SASAKIAN MANIFOLDS, HODGE DECOMPOSITION AND MILNOR ALGEBRAS

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Abstract. In this paper we show that Hodge numbers $h^{p,q}$ of a compact Sasakian $(2n + 1)$ -manifold M , $2n + 1 \geq 5$, are given by the dimension of certain linear subspaces of the Milnor algebras $\mathbb{C}[z_i]/(\partial f/\partial z_i)$, when M is a link of an isolated singularity associated with a weighted homogeneous polynomial $f = f(z_1, \dots, z_{n+2})$.

1. Introduction

1.1.

Let $f = f(z)$ be a polynomial in $z = (z_1, \dots, z_{n+2})$ of $\mathbb{C}^{n+2}$, $n \geq 2$. $f = f(z)$ is said to be weighted homogeneous, of degree d and with weight $\mathbf{w} = (w_1, \dots, w_{n+2})$, $w_i \in \mathbb{N}$, $i = 1, \dots, n + 2$, if it is a linear combination of monomials $z_1^{a_1} z_2^{a_2} \cdots z_{n+2}^{a_{n+2}}$ satisfying $w_1 a_1 + \cdots + w_{n+2} a_{n+2} = d$.

The zero locus of $f = f(z)$, $V_f = \{z \in \mathbb{C}^{n+2} \mid f(z) = 0\}$, which is an algebraic variety, admits in general the origin as a singularity.

Assume the origin is an isolated singularity so that we intersect V_f with a small hypersphere S_ε^{2n+3} centered at the origin to obtain a compact smooth $(2n+1)$ -manifold

$$K_f = V_f \cap S_\varepsilon^{2n+3},$$

called a link of the singularity. The link K_f is equipped with a Sasakian structure. Furthermore, the $\mathbb{C}^\times$ -action on $\mathbb{C}^{n+2}$, $(t, z) \mapsto t \cdot z = (t^{w_1} z_1, \dots, t^{w_{n+2}} z_{n+2})$, induces an S^1 -action of contact-isometries on K_f . This S^1 -action is almost free, namely, it is free but admits finitely many distinct isotropy subgroups contained in some finite subgroup Γ of S^1 , so that K_f admits a double fibration over a complex V -manifold, a complex hypersurface in the weighted projective space $\mathbb{P}^{n+1}(\mathbf{w})$,

$$\pi : K_f \longrightarrow K_f/\Gamma \longrightarrow K_f/S^1 = (V_f \setminus \{0\})/\mathbb{C}^\times.$$

The purpose of this paper is to give an explicit formula for the Hodge numbers of a compact Sasakian manifold associated with a weighted homogeneous polynomial.

Even the study of weighted homogeneous polynomials is an interesting subject in mathematical physics, as it is associated with Frobenius algebras and Frobenius manifolds, given by the work of Dubrovin [7]; we will concentrate ourselves on investigating a link of the singularity and the Hodge decomposition structure inherent in the link.

Consider the Hodge decomposition theorem for harmonic forms on the link K_f . So, recall briefly the harmonic theory of a compact Sasakian manifold established by Tanaka. In his lecture note [23] he discussed it over a normal strongly pseudo-convex CR manifold. However, a normal strongly pseudo-convex CR structure in CR geometry is equivalent to the notion of Sasakian structure in contact geometry, as we will see in Theorem 2.2.

Let ψ be a harmonic k -form on such a $(2n + 1)$ -manifold M , $k \leq n$. Then it satisfies $i(\xi)\psi = 0$, so it is in $\Gamma(\Lambda^k(E)) = \Gamma(M; \Lambda^k(E))$ for the contact bundle $E = \text{Ker } \eta = \{X \in TM \mid \eta(X) = 0\} \subset TM$.

We split $\Lambda^k(E_{\mathbb{C}})$ as $\Lambda^k(E_{\mathbb{C}}) = \bigoplus_{p+q=k} \Lambda^{p,q}(E)$ for $E_{\mathbb{C}} = E \otimes \mathbb{C}$ and define the so-called contact Dolbeault operator $\bar{\partial} : \Gamma(\Lambda^{p,q}(E)) \longrightarrow \Gamma(\Lambda^{p,q+1}(E))$, introduced in [23], so that we have the Hodge decomposition theorem as follows.

THEOREM 1.1. (Tanaka [23]) *Let $\mathbb{H}^k(M)$ be the space of harmonic k -forms on a compact Sasakian $(2n + 1)$ -manifold. Then, for $k \leq n$,*

$$\mathbb{H}^k(M) = \bigoplus_{p+q=k} \mathbb{H}^{p,q}(M), \quad \overline{\mathbb{H}}^{q,p}(M) = \mathbb{H}^{p,q}(M)$$

in terms of the space $\mathbb{H}^{p,q}(M)$ of $\bar{\partial}$ -harmonic (p, q) -forms on M .

To give the explicit formula of Hodge number $h^{p,q} = \dim_{\mathbb{C}} \mathbb{H}^{p,q}$ for the link K_f of the singularity we take account of the double fibration of K_f over $V_f^* = (V_f \setminus \{0\})/\mathbb{C}^\times$ and the mixed Hodge structure associated with the Hodge decomposition on V_f^* . Then the Gysin sequence for $K_f/\Gamma \longrightarrow V_f^*$ yields that $h^{p,q} = \dim_{\mathbb{C}} \mathbb{H}_0^{p,q}(V_f^*)$, as in Theorem 5.3. Here $\mathbb{H}_0^{p,q}(V_f^*)$ is the space of primitive harmonic (p, q) -forms on V_f^* and is representable from works of Steenbrink in [21], by using the Milnor algebra. Here the Milnor algebra $\mathbb{M}_f$ is the quotient of the polynomial ring $\mathbb{C}[z_1, \dots, z_{n+2}]$:

$$\mathbb{M}_f = \mathbb{C}[z_1, \dots, z_{n+2}] \bigg/ \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+2}} \right)$$

with degree $\deg(z_i) = w_i$, $i = 1, \dots, n+2$, whose dimension as a vector space is $(w_1 - 1) \cdots (w_{n+2} - 1)$.

THEOREM 1.2. *Let K_f be the link of the singularity associated with a weighted homogeneous polynomial $f = f(z)$ in $(n+2)$ -complex variables z . Then, for (p, q) , $p + q = n$,*

$$h^{p,q}(K_f) = \dim_{\mathbb{C}} (\mathbb{M}_f)_{\ell}, \quad \ell = (p+1)d - \sum_i w_i,$$

where $(\mathbb{M}_f)_{\ell}$ denotes the linear subspace of $\mathbb{M}_f$ consisting of degree ℓ elements.

From the theory of Milnor fibration [15] the link K_f is $(n-1)$ -connected so that the only non-trivial Betti numbers are the middle ones, $b_n = b_{n+1}$. Therefore, $h^{p,q}(K_f) = 0$ for $p + q \neq n, n+1$.

