 14, 28)(12, 25, 13, 39)(16, 30)(17, 33)(19, 31)(20, 21)(22, 23, 27, 38)(24, 41, 37, 36)$ $(26, 40)(34, 35), (2, 5)(6, 7)(9, 42, 14, 28)(12, 25, 13, 39)(16, 31)(17, 30)(18, 21)$ $(19, 33)(22, 23, 27, 38)(24, 41, 37, 36)(26, 40)(32, 35), (10, 11, 26)(12, 28, 39)(13, 24, 38)(14, 27, 22)$ $(23, 36, 42)(25, 41, 37), (9, 37, 10, 42, 27)(11, 22, 36, 13, 14)(12, 26, 41, 28, 24)(23, 40, 39, 25, 38),$ $(8, 10)(9, 42, 27, 37)(12, 25, 41, 38)(13, 28, 36, 24)(14, 39, 22, 23)(26, 40), (10, 39)(11, 41)(12, 27)$ $(14, 25)(22, 26)(23, 36)(28, 37)(40, 42), (11, 26, 40)(12, 22, 36)(13, 38, 24)(14, 39, 25)$ $(23, 41, 28)(27, 37, 42), (1, 35)(2, 21)(3, 31)(5, 17)(16, 20)(18, 33)(19, 30)(32, 34), (1, 19)(2, 34)$ $(3, 33)(5, 20)(9, 25)(12, 42)(13, 28)(14, 39)(16, 17)(18, 31)(21, 32)(22, 23)(24, 36)(27, 38)$ $(30, 35)(37, 41), (1, 18)(2, 16)(3, 30)(5, 32)(9, 25)(12, 42)(13, 28)(14, 39)(17, 34)(19, 31)(20, 21)$ $(22, 23)(24, 36)(27, 38)(33, 35)(37, 41)$
$C_{42,2}$	18432	$(10, 37)(11, 19)(12, 40)(14, 27)(18, 26)(28, 42)(29, 39)(36, 41), (10, 11)(12, 14)(18, 42)(19, 37)$ $(26, 28)(27, 40)(29, 36)(39, 41), (10, 12)(11, 14)(18, 36)(19, 27)(26, 41)(28, 39)(29, 42)(37, 40),$ $(1, 36, 16, 37)(2, 27, 22, 42)(3, 14, 32, 28)(4, 13)(5, 41, 30, 10)(6, 38)(7, 20)(8, 15)(9, 21)$ $(11, 17, 39, 23)(12, 34, 26, 31)(18, 25, 40, 33)(19, 35, 29, 24), (10, 14, 39)(11, 19, 18)(12, 42, 27)$ $(13, 20, 38)(28, 36, 29)(37, 41, 40), (1, 23)(2, 34)(3, 33)(5, 24)(16, 17)(22, 31)(25, 32)(30, 35),$ $(1, 5, 3, 2)(6, 7)(15, 21)(16, 35, 32, 31)(17, 30, 25, 22)(23, 24, 33, 34), (10, 29)(11, 36)(12, 42)$ $(14, 18)(19, 41)(26, 27)(28, 40)(37, 39), (1, 17)(2, 31)(3, 25)(5, 35)(16, 23)(22, 34)(24, 30)(32, 33),$ $(1, 30, 31)(2, 16, 35)(3, 32, 25)(4, 6, 7)(5, 22, 17)(23, 24, 34), (1, 5)(2, 3)(16, 30)(17, 35)(22, 32)$ $(23, 24)(25, 31)(33, 34), (1, 2)(3, 5)(16, 22)(17, 31)(23, 34)(24, 33)(25, 35)(30, 32)$

TABLE 2. The automorphism groups of $C_{44,j}$ ($j = 1, \dots, 9$).

