MAXIMAL SURFACES WITH SIMPLE ENDS

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0. Introduction

A maximal surface in the three-dimensional Minkowski space $\mathbf{L}^3$ is a space-like surface with zero mean curvature. It is well-known that there is a Weierstrass-type representation for maximal surfaces, that is, a representation formula in terms of holomorphic data similar to the Weierstrass representation for minimal surfaces in the Euclidean 3-space $\mathbf{E}^3$ (Lemma 1.3). Using the holomorphic data (g, ω) of the Weierstrass-type representation for a maximal surface, we see that the induced metric degenerates on the set $\{|g| = 1\}$. On the other hand, it is well-known that there exist no complete maximal surfaces in $\mathbf{L}^3$ except space-like planes. So we consider a class of surfaces which admit degenerate points of the induced metric (Definition 2.1).

For a minimal surface in $\mathbf{E}^3$, the following fact is well-known: Let $f: \Delta^* \rightarrow \mathbf{E}^3$ be a conformal minimal immersion of a punctured disc $\Delta^* = \{z \in \mathbb{C}; 0 < |z| < 1\}$ into $\mathbf{E}^3$ which is complete at the origin 0. Then $\text{ord}_{z=0} ds^2 = -2$ holds