Furthermore, the middle Betti numbers have the following arithmetic formula [16, 18]:

$$b_n(K_f) = \sum_{s=0}^{n+2} \sum_I (-1)^{n-s} \frac{q_{i_1} \cdots q_{i_s}}{[u_{i_1}, \dots, u_{i_s}]}.$$

Here, $I = (i_1, \dots, i_s)$ is an s -tuple of indices $1 \leq i_1 < \dots < i_s \leq n+2$ and $[u_{i_1}, \dots, u_{i_s}]$ denotes the least common multiple of integers $u_{i_1}, \dots, u_{i_s}$ (where integers q_i and u_i are defined as $q_i = d/w_i$ and as $q_i = u_i/v_i$ with $(u_i, v_i) = 1$).

Since $b_n = \sum_{p=0}^n h^{p,n-p}$, we have the following corollary through the intermediary of the Betti number b_n .

COROLLARY 1.3. *The summation of certain vector space dimensions*

$$\sum_{p=0}^n \dim_{\mathbb{C}} (\mathbb{M}_f)_{(p+1)d - \sum_i w_i}$$

and the arithmetic formula

$$\sum_{s=0}^{n+2} \sum_I (-1)^{n-s} \frac{q_{i_1} \cdots q_{i_s}}{[u_{i_1}, \dots, u_{i_s}]}$$

coincide.

The arithmetic formula asserts that the Betti numbers are completely written in the weights $\mathbf{w} = (w_1, \dots, w_{n+2})$ and the degree d . Therefore, we pose the following.

Problem. Does $h^{p,q}(K_f)$ depend only on the degree d and the weights, at least when $\dim K_f = 5$?

The motivation of our study is to shed light directly on the Hodge decomposition theorem for a compact Sasakian 5-manifold M and hence on the self-duality on M . The self-duality in four-dimensions, which is a significantly important geometrical object, is also valid in five-dimensional contact geometry as one has

$$\Lambda^2(E) = \Lambda_+^2(E) \oplus \Lambda_-^2(E),$$

which yields from Tachibana's theorem the self-duality in harmonic form level [22],

$$\mathbb{H}^2(M) = \mathbb{H}_+^2(M) \oplus \mathbb{H}_-^2(M).$$

One additional remark is that, similarly to the self-duality for a metric (or conformal structure) in four-dimensions, we can say a contact (or Sasakian) 5-manifold to be self-dual relative to the Weyl curvature tensor [10]. On such a 5-manifold we define a CR twistor space over M , which is a CR analogue of the Penrose twistor space [11, 12]. Contact metric (or Sasakian) manifolds are equipped with a much finer structure, an almost complex structure $\varphi|_E$ on E , so one has the splitting $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$ and hence $\Lambda^2(E_{\mathbb{C}}) = (\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1})(E)$. Tanaka's decomposition theorem asserts that $\mathbb{H}^2(M) \otimes \mathbb{C} = \mathbb{H}^{2,0} \oplus \mathbb{H}^{0,2} \oplus \mathbb{H}^{1,1}$ with $\mathbb{H}_+^2(M) \otimes \mathbb{C} = \mathbb{H}^{2,0} \oplus \mathbb{H}^{0,2}$, $\mathbb{H}_-^2(M) \otimes \mathbb{C} = \mathbb{H}^{1,1}$, so, in general, numerically $b_+(M) = 2h^{2,0}(M)$, $b_-(M) = h^{1,1}(M)$.

When M is a principal circle bundle over a degree d complex hypersurface $V_d^* = \{[z] \mid \sum_i z_i^d = 0\} \subset \mathbb{P}^3(\mathbb{C})$, namely a Boothby–Wang–Hatakeyama fibration over V_d^* , $b_+(M)$ and $b_-(M)$, respectively, coincide with b_+ and b_- of the base space; the latter are written as degree-three polynomials in d , and are given by $b_+ = (d^3 - 6d^2 + 11d - 3)/3$ and $b_- = (2d^3 - 6d^2 + 7d - 3)/3$ (see [14]). Inspired by these formulae, we focus on the class of Sasakian manifolds of dimension $2n+1 \geq 5$ associated with weighted homogeneous polynomials, which is fairly wider than the class associated with homogeneous polynomials.

2. Sasakian structure

2.1.

The link K_f of the isolated singularity admits as an odd-dimensional manifold a Sasakian structure. In this section we will, therefore, give a brief sketch of the basic theory of Sasakian structure.

Let M be a smooth $(2n+1)$ -manifold. A triple (η, ξ, φ) consisting of a 1-form η , a vector field ξ together with a $(1, 1)$ -tensor φ in M is called a *contact structure* when they satisfy $\eta \wedge (d\eta)^n \neq 0$, $\eta(\xi) = 1$, $d\eta(\xi, X) = 0$ and $\varphi^2(X) = -X + \eta(X)\xi$.

A contact structure (η, ξ, φ) together with a metric g is called *contact metric structure* if they satisfy further that $g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \varphi(Y)) = \frac{1}{2} d\eta(X, Y)$.

Definition 2.1. A contact metric structure (η, ξ, φ, g) is *Sasakian* if the almost complex structure J defined on the product $M \times \mathbb{R}$,

$$J\left(X + a\frac{d}{dt}\right) = \varphi(X) - a\xi + \eta(X)\frac{d}{dt}, \quad X \in TM, a \in \mathbb{R}, \quad (1)$$

is integrable.

A Sasakian structure is an odd-dimensional analogue of a Kähler structure. In fact, as in [2], we have the following theorem.

THEOREM 2.1. *A contact metric structure (η, ξ, φ, g) is Sasakian if and only if*

$$(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad (2)$$

where ∇ is the Levi-Civita connection of g .

Note that on a Sasakian manifold $\mathcal{L}_\xi \phi = 0$ and hence ξ is a Killing field with unit norm (see [2, Section 5, IV]).

The *contact bundle* $E = \ker \eta$, a subbundle of TM with the restriction of φ , becomes a complex Hermitian vector bundle with a Hermitian fiber metric $g|_E$, the restriction of g .

Split the complexified contact bundle $E_{\mathbb{C}} = E \otimes \mathbb{C}$ as $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$ with respect to φ . Then we have the following.

Definition 2.2. A contact metric structure (η, ξ, φ, g) is said to be a *strongly pseudo-convex CR structure* when $\Gamma(M; E^{1,0})$ is closed under the bracket operation $[\cdot, \cdot]$. A strongly pseudo-convex CR structure is said to be *normal* when ξ is CR holomorphic, i.e. $[\xi, X] \in \Gamma(M; E^{1,0})$ for any $X \in \Gamma(M; E^{1,0})$. For normal strongly pseudo-convex CR structures refer to [23] and [26].

The following is a folklore theorem.