$C_{44,j}$	Order	Generators
$C_{44,1}$	288	$(11, 43, 29)(12, 21, 31)(13, 15, 33)(14, 28, 41)(20, 44, 30)(39, 42, 40), (4, 8)(6, 25)(7, 24)(9, 27)$ $(11, 43)(14, 30)(15, 33)(17, 26)(20, 41)(21, 31)(22, 32)(28, 44)(34, 37)(39, 40), (3, 16)(4, 17)(5, 18)$ $(6, 24)(7, 25)(8, 26)(9, 27)(11, 21)(12, 29)(14, 20)(19, 38)(28, 44)(30, 41)(31, 43), (3, 4)(6, 24)$ $(7, 27)(9, 25)(11, 21)(12, 29)(14, 20)(16, 17)(19, 38)(22, 36)(28, 44)(30, 41)(31, 43)(34, 35),$ $(2, 23)(3, 16)(4, 17)(5, 18)(8, 26)(22, 34)(32, 37)(35, 36), (3, 8, 4)(6, 7, 9)(16, 26, 17)(22, 36, 32)$ $(24, 25, 27)(34, 35, 37), (1, 10)(2, 23)(6, 24)(7, 25)(9, 27)(13, 42)(14, 20)(15, 40)(22, 34)(28, 44)$ $(30, 41)(32, 37)(33, 39)(35, 36)$
$C_{44,2}$	768	$(3, 22)(4, 25)(5, 37)(6, 17)(7, 8)(16, 32)(18, 34)(24, 26), (11, 12)(13, 15)(19, 38)(21, 29)(28, 41)$ $(30, 44)(31, 43)(40, 42), (3, 18)(4, 8)(5, 16)(6, 24)(7, 25)(17, 26)(22, 34)(32, 37), (3, 17)(4, 16)$ $(5, 8)(6, 24)(10, 39)(11, 44)(12, 30)(13, 15)(18, 26)(21, 28)(22, 34)(29, 41)(31, 43)(35, 36),$ $(3, 17, 32)(4, 16, 37)(5, 6, 18)(8, 26, 24)(22, 25, 34)(27, 36, 35), (2, 23)(3, 8, 4, 18)(5, 24, 17, 32)$ $(6, 16, 37, 26)(7, 25, 34, 22)(27, 35), (1, 33)(2, 23)(3, 4)(5, 26)(6, 24)(8, 18)(11, 28)(12, 41)(16, 17)$ $(21, 44)(29, 30)(31, 43)(32, 37)(40, 42), (3, 8)(4, 18)(5, 26)(6, 32)(7, 22)(16, 17)(24, 37)(25, 34),$ $(3, 5)(4, 17)(6, 25)(7, 24)(8, 26)(16, 18)(22, 37)(32, 34)$
$C_{44,3}$	384	$(11, 12)(13, 15)(19, 38)(21, 29)(28, 41)(30, 44)(31, 43)(40, 42), (4, 8, 18)(6, 25, 35)(7, 36, 24)$ $(9, 37, 22)(16, 26, 17)(27, 32, 34), (5, 6)(7, 9)(16, 37)(17, 32)(22, 27)(24, 26)(25, 35)(34, 36),$ $(5, 37)(6, 16)(7, 27)(9, 22)(17, 24)(25, 36)(26, 32)(34, 35), (1, 33)(10, 39)(11, 21)(12, 29)(13, 15)$ $(28, 44)(30, 41)(40, 42), (1, 10)(5, 35, 6, 25)(7, 16, 9, 37)(8, 18)(13, 42)(14, 20)(15, 40)$ $(17, 36, 32, 34)(22, 26, 27, 24)(28, 44)(30, 41)(33, 39), (5, 25)(6, 35)(7, 17)(9, 32)(16, 34)(22, 26)$ $(24, 27)(36, 37), (5, 24)(6, 26)(7, 36)(9, 34)(16, 32)(17, 37)(22, 35)(25, 27)$

TABLE 2. Continued.