THEOREM 2.2. *A contact metric structure is Sasakian if and only if it is normal strongly pseudo-convex CR.*

Proof. Assume that the almost complex structure J on $M \times \mathbb{R}$ is integrable. Then

$$[U, V] \in \Gamma(M \times \mathbb{R}; T^{1,0}(M \times \mathbb{R})) \quad (3)$$

for any $U, V \in \Gamma(M \times \mathbb{R}; T^{1,0}(M \times \mathbb{R}))$. Note that

$$T^{1,0}(M \times \mathbb{R}) = E^{1,0} \oplus \mathbb{C} \left(\xi - \sqrt{-1} \frac{d}{dt} \right) \quad (4)$$

and set $U = X + (\xi - \sqrt{-1} d/dt)$, $V = Y + (\xi - \sqrt{-1} d/dt)$, $X, Y \in \Gamma(M; E^{1,0})$. Then, from (3), $[X, Y]$ and $[\xi, X]$ are both in $\Gamma(M; E^{1,0})$ so that the structure is a normal strongly pseudo-convex CR. The converse is easily shown. $\square$

2.2. Almost free S^1 -action

We call a Sasakian structure (η, ξ, ϕ, g) S^1 -Sasakian when the vector field ξ generates an S^1 -action. Since $\mathcal{L}_\xi \eta = 0$ and $\mathcal{L}_\xi \phi = 0$, this S^1 -action consists of isometric contact transformations of M . When this S^1 -action is free, it follows from the classical results of Boothby and Wang [4] and Hatakeyama [9] that the Sasakian structure on M arises from a circle bundle with a connection η over a complex Hodge manifold (M', J, G) , so that $g = \pi^*G + \eta \otimes \eta$, $\phi = \tilde{\pi}_* \circ J \circ \pi_*$ ($\pi : M \rightarrow M'$ is the projection over the space M' of all S^1 -orbits in M and $\tilde{\pi}_*$ is the horizontal lift) and ξ is the vertical vector field induced from the right action. For this argument we need only the fact that every orbit of ξ is a circle and do not need the compactness of M (see the proof of the theorem of Boothby and Wang given in [2] and also in [24]). Note that an S^1 -Sasakian structure with the S^1 -action which is free is obviously regular contact.

On the other hand, we have another notion: almost regular contact metric structure ([25], see also [3, p. 29]). Associated to this notion we give the following definition.

Definition 2.3. The S^1 -action of an S^1 -Sasakian structure is said to be *almost free* if the action has no fixed points but the finitely many distinct isotropy subgroups are contained in some finite subgroup Γ of S^1 .

Thomas obtained the following result [25]. Let (M, η, ξ, ϕ, g) be a compact Sasakian manifold or, more generally, a compact contact manifold. If it is almost regular, then the field ξ generates an almost free S^1 -action on M and *vice versa*.

If the S^1 -action of a Sasakian manifold M is almost free, then M admits a double fibration, first by taking the quotient by the finite subgroup Γ and then by the free action of $S^1/\Gamma \cong S^1$ induced from the original S^1 -action [25]:

$$M \xrightarrow{\pi_1} M/\Gamma \xrightarrow{\pi_2} M/S^1 = (M/\Gamma)/(S^1/\Gamma). \quad (5)$$

The first fibration is a ramified covering and the second is a proper S^1 -fibration, so that we get the projection $\pi : M \rightarrow M/S^1$ as the composition $\pi = \pi_2 \circ \pi_1$.

A link K_f of the singularity associated with a weighted homogeneous polynomial f carries the S^1 -Sasakian structure which is almost free, as explained in Lemma 3.1.

Now assume that $(M^{2n+1}, \eta, \xi, \varphi, g)$ is an S^1 -Sasakian manifold for which the S^1 -action generated by ξ is almost free. The isotropy subgroup Γ_x at $x \in M$ is a finite subgroup of S^1 . We can then apply the isotropy invariant slice lemma (see, for example, [5, II]) to the S^1 -orbit through x so that the quotient space M/S^1 becomes a real V -manifold, i.e. an orbifold of dimension $2n$. In fact, each point $[x] \in M/S^1$ has a neighborhood isomorphic to U/Γ_x , where $U = \{W \in E_x \mid \|W\| < \varepsilon\}$ is an ε -ball in E_x , a slice at x . The tangent space $T_{[x]}M/S^1$ is then identified with E_x , $x \in \pi^{-1}([x])$. For this argument we used the fact that the S^1 -action is contact, so E_x is mapped onto $E_{t \cdot x}$ linearly isomorphically by $t \in S^1$.

We now define a Riemannian metric G on M/S^1 , by making use of the identification $E_x \cong T_{[x]}M/S^1 : X \mapsto \underline{X}$:

$$G_{[x]}(\underline{X}, \underline{Y}) = g_x(X, Y) \quad (6)$$

for $\underline{X}, \underline{Y} \in T_{[x]}M/S^1$, identified with $X, Y \in E_x$. G is well-defined, since the S^1 -action is isometric. Then $\pi : (M, g) \longrightarrow (M/S^1, G)$ turns out to be a Riemannian submersion.

Moreover, we can define on M/S^1 an almost complex structure J just like the free S^1 -action case as follows:

$$J_{[x]}(\underline{X}) = \underline{\varphi(X)}. \quad (7)$$

Therefore, G is J -Hermitian, i.e. $G(J\underline{X}, J\underline{Y}) = G(\underline{X}, \underline{Y})$, and furthermore the Kähler form $\Omega = \Omega_G$ defined by $\Omega(\underline{X}, \underline{Y}) = G(J\underline{X}, \underline{Y})$ is a closed 2-form, since the following is obviously seen:

$$\pi^*\Omega = d\eta. \quad (8)$$

Thus we obtain the following.

THEOREM 2.3. *If the S^1 -action of an S^1 -Sasakian manifold is almost free, then the quotient space M/S^1 admits a real V -manifold structure with a Riemannian metric G such that $\pi : M \longrightarrow M/S^1$ is a Riemannian submersion. Moreover, the space M/S^1 carries an almost complex structure J such that G is J -Hermitian and the Kähler form Ω is closed.*

2.3. The Hodge decomposition

Let $\mathbb{H}^k(M)$ be the space of harmonic k -forms on a compact Sasakian $(2n+1)$ -manifold $(M, \eta, \xi, \varphi, g)$:

$$\mathbb{H}^k(M) = \{\psi \in \Gamma(\Lambda^k(TM^* \otimes \mathbb{C})) \mid d\psi = 0, d^*\psi = 0\}. \quad (9)$$

To obtain the Hodge decomposition of $\mathbb{H}^k(M)$ we follow Sections 12 and 13 of [23] and give an outline sketch of the detailed argument.