$C_{44,j}$	Order	Generators
$C_{44,4}$	1152	$(5, 7)(6, 9)(16, 22)(17, 25)(24, 36)(26, 34)(27, 37)(32, 35), (2, 23)(4, 8)(5, 34, 27, 32)(6, 26, 22, 35)$ $(7, 36, 37, 17)(9, 24, 16, 25), (5, 27)(6, 22)(7, 37)(9, 16)(17, 36)(24, 25)(26, 35)(32, 34),$ $(5, 25)(6, 35)(7, 17)(9, 32)(16, 34)(22, 26)(24, 27)(36, 37), (10, 14)(13, 15)(19, 38)(20, 39)(21, 30)$ $(29, 44)(31, 42)(40, 43), (5, 9)(6, 7)(16, 27)(17, 35)(22, 37)(24, 34)(25, 32)(26, 36),$ $(1, 12, 11)(10, 42, 40)(13, 44, 21)(14, 31, 43)(15, 29, 30)(28, 41, 33), (1, 11)(10, 43)(13, 21)(14, 40)$ $(15, 30)(20, 39)(31, 42)(33, 41), (3, 4, 8)(5, 7, 9)(16, 25, 36)(17, 34, 27)(22, 35, 26)(24, 37, 32)$
$C_{44,5}$	69120	$(3, 32)(4, 37)(6, 18)(7, 9)(8, 24)(22, 35)(25, 36)(27, 34), (10, 23)(11, 12)(19, 38)(20, 39)(21, 43)$ $(29, 31)(30, 40)(42, 44), (5, 36)(6, 34)(7, 24)(9, 26)(16, 35)(17, 27)(22, 32)(25, 37), (5, 35)(6, 25)$ $(7, 32)(9, 17)(16, 36)(22, 24)(26, 27)(34, 37), (4, 18, 8)(5, 24, 7)(6, 22, 17)(9, 35, 27)$ $(16, 37, 34)(25, 26, 32), (1, 28)(3, 26)(4, 17)(5, 8)(6, 32)(9, 36)(10, 14)(12, 31)(15, 44)(16, 18)$ $(20, 40)(23, 41)(25, 34)(33, 42), (11, 29, 43)(12, 31, 21)(13, 33, 15)(14, 41, 28)$ $(20, 30, 44)(39, 40, 42), (5, 37)(6, 16)(7, 27)(9, 22)(17, 24)(25, 36)(26, 32)(34, 35), (5, 24)(6, 26)$ $(7, 36)(9, 34)(16, 32)(17, 37)(22, 35)(25, 27), (10, 23)(11, 29)(12, 31)(14, 41)(15, 33)(20, 40)$ $(30, 39)(42, 44), (1, 14)(3, 26)(4, 17)(5, 8)(6, 32)(9, 36)(10, 28)(13, 39)(15, 20)(16, 18)(23, 41)$ $(25, 34)(29, 43)(40, 44)$
$C_{44,6}$	1152	$(3, 8)(5, 32, 22, 36)(6, 34, 27, 17)(7, 9, 16, 37)(11, 43)(14, 30)(15, 33)(20, 41)(21, 31)(24, 25, 35, 26)$ $(28, 44)(39, 40), (5, 22)(6, 27)(7, 16)(9, 37)(17, 34)(24, 35)(25, 26)(32, 36), (5, 37)(6, 16)(7, 27)$ $(9, 22)(17, 24)(25, 36)(26, 32)(34, 35), (3, 4)(5, 9, 32, 25)(6, 34, 17, 27)(7, 16, 35, 24)(11, 43)(14, 30)$ $(15, 33)(20, 41)(21, 31)(22, 26, 36, 37)(28, 44)(39, 40), (11, 12)(13, 15)(19, 38)(21, 29)(28, 41)$ $(30, 44)(31, 43)(40, 42), (1, 10)(11, 43)(13, 42)(14, 41)(15, 39)(20, 30)(21, 31)(33, 40),$ $(11, 43, 29)(12, 21, 31)(13, 15, 33)(14, 28, 41)(20, 44, 30)(39, 42, 40), (4, 8)(5, 24, 36, 7)$ $(6, 27, 34, 17)(9, 25, 26, 37)(11, 43)(14, 30)(15, 33)(16, 32, 35, 22)(20, 41)(21, 31)(28, 44)(39, 40),$ $(5, 26)(6, 24)(7, 34)(9, 36)(16, 17)(22, 25)(27, 35)(32, 37), (5, 34)(6, 36)(7, 26)(9, 24)(16, 25)$ $(17, 22)(27, 32)(35, 37)$

TABLE 2. Continued.