Assume $\dim M = 2n + 1 \geq 5$ and consider

$$\Lambda^k(E_{\mathbb{C}}) = \bigoplus_{p+q=k} \Lambda^{p,q}(E). \quad (10)$$

$\Lambda^k(E_{\mathbb{C}})$ and hence $\Lambda^{p,q}(E)$ admits the Hermitian inner product $\langle \cdot, \cdot \rangle_{E_{\mathbb{C}}}$, induced from $g|_E$ of E . We define several operators, as on a Kähler manifold: the star operator $\star_E$, L , the exterior product with the 2-form $-d\eta$, and Λ , the formal adjoint of L :

$$\begin{aligned} \star_E : \Lambda^k(E_{\mathbb{C}}) &\longrightarrow \Lambda^{2n-k}(E_{\mathbb{C}}), & \phi \wedge \overline{\star_E \psi} &= \frac{1}{n!} \langle \phi, \psi \rangle_E (d\eta)^n, \\ L : \Lambda^k(E_{\mathbb{C}}) &\longrightarrow \Lambda^{k+2}(E_{\mathbb{C}}), & L\psi &= (-d\eta) \wedge \psi, \\ \Lambda : \Lambda^k(E_{\mathbb{C}}) &\longrightarrow \Lambda^{k-2}(E_{\mathbb{C}}), & \Lambda\psi &= (-1)^k \star_E (L(\star_E \psi)). \end{aligned} \quad (11)$$

Furthermore, we define

$$N = \sqrt{-1} \mathcal{L}_{\xi} : \Gamma(\Lambda^k(E_{\mathbb{C}})) \longrightarrow \Gamma(\Lambda^k(E_{\mathbb{C}})) \quad (12)$$

with respect to the Lie derivative in the vector field ξ . These are real operators. From the normality of the CR structure we see that $[\xi, \Gamma(E)] \subset \Gamma(E)$ and hence N preserves the (p, q) -bidegree of form ψ in $\Lambda^k(E_{\mathbb{C}})$. N is self-adjoint with respect to the inner product,

$$\langle N(\phi), \psi \rangle_{E_{\mathbb{C}}} = \langle \phi, N(\psi) \rangle_{E_{\mathbb{C}}}, \quad (13)$$

and N commutes with the operators $\star_E$, L and Λ .

THEOREM 2.4. (Tachibana [22]) *Let $k \leq n$. A harmonic k -form ψ on M satisfies:*

- (i) $i(\xi)\psi = 0$, i.e., $\psi \in \Gamma(\Lambda^k(E))$;
- (ii) $\Lambda\psi = 0$.

Since ξ is Killing, $N\psi = \sqrt{-1} \mathcal{L}_{\xi} \psi = 0$ for a harmonic form ψ so it follows that, if $k \leq n$, $\mathbb{H}^k(M)$ is a linear subspace of $\{\psi \in \Gamma(M; \Lambda^k(E)) \mid \Lambda\psi = 0, N\psi = 0\}$.

This theorem partially suggests that we focus on the bundles $\Lambda^k(E_{\mathbb{C}})$, $\Lambda^{p,q}(E)$ and define Dolbeault operators, analogously to complex manifolds.

Consider the exterior derivative $d\psi$ for $\psi \in \Gamma(\Lambda^{p,q}(E))$. Then there exist uniquely $\partial\psi \in \Gamma(\Lambda^{p+1,q}(E))$ and $\bar{\partial}\psi \in \Gamma(\Lambda^{p,q+1}(E))$ such that

$$d\psi \equiv \partial\psi + \bar{\partial}\psi \quad \text{modulo } \eta. \quad (14)$$

The reason for this is that if $\{\theta^1, \dots, \theta^n\}$ is a local frame field of $\Lambda^{1,0}(E)$, then from the integrability of the CR structure each $d\theta^i$ admits no components in $\bar{\theta}^j \wedge \bar{\theta}^k$, i.e.

$$d\theta^i \in \Gamma(\Lambda^{2,0} \oplus \Lambda^{1,1})(E) \quad \text{modulo } \eta. \quad (15)$$

Since any $\psi \in \Gamma(\Lambda^{p,q}(E))$ is spanned by $\theta^{i_1} \wedge \dots \wedge \theta^{i_p} \wedge \bar{\theta}^{j_1} \wedge \dots \wedge \bar{\theta}^{j_q}$, the operators $\partial, \bar{\partial}$ are inductively defined,

$$\partial : \Gamma(\Lambda^{p,q}(E)) \longrightarrow \Gamma(\Lambda^{p+1,q}(E)), \quad (16)$$

$$\bar{\partial} : \Gamma(\Lambda^{p,q}(E)) \longrightarrow \Gamma(\Lambda^{p,q+1}(E)), \quad (17)$$

so that one has $d\psi \equiv \partial\psi + \bar{\partial}\psi$ modulo η . Obviously $\bar{\partial}\bar{\partial}\psi = \bar{\partial}^2\psi$.

In terms of the L^2 -inner product $(\cdot, \cdot)_E$, $(\phi, \psi)_E = \int_M \langle \phi, \psi \rangle_E (\eta \wedge (d\eta)^n)$, in the complex vector space $\Gamma(\Lambda^{p,q}(E))$, the operators $\bar{\partial}, \partial$ have the formal adjoints ϑ and $\bar{\vartheta}$, respectively; therefore $\vartheta = \bar{\partial}^*$ and $\bar{\vartheta} = \partial^*$ so that

$$\vartheta = -\star_E \circ \bar{\partial} \circ \star_E, \quad (18)$$

$$\bar{\vartheta} = -\star_E \circ \partial \circ \star_E. \quad (19)$$

LEMMA 2.5. $\{\Gamma(\Lambda^{p,q}(E)), \partial, \bar{\partial}\}$ constitutes a double complex in a generalized sense, that is:

(i)

$$\partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad \partial\bar{\partial} + \bar{\partial}\partial = -\sqrt{-1} L \circ N; \quad (20)$$

(ii)

$$\vartheta^2 = \bar{\vartheta}^2 = 0 \quad \text{and} \quad \vartheta\bar{\vartheta} + \bar{\vartheta}\vartheta = \sqrt{-1} \Lambda \circ N. \quad (21)$$

The most crucial identities in Kähler geometry, called the Hodge identities, are [8, 17]

$$[\bar{\partial}, \Lambda] = -\sqrt{-1} \bar{\vartheta}, \quad [\partial, \Lambda] = \sqrt{-1} \vartheta \quad (22)$$

or equivalently

$$[\vartheta, L] = -\sqrt{-1} \partial, \quad [\bar{\vartheta}, L] = \sqrt{-1} \bar{\partial} \quad (23)$$

which also hold on a compact Sasakian manifold, as shown in [23].

Now we define the complex Laplacians

$$\square = \vartheta \bar{\partial} + \bar{\partial} \vartheta, \quad \bar{\square} = \bar{\vartheta} \partial + \partial \bar{\vartheta} \quad (24)$$

on $\Gamma(\Lambda^k(E_{\mathbb{C}}))$. Then, the Hodge identities yield the following beautiful relation between complex Laplacians.