$C_{44,j}$	Order	Generators
$C_{44,7}$	103680	$(5, 7)(6, 9)(16, 22)(17, 25)(24, 36)(26, 34)(27, 37)(32, 35), (5, 22)(6, 27)(7, 16)(9, 37)(17, 34)$ $(24, 35)(25, 26)(32, 36), (5, 37)(6, 16)(7, 27)(9, 22)(17, 24)(25, 36)(26, 32)(34, 35),$ $(11, 12)(13, 15)(19, 38)(21, 29)(28, 41)(30, 44)(31, 43)(40, 42), (5, 32)(6, 17)(7, 35)(9, 25)(16, 24)$ $(22, 36)(26, 37)(27, 34), (1, 12)(14, 15)(19, 21)(20, 43)(28, 41)(29, 42)(30, 39)(38, 40),$ $(4, 8, 18)(5, 16, 17)(6, 34, 9)(7, 35, 37)(22, 27, 24)(25, 36, 32), (1, 42, 40)(10, 28, 41)(11, 21, 19)$ $(12, 38, 29)(13, 15, 14)(30, 44, 39), (2, 33)(5, 6, 22, 27)(7, 9, 16, 37)(8, 18)$ $(17, 24, 34, 35)(25, 36, 26, 32), (2, 23)(5, 9, 37, 22)(6, 34, 16, 35)(7, 17, 27, 24)(8, 18)(25, 26, 36, 32),$ $(3, 32, 18, 24)(4, 6, 8, 37)(5, 35, 26, 9)(7, 22)(16, 27, 17, 36)(25, 34), (5, 24, 9, 34)(6, 27, 7, 16)(8, 18)$ $(17, 37, 35, 22)(23, 33)(25, 26, 32, 36), (1, 11)(13, 14)(19, 42)(20, 31)(21, 40)(28, 41)(29, 38)(39, 44)$
$C_{44,8}$	55296	$(10, 40)(11, 21)(12, 44)(14, 31)(20, 43)(28, 29)(30, 41)(39, 42), (10, 30)(11, 42)(12, 43)(14, 29)$ $(20, 44)(21, 39)(28, 31)(40, 41), (2, 33, 23)(3, 37, 22)(4, 16, 8)(6, 18, 7)(17, 32, 24)(25, 34, 26),$ $(3, 4)(5, 17)(6, 37)(7, 34)(8, 18)(16, 26)(22, 25)(24, 32), (3, 25)(4, 22)(5, 6)(7, 18)(8, 34)(16, 24)$ $(17, 37)(26, 32), (3, 8)(4, 18)(5, 26)(6, 32)(7, 22)(16, 17)(24, 37)(25, 34), (3, 5, 37)(4, 26, 32)$ $(6, 8, 16)(7, 34, 25)(17, 24, 18)(27, 36, 35), (10, 42)(11, 30)(12, 29)(14, 43)(20, 31)(21, 41)$ $(28, 44)(39, 40), (1, 15, 13)(11, 14, 12)(20, 29, 21)(28, 30, 31)(39, 42, 40)(41, 43, 44),$ $(3, 16, 24, 25)(4, 37, 32, 8)(5, 18, 7, 6)(17, 22, 34, 26)(23, 33)(27, 35), (1, 13)(11, 14)(19, 38)(20, 21)$ $(28, 44)(30, 43)(31, 41)(39, 40), (10, 29)(11, 43)(12, 42)(14, 30)(20, 21)(28, 40)(31, 41)(39, 44),$ $(2, 23)(3, 26, 18, 17)(4, 16, 8, 5)(6, 25, 24, 7)(10, 12, 31, 21)(11, 42, 44, 20)(13, 15)(14, 41, 40, 29)$ $(22, 32, 34, 37)(28, 43, 30, 39), (3, 26)(4, 16)(5, 8)(6, 34)(7, 37)(17, 18)(22, 24)(25, 32)$

TABLE 2. Continued.