LEMMA 2.6. (Tanaka [23])

$$\square \psi = \bar{\square} \psi + (n - k) N \psi, \quad \psi \in \Gamma(\Lambda^k(E_{\mathbb{C}})). \quad (25)$$

Define for $\lambda \in \mathbb{R}$

$$\mathcal{H}_{\bar{\partial}, \lambda}^{p, q} = \{\psi \in \Gamma(\Lambda^{p, q}(E)) \mid \square \psi = 0, N \psi = \lambda \psi\}. \quad (26)$$

The space $\mathcal{H}_{\bar{\partial}, \lambda}^{p, q}$ is closely related, when $\lambda = 0$, to the space $\mathbb{H}^k(M)$. Indeed we follow [23] to define the subspaces

$$\begin{aligned} \mathcal{K}_{\Lambda}^{p, q}(M) &= \{\psi \in \mathcal{H}_{\bar{\partial}, 0}^{p, q}(M) \mid \Lambda \psi = 0\} \\ &= \{\psi \in \Gamma(\Lambda^{p, q}(E)) \mid \square \psi = 0, N \psi = 0, \Lambda \psi = 0\} \end{aligned} \quad (27)$$

and

$$\begin{aligned} \mathcal{K}_L^{p, q}(M) &= \{\psi \in \mathcal{H}_{\bar{\partial}, 0}^{p, q}(M) \mid L \psi = 0\} \\ &= \{\psi \in \Gamma(\Lambda^{p, q}(E)) \mid \square \psi = 0, N \psi = 0, L \psi = 0\}. \end{aligned} \quad (28)$$

Then

$$\mathcal{K}_L^{p, q}(M) = \sharp_E \mathcal{K}_{\Lambda}^{n-p, n-q}(M) \quad (29)$$

holds for the operator $\sharp_E$ defined by $\sharp_E \psi = \overline{\star_E \psi}$ and we have eventually the Hodge decomposition theorem.

THEOREM 2.7. (Tanaka [23]) *Let (M, η, ξ, ϕ, g) be a compact Sasakian $(2n + 1)$ -manifold. Then the space $\mathbb{H}^k(M)$ is decomposed as*

(i)

$$\mathbb{H}^k(M) = \bigoplus_{p+q=k} \mathcal{K}_{\Lambda}^{p, q}(M), \quad k \leq n, \quad (30)$$

(ii)

$$\mathbb{H}^k(M) = \bigoplus_{p+q=k} \eta \wedge \mathcal{K}_L^{p, q}(M), \quad k \geq n + 1. \quad (31)$$

So, set $\mathbb{H}^{p,q}(M) = \mathcal{K}_\Lambda^{p,q}(M)$, when $p + q \leq n$. The above theorem then asserts for $k \leq n$ that $\mathbb{H}^{p,q}(M)$ is the space of harmonic (p, q) -forms on M and the space $\mathbb{H}^k(M)$ admits the Hodge decomposition:

$$\begin{aligned} \mathbb{H}^k(M) &= \bigoplus_{p+q=k} \mathbb{H}^{p,q}(M), \\ \overline{\mathbb{H}}^{p,q}(M) &= \mathbb{H}^{q,p}(M), \quad k \leq n \end{aligned} \tag{32}$$

Remark. For $k \geq n + 1$, set $\mathbb{H}^{p,q}(M) = \eta \wedge \mathcal{K}_L^{p-1,q}(M)$, $p + q = k$. Then one has $\mathbb{H}^k(M) = \bigoplus_{p+q=k} \mathbb{H}^{p,q}(M)$, but with the property $\overline{\mathbb{H}}^{p,q}(M) = \mathbb{H}^{q+1,p-1}(M)$.

3. The S^1 -Sasakian structure of K_f

3.1. $\mathbb{C}^\times$ -action

Let

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^{n+2} &\longrightarrow \mathbb{C}^{n+2}, \\ (t, z) &\mapsto t \cdot z = (t^{w_1} z_1, \dots, t^{w_{n+2}} z_{n+2}) \end{aligned}$$

be the weighted $\mathbb{C}^\times$ -action on $\mathbb{C}^{n+2}$. Then, the weighted homogeneous polynomial f satisfies $f(t \cdot z) = t^d f(z)$ so that the variety V_f is $\mathbb{C}^\times$ -invariant. The link K_f of the singularity is also invariant under the action of $S^1 = \{e^{\sqrt{-1}\theta} \mid \theta \in \mathbb{R}\}$.

Assume that $\text{GCD}\{w_1, \dots, w_{n+2}\} = 1$ which guarantees effectiveness of the $\mathbb{C}^\times$ -action on V_f and of the S^1 -action on K_f .

LEMMA 3.1. (Dimca [6]) *The isotropy subgroup Γ_z , $z \in K_f$, is the multiplicative group of $w(z)$ -roots of unity,*

$$\Gamma_z = \{\zeta \in S^1 \mid \zeta^{w(z)} = 1\}, \tag{33}$$

where $w(z) = \text{GCD}\{w_i \mid z_i \neq 0\}$ for $z = (z_1, \dots, z_{n+2})$.

The proof is simply arithmetic, so we omit it. From this lemma it follows that the S^1 -action on K_f is almost free.

3.2. S^1 -Sasakian structure on K_f

To introduce a Sasakian structure on the link K_f , we define instead, by virtue of Theorem 2.1, a normal strongly pseudo-convex CR structure on K_f .

Let $E^{1,0}$ be the subbundle of $TK_f \otimes \mathbb{C}$ defined by

$$E_z^{1,0} = \{X \in T_z^{1,0}\mathbb{C}^{n+2} \mid Xf = 0, X\zeta = 0\}, \quad z \in K_f \quad (34)$$

and $E^{0,1}$ its complex conjugate; $E^{0,1} = \overline{E}^{1,0}$ (where $\zeta = \sum_i |z_i|^2 - \varepsilon^2$ is a smooth function defining the sphere S_ε^{2n+3}). We see that $\dim_{\mathbb{C}} E^{1,0} = n$.

It is easily seen that $\Gamma(E^{1,0})$ is closed under the bracket operation $[\cdot, \cdot]$. We define the vector field ξ as $\xi_z = d/d\theta(e^{-\sqrt{-1}\theta} \cdot z)|_{\theta=0}$, just as the vector field on $\mathbb{C}^{n+2}$ induced from the S^1 -action. Then

$$\xi = \sqrt{-1} \sum_{i=1}^{n+2} w_i \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right). \quad (35)$$

Since the S^1 -action leaves both V_f and S_ε^{2n+3} invariant, one can see that the restriction of ξ to K_f is a vector field, which we denote by the same letter, and that it satisfies

$$[\xi, \Gamma(E^{1,0})] \subset \Gamma(E^{1,0}). \quad (36)$$

On K_f , $\sum_i (z_i d\bar{z}_i + \bar{z}_i dz_i) = 0$, i.e. $\sum_i \bar{z}_i dz_i = -\sum_i z_i d\bar{z}_i$ holds. So we set a 1-form η as

$$\eta = \sqrt{-1} \left(\sum_j w_j |z_j|^2 \right)^{-1} \sum_i \bar{z}_i dz_i. \quad (37)$$

Then we have $\eta(\xi) = 1$.

Furthermore, the 2-form $d\eta$, which is represented as

$$\begin{aligned} d\eta &= -\frac{\sqrt{-1}}{(\sum_j w_j |z_j|^2)} \sum_i dz_i \wedge d\bar{z}_i \\ &\quad - \frac{\sqrt{-1}}{(\sum_j w_j |z_j|^2)^2} d\left(\sum_j w_j |z_j|^2\right) \wedge \left(\sum_i \bar{z}_i dz_i\right), \end{aligned} \quad (38)$$

reduces to

$$-\frac{\sqrt{-1}}{(\sum_j w_j |z_j|^2)} \sum_i dz_i \wedge d\bar{z}_i, \quad (39)$$

when restricted to $E^{1,0}$, since $(\sum_i \bar{z}_i dz_i)(X) = X\zeta = 0$ for $X \in E^{1,0}$. So, the form $d\eta$ yields a positive definite Hermitian form, called the Levi form on $E^{1,0}$:

$$\mathbb{L}(X, \bar{Y}) = \frac{1}{(\sum_j w_j |z_j|^2)} \sum_{i=1}^{n+2} (Xz_i)(\overline{Yz_i}), \quad X, Y \in E^{1,0}. \quad (40)$$

One can easily see that $d\eta(\xi, \cdot) = 0$.

The direct sum $E^{1,0} \oplus E^{0,1}$ defines a real subbundle E of the real tangent bundle TK_f at z of K_f , such that the complexification of E is

$$E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1},$$

together with an endomorphism φ of E satisfying $\varphi|_{E^{1,0}} = \sqrt{-1} \text{ id}$ and $\varphi|_{E^{0,1}} = -\sqrt{-1} \text{ id}$. One then has, at z

$$\mathbb{L}(X, \bar{Y}) = d\eta(\varphi(X), \bar{Y}), \quad X, Y \in E_z^{1,0}, \quad (41)$$

and one obtains a smooth Riemannian metric g on K_f as $g(\xi, \xi) = 1$ and $g(\xi, X) = 0$ for any X of E_z and $2g(X, Y) = \mathbb{L}(X^{1,0}, \bar{Y}^{1,0}) + \mathbb{L}(Y^{1,0}, \bar{X}^{1,0})$ for real vectors $X = X^{1,0} + \bar{X}^{1,0}$, $Y = Y^{1,0} + \bar{Y}^{1,0}$, so that $g(X, \varphi Y) = \frac{1}{2}d\eta(X, Y)$.

Therefore, it follows from the theorem of Tanaka (see [23, 11.2, III]) that (η, ξ, φ) together with the induced metric g yields on K_f a normal strongly pseudo-convex CR structure.

Therefore, we are able to summarize the above arguments as follows.

THEOREM 3.2. *The link $K_f = V_f \cap S_{\varepsilon}^{2n+3}$ associated with a weighted homogeneous singularity carries an S^1 -action and is equipped with a normal strongly pseudo-convex CR structure, namely, a Sasakian structure (η, ξ, φ, g) . This S^1 -action is almost free and consists of isometric contact transformations of the Sasakian structure.*

Note that Sasaki and Hsu [19] introduced directly a Sasakian structure on K_f for a weighted homogeneous polynomial f of Brieskorn–Pham type.

3.3. The Hodge structure on K_f/S^1

The S^1 -action of K_f is almost free and so we write the projection $\pi : K_f \rightarrow K_f/S^1$ as the composition of the fibrations $\pi_1 : K_f \rightarrow K_f/\Gamma$ and $\pi_2 : K_f/\Gamma \rightarrow K_f/S^1$. Here Γ is a finite subgroup of S^1 containing all the $\Gamma_z, z \in K_f$.

The quotient space K_f/S^1 is described as $(V_f \setminus \{0\})/\mathbb{C}^\times$, a complex hypersurface in $\mathbb{P}^{n+1}(\mathbf{w})$. Here the space $\mathbb{P}^{n+1}(\mathbf{w}) = (\mathbb{C}^{n+2} \setminus \{0\})/\mathbb{C}^\times$ is the weighted projective space of weight $\mathbf{w} = (w_1, \dots, w_{n+2})$, obtained by the weighted $\mathbb{C}^\times$ -action. We write the weighted projective space as the quotient of the ordinary complex projective space,

$$\mathbb{P}^{n+1}(\mathbf{w}) = \mathbb{P}^{n+1}(\mathbb{C})/\Gamma(\mathbf{w}), \quad (42)$$

from which $\mathbb{P}^{n+1}(\mathbf{w})$ is a complex V -manifold. Here $\Gamma(\mathbf{w}) = \Gamma(w_1) \times \dots \times \Gamma(w_{n+2})$ is the product group of $\Gamma(w_i) = \{\zeta \mid \zeta^{w_i} = 1\}, i = 1, \dots, n+2$.

Denote by V_f^* the weighted complex hypersurface K_f/S^1 .

From Theorem 2.3, the quotient space $V_f^* = K_f/S^1$ is a V -manifold with an almost Hermitian structure (J, G) so that the projection $\pi : K_f \longrightarrow V_f^*$ is a Riemannian submersion.

We next assert is that the almost Hermitian structure (J, G) which V_f^* carries is a Hodge structure in the sense of a V -manifold, so that the J is the complex structure canonically induced from the affine hypersurface $V_f \subset \mathbb{C}^{n+2}$ and hence the differential map $\pi_* : E_z^{1,0} \longrightarrow T_{[z]}^{1,0} V_f^*$ is complex linear with respect to φ and J at any $z \in K_f$.

To assert this we write the holomorphic tangent space $T_z^{1,0} V_f, z \in K_f$,

$$T_z^{1,0} V_f = E_z^{1,0} \oplus \mathbb{C} \xi^{1,0}, \quad (43)$$

where $\xi^{1,0}$ is the weighted holomorphic Euler vector field, induced from the weighted holomorphic $\mathbb{C}^\times$ -action,

$$\xi^{1,0} = \left(\sum_i w_i z_i \frac{\partial}{\partial z_i} \right) \Big|_{V_f}. \quad (44)$$

The holomorphic tangent space of V_f^* with respect to the canonically induced complex structure is, by definition, the quotient space $T_z^{1,0} V_f$ by the $\mathbb{C}$ -linear subspace $\mathbb{C} \xi^{1,0}$, and hence coincides with $E_z^{1,0}$. Therefore, this implies that the almost complex structure J is exactly the one canonically induced from $V_f \subset \mathbb{C}^{n+2}$. Summing up these arguments, we have the following theorem.

THEOREM 3.3. *The quotient space $K_f/S^1 = V_f^*$ is a complex V -manifold with a Hodge structure (J, G) such that J is canonically induced from the projection $V_f \longrightarrow V_f^*$ and π is a Riemannian submersion and satisfies $\pi^* \Omega = d\eta$.*

Remarks.

- (1) Since $\Gamma(\mathbf{w})$ acts on $\mathbb{P}^{n+1}(\mathbb{C})$ bi-holomorphically and isometrically, $\mathbb{P}^{n+1}(\mathbf{w})$ admits the Fubini–Study metric as a V -manifold. However, our metric G defined on V_f^* does not, in general, coincide with the metric induced from the embedding $\iota : V_f^* \hookrightarrow \mathbb{P}^{n+1}(\mathbf{w})$.
- (2) The complex V -manifold structure which V_f^* carries is dealt by Steenbrink (see [21, Lemma 1], where he treated the case of weighted homogeneous polynomials of special type $f = h(z_1, \dots, z_{n+1}) + z_{n+2}^d$).