$C_{44,j}$	Order	Generators
$C_{44,9}$	27648	$(10, 43)(13, 21)(14, 40)(15, 30)(19, 44)(20, 31)(29, 38)(39, 42),$ $(24, 35)(25, 26)(32, 36),$ $(1, 12, 11)(10, 40, 19)(13, 43, 39)(14, 21, 20)(15, 30, 38)(31, 42, 44),$ $(5, 17)(6, 32)(7, 25)(9, 35)(16, 26)(22, 34)(24, 37)(27, 36),$ $(5, 27)(6, 22)(7, 37)(9, 16)(17, 36)$ $(24, 25)(26, 35)(32, 34),$ $(4, 8)(5, 25)(7, 16)(9, 36)(17, 34)(22, 26)(23, 33)(32, 37),$ $(1, 11)(10, 44, 30, 39)(13, 29, 40, 20)(14, 31, 21, 38)(15, 42, 43, 19)(28, 41),$ $(2, 33)(4, 8)(5, 24)$ $(6, 16)(9, 35)(17, 37)(22, 34)(26, 32),$ $(10, 14)(13, 15)(19, 38)(20, 39)(21, 30)(29, 44)$ $(31, 42)(40, 43),$ $(3, 4, 8)(6, 9, 7)(16, 17, 26)(22, 32, 36)(24, 27, 25)(34, 37, 35),$ $(2, 23)(4, 8)(6, 7)$ $(17, 26)(22, 37)(24, 25)(32, 34)(35, 36),$ $(5, 16)(6, 37)(7, 22)(9, 27)(17, 26)(24, 32)(25, 34)(35, 36),$ $(10, 39)(13, 38)(14, 20)(15, 19)(21, 29)(30, 44)(31, 40)(42, 43),$ $(10, 30)(13, 40)(14, 21)(15, 43)$ $(19, 42)(20, 29)(31, 38)(39, 44)$

TABLE 3. The vectors that define $C_{42,1}$ and $C_{42,2}$.

$C_{42,i}$	$d(C_{42,i})$	Defining vector
$C_{42,1}$	6	[2, 0, 0, 0, 0, 1, 1, 9, A, 6, 4]
$C_{42,2}$	8	[2, 0, 0, 0, 0, 0, D, 1, 9, 2, 6]

TABLE 4. The vectors that define $C_{44,j}$ ($j = 1, \dots, 9$).

$C_{44,j}$	$d(C_{44,j})$	Defining vector
$C_{44,1}$	6	[8, 0, 0, 0, 0, 2, 0, 0, E, C, B]
$C_{44,2}$		[8, 0, 0, 0, 0, 3, 9, 2, 5, A, A]
$C_{44,3}$		[8, 0, 0, 0, 0, 2, D, F, 1, 5, 3]
$C_{44,4}$		[8, 0, 0, 0, 0, 2, 0, 0, 8, 7, F]
$C_{44,5}$		[8, 0, 0, 0, 0, 3, 4, C, D, 3, 2]
$C_{44,6}$		[8, 0, 0, 0, 0, 3, B, C, 6, C, 5]
$C_{44,7}$		[8, 0, 0, 0, 0, 3, A, E, 6, 9, 0]
$C_{44,8}$		[8, 0, 0, 0, 0, 2, E, 8, B, E, C]
$C_{44,9}$		[8, 0, 0, 0, 0, 2, 3, 0, 3, 5, 9]

TABLE 5. New extremal binary codes with weight enumerator $W_{42,1}(y)$.

C'	β	Defining vector	Neighbor of
C'_1	17	[2, 0, 0, 0, 0, 0, A, 4, 2, A, 1]	$C_{42,1}$
C'_2	19	[2, 0, 0, 0, 0, 0, D, A, 3, 1, 7]	$C_{42,1}$
C'_3	21	[2, 0, 0, 0, 0, 0, A, 4, 2, A, 1]	C'_1
C'_4	28	[2, 0, 0, 0, 0, 1, 6, D, F, C, D]	$C_{42,2}$

$C_{42,2}$, respectively. Examples of C'_i ($i = 1, \dots, 4$) are listed in Table 5. Thus we have the following.