The final remark of this section is on the free-action points of K_f . Denote by $\check{K}_f = \{z \in K_f \mid G_z = e\}$ the set of free-action points of K_f . $\check{K}_f$ is an open dense

subset of K_f . The projection π then yields a principal S^1 -bundle when one restricts it to $\check{K}_f$,

$$\pi : \check{K}_f \longrightarrow \check{V}_f^*, \quad (45)$$

over a smooth hypersurface part $\check{V}_f^* = \pi(\check{K}_f) \subset \pi(K_f)$ in $\mathbb{P}^{n+1}(\mathbf{w})$. Even though $\check{K}_f$ is non-compact as remarked in 2.2, the Sasakian structure (η, ξ, φ, g) restricted to $\check{K}_f$ is correctly a Boothby–Wang–Hatakeyama fibration of the Hodge structure of $\check{V}_f^*$, the restriction of the Hodge structure (J, G) of V_f^* .

We note further that the differential map $\pi_* : E_z \longrightarrow T_{\pi(z)}V_f^*$, at any $z \in K_f$, yields an isomorphism between $\Lambda^{p,q}(E)$ and the bundle $\Lambda^{p,q}(V_f^*)$ of (p, q) -forms on V_f^* , even though V_f^* is a V -manifold.

4. Milnor fibration

4.1.

Following Milnor [15], define the map, called the Milnor fibration,

$$\Phi : S_\varepsilon^{2n+3} \setminus K_f \longrightarrow S^1; \quad z \mapsto f(z)/|f(z)|. \quad (46)$$

Each fiber $F_\theta = \Phi^{-1}(e^{\sqrt{-1}2\pi\theta}) \subset S_\varepsilon \setminus K_f$, $e^{\sqrt{-1}2\pi\theta} \in S^1$, is a smooth, parallelizable $(2n+2)$ -manifold with boundary $\partial F_\theta = K_f$. Since the origin 0 is an isolated singularity of V_f , K_f is $(n-1)$ -connected and F_θ has the homotopy type of a bouquet of $(n+1)$ -spheres. Let $F = F_1$ and define $h : F = F_1 \longrightarrow F = F_0$, the monodromy map, so that the product space $F \times [0, 2\pi]$ yields $S_\varepsilon \setminus K_f$ by identifying $F \times 0$ and $F \times 2\pi$ by the h .

Consider the induced homomorphism

$$h_* : H_{n+1}(F; \mathbb{C}) \longrightarrow H_{n+1}(F; \mathbb{C}) \quad (47)$$

of the $(n+1)$ th homology group of F . The Wang sequence associated with the Milnor fibration reduces to the short exact sequence

$$0 \longrightarrow H_{n+1}(K_f) \longrightarrow H_{n+1}(F) \xrightarrow{I-h_*} H_{n+1}(F) \longrightarrow H_n(K_f) \longrightarrow 0 \quad (48)$$

and it is asserted that

$$H_{n+1}(K_f) = \text{Ker}(I - h_*), \quad H_n(K_f) = \text{Coker}(I - h_*) \quad (49)$$

and $H_{n+1}(F)$ is free abelian and of rank $\mu = (w_1 - 1)(w_2 - 1) \cdots (w_{n+2} - 1)$.

Milnor proves further that the fiber F is diffeomorphic to a non-singular variety

$$F' = \{z \in \mathbb{C}^{n+2} \mid f(z) = 1\} \quad (50)$$

and the h may be chosen as

$$h' : F' \longrightarrow F', \quad h'(z) = (\zeta^{w_1} z_1, \dots, \zeta^{w_{n+2}} z_{n+2}), \quad (51)$$

where $\zeta = \exp(2\pi\sqrt{-1}/d)$. Therefore, the linear transformation $h'_* : H_{n+1}(F') \longrightarrow H_{n+1}(F')$ has finite period and it can be diagonalized by a suitable basis. The multiplicity k of the solution $t = 1$ in the characteristic polynomial $\Delta(t) = \det(tI - h'_*)$ therefore gives the rank of $H_{n+1}(K_f)$, i.e. the Betti number $b_n = b_{n+1}$.

The multiplicity k can be computed by the divisor's argument and Weil's ζ -function argument [16, 18]). Therefore, one has the following theorem.

THEOREM 4.1. [16, 18]

$$b_n(K_f) = \sum_{s=0}^{n+2} \sum_I (-1)^{n-s} \frac{q_{i_1} \times \cdots \times q_{i_s}}{[u_{i_1}, \dots, u_{i_s}]}. \quad (52)$$

5. The main theorem

5.1.

The S^1 -action of the Sasakian manifold K_f is almost free so that from the argument of Thomas one has the double fibration,

$$K_f \xrightarrow{\pi_1} K_f/\Gamma \xrightarrow{\pi_2} V_f^* = K_f/S^1 = (V_f \setminus \{0\})/\mathbb{C}^\times. \quad (53)$$

Here π_1 is a branched covering and π_2 is a projection by a free $S^1/\Gamma (\cong S^1)$ -action. Furthermore, V_f^* is a complex hypersurface in the weighted projective $(n+1)$ -space $\mathbb{P}^{n+1}(\mathbf{w})$. V_f^* is in general a V -manifold, as we saw in Section 3. Note that the Γ -quotient K_f/Γ is also a real V -manifold.

LEMMA 5.1. *The n th cohomology group $H^n(K_f/\Gamma; \mathbb{R})$ is isomorphic to $\mathbb{H}^n(K_f)$ of K_f through π_1 ,*

$$H^n(K_f/\Gamma; \mathbb{R}) \cong \mathbb{H}^n(K_f). \quad (54)$$

Proof. Since the space K_f/Γ is a finite quotient of a smooth manifold, one obtains $H^n(K_f/\Gamma; \mathbb{R}) \cong \mathbb{H}^n(K_f)^\Gamma$, the space of Γ -invariant harmonic n -forms on K_f (see [5, 2.4 Theorem, III]). The S^1 -action is isometric so that any harmonic form on K_f is Γ -invariant. So $\mathbb{H}^n(K_f)^\Gamma = \mathbb{H}^n(K_f)$. $\square$

5.2. Gysin sequence

Next consider the S^1 -fibration $\pi_2 : K_f / \Gamma \longrightarrow K_f / S^1 = V_f^*$ to which we can apply the Gysin exact sequence of the cohomology groups [5],

$$\begin{aligned} \longrightarrow H^{n-1}(K_f / \Gamma) &\xrightarrow{\int} H^{n-2}(V_f^*) \xrightarrow{b} H^n(V_f^*) \\ &\xrightarrow{\pi_2^*} H^n(K_f / \Gamma) \xrightarrow{\int} H^{n-1}(V_f^*) \xrightarrow{b} . \end{aligned} \quad (55)$$

Here $b : H^{n-2}(V_f^*) \longrightarrow H^n(V_f^*)$ is the map $[\alpha] \mapsto [\alpha] \wedge [\Omega]$ where Ω is the Kähler form on V_f^* , defined from the Hodge structure (J, G) of V_f^* , representing the curvature form of the S^1 -fibration π_2 . The map $\int$ is an integrating map along fibers so $\int$ reduces to the zero map, since harmonic forms in K_f / Γ or in K_f have no fiber directional part, i.e. no ξ -directional part. See Theorem 2.4. Therefore, the map b is injective and we have

$$H^n(V_f^*) \cong H_0^n(V_f^*) \oplus H^{n-2}(V_f^*). \quad (56)$$

Here, $H_0^n(V_f^*) = \{[\alpha] \in H^n(V_f^*) \mid b^*[\alpha] = 0\}$ is the primitive cohomology group of V_f^* , naturally isomorphic to $H^n(V_f^*)/\text{Im } b$. We therefore get the isomorphism via the Gysin sequence.

LEMMA 5.2.

$$\pi_2^* : H_0^n(V_f^*) \xrightarrow{\cong} H^n(K_f / \Gamma) (\cong \mathbb{H}^n(K_f)) \quad (57)$$

is an isomorphism.

5.3.

Since V_f^* is a complex V -manifold, the Hodge–de Rham–Kodaira harmonic theory can also be applied to V_f^* [1]. Therefore, $H_0^n(V_f^*)$, which is regarded as the space $\mathbb{H}_0^n(V_f^*)$ of primitive harmonic forms on V_f^* , splits into the direct sum of the space $\mathbb{H}_0^{p,q}(V_f^*)$ of primitive harmonic (p, q) -forms on V_f^* ,

$$H_0^n(V_f^*) \cong \bigoplus_{p+q=n} \mathbb{H}_0^{p,q}(V_f^*), \quad (58)$$

where a harmonic form ψ is primitive provided it satisfies $\Lambda_{\Omega_G} \psi = 0$, and $\mathbb{H}_0^{p,q}(V_f^*) = \{\bar{\partial}$ -harmonic (p, q) form $\psi^{p,q}$ on $V_f^* \mid \Lambda_{\Omega_G} \psi^{p,q} = 0\}$.

THEOREM 5.3. For non-negative integers p, q with $p + q = n$,

$$\mathbb{H}^{p,q}(K_f) \cong \mathbb{H}_0^{p,q}(V_f^*). \quad (59)$$

Proof. A primitive harmonic n -form ψ on V_f^* is written as the direct sum of harmonic (p, q) -forms, $\psi = \sum_{p+q=n} \psi^{p,q}$, $\psi^{p,q} \in \mathbb{H}_0^{p,q}(V_f^*)$. Thus the lift of ψ by π is $\pi^*\psi = \sum_{p+q=n} \pi^*\psi^{p,q}$. From the property of the projection π , $\pi^*\psi^{p,q}$ is a (p, q) -form on K_f with no ξ -component. Since π is a Riemannian submersion with totally geodesic fibers, $\pi^*\psi^{p,q}$ is also harmonic in K_f , even π is a submersion onto a V -manifold. On the other hand, it is easily seen that π^* is injective. So, $\dim_{\mathbb{C}} \mathbb{H}_0^{p,q}(V_f^*) \leq \dim_{\mathbb{C}} \mathbb{H}^{p,q}(K_f)$. Therefore, by counting the dimension of both the cohomology groups $\mathbb{H}_0^n(V_f^*)$ and $\mathbb{H}^n(K_f)$, which must be the same from Lemma 5.2, we have the equality in Theorem 5.3. $\square$

5.4. The mixed Hodge numbers

Hodge–de Rham–Kodaira’s theorem and Dolbeault–Kodaira’s theorem, which work on a complex V -manifold, do not depend on the choice of Kähler metric ([1, Theorems H, K]). So, we can identify $\mathbb{H}_0^{p,q}(V_f^*)$ with $H^{p,q}(V_f^*)/\text{Im } j^n$, where $j^n : H^n(\mathbb{P}^{n+1}(\mathbf{w})) \rightarrow H^n(V_f^*)$ is the monomorphism induced by the inclusion $j : V_f^* \rightarrow \mathbb{P}^{n+1}(\mathbf{w})$ (see [6, (B25)Lemma/Definition, Appendix B]). The following key theorem allows us to compute the mixed Hodge numbers $h_0^{p,q}(V_f^*) = \dim_{\mathbb{C}} H_0^{p,q}(V_f^*)$ of the primitive cohomology group $H_0^n(V_f^*)$.

THEOREM 5.4. [20, 21]

$$h_0^{p,q}(V_f^*) = \dim_{\mathbb{C}} (\mathbb{M}_f)_{\ell}, \quad (60)$$

where $\mathbb{M}_f$ is the Milnor algebra associated to the weighted homogeneous polynomial f , introduced in Section 1, and $\ell = (p+1)d - \sum_i w_i$.

See also [6, (B34)Theorem, Appendix B].

The proof of Theorem 1.2, which is our final goal, is immediate from Theorems 5.3 and 5.4.

5.5. Numerical examples

Consider $f = z_1^6 + z_2^6 + z_3^6 + z_4^2$, a polynomial of Brieskorn–Pham type. f is weighted homogeneous, of degree $d = 6$ and weights $\mathbf{w} = (1, 1, 1, 3)$.

The link K_f is five-dimensional and $b_2 = b_3 = 21$, $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 19$. So $b_+ = 2h^{2,0} = 2$ and $b_- = 19$, and hence we get $b_2 = 2h^{2,0} + h^{1,1}$ and the contact-signature $\tau = -17$. See [10] for b^+ and b^- and the contact signature τ of a compact Sasakian 5-manifold.

We compute in fact the Betti number via Theorem 4.1, namely the formula of Milnor–Orlik, as

$$b_2 = 1 - \sum_i \frac{q_i}{[q_i]} + \sum_{i,j} \frac{q_i q_j}{[q_i, q_j]}$$

whose value is $1 - 4 + 24 = 21$, since $(q_i) = (u_i) = (6, 6, 6, 2)$ with all $v_i = 1$. On the other hand, we have

$$\left(\frac{\partial f}{\partial z_i} \right) = (6z_1^5, 6z_2^5, 6z_3^5, 2z_4) \quad (61)$$

so that the Milnor algebra $\mathbb{M}_f$ is $\mathbb{C}[z_1, z_2, z_3]/(z_1^5, z_2^5, z_3^5)$. Then, $(\mathbb{M}_f)_0$ is one-dimensional so $h^{2,0} = 1$. The subspace $(\mathbb{M}_f)_{2d-\sum_i w_i} = (\mathbb{M}_f)_{16}$ has dimension 19. So, $h^{1,1} = 19$.

Another example is $f = z_1^2 + z_2^3 + z_3^3 + z_4^4 + z_5^6$. Its degree d is 12 and the weights are $(6, 4, 4, 3, 2)$. The link in this case is seven-dimensional and, as we see in [13], $b_3 = b_4 = 2$ and $h^{3,0} = 0, h^{2,1} = 1$.

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