PROPOSITION 1. *There exist extremal self-dual $[42, 21, 8]$ codes with weight enumerator $W_{42,1}(y)$ for $\beta = 17, 19, 21, 28$.*

By using $C_{44,1}$ and the defining vector [8, 0, 0, 0, 0, 2, E, F, 0, A, 4], we found the code C''_1 with the weight enumerator $W_{44,1}$ for a new value $\beta = 49$. Similarly, we

TABLE 6. New extremal binary codes with weight enumerator $W_{44,1}(y)$.

C''	β	Defining vector	Neighbor of
C''_1	49	$[8, 0, 0, 0, 0, 2, E, F, 0, A, 4]$	$C_{44,1}$
C''_2	51	$[8, 0, 0, 0, 0, 3, 9, 2, 5, A, A]$	$C_{44,2}$
C''_3	53	$[8, 0, 0, 0, 0, 3, 5, 2, 3, A, A]$	$C_{44,3}$
C''_4	55	$[8, 0, 0, 0, 0, 2, 4, 3, 2, 6, 9]$	$C_{44,4}$
C''_5	57	$[8, 0, 0, 0, 0, 3, B, 6, 8, 4, 0]$	$C_{44,5}$
C''_6	59	$[8, 0, 0, 0, 0, 2, E, E, 5, 2, 4]$	$C_{44,1}$
C''_7	63	$[8, 0, 0, 0, 0, 3, 4, C, D, 3, 2]$	$C_{44,6}$
C''_8	65	$[8, 0, 0, 0, 0, 2, E, 2, B, E, 9]$	C''_{11}
C''_9	67	$[8, 0, 0, 0, 0, 3, 3, 8, 4, 9, B]$	$C_{44,7}$
C''_{10}	68	$[8, 0, 0, 0, 0, 2, 1, 8, 6, A, F]$	C''_5
C''_{11}	72	$[8, 0, 0, 0, 0, 2, E, 2, B, E, 9]$	$C_{44,8}$
C''_{12}	86	$[8, 0, 0, 0, 0, 2, E, 8, B, E, C]$	$C_{44,9}$

TABLE 7. New extremal binary codes with weight enumerator $W_{44,2}(y)$.

C'''	β	Defining vector	Neighbor of
C'''_1	49	$[8, 0, 0, 0, 0, 3, A, 4, D, 2, C]$	$C_{44,9}$
C'''_2	51	$[8, 0, 0, 0, 0, 3, 5, E, 5, 2, 1]$	$C_{44,9}$
C'''_3	59	$[8, 0, 0, 0, 0, 2, 1, 8, 6, E, 4]$	$C_{44,7}$

found the codes C''_i ($i = 2, \dots, 12$) for $\beta = 51, 53, 55, 57, 59, 63, 65, 67, 68, 72, 86$ from $C_{44,2}, C_{44,3}, C_{44,4}, C_{44,5}, C_{44,1}, C_{44,6}, C''_{11}, C_{44,7}, C''_5, C_{44,8}, C_{44,9}$, respectively. Examples of C''_i ($i = 1, \dots, 12$) are listed in Table 6. Thus we have the following.

PROPOSITION 2. *There exist extremal self-dual $[44, 22, 8]$ codes with weight enumerator $W_{44,1}(y)$ for $\beta = 49, 51, 53, 55, 57, 59, 63, 65, 67, 68, 72, 86$.*

By using the code $C_{44,9}$ and the defining vector $[8, 0, 0, 0, 0, 3, A, 4, D, 2, C]$, we found the code C'''_1 with the weight enumerator $W_{44,2}$ for a new value $\beta = 49$. Similarly we found the codes C'''_2, C'''_3 from $C_{44,9}, C_{44,7}$, respectively. Examples of C'''_i ($i = 1, 2, 3$) are listed in Table 7. Thus we have the following.

PROPOSITION 3. *There exist extremal self-dual $[44, 22, 8]$ codes with weight enumerator $W_{44,2}(y)$ for $\beta = 49, 51, 59$.*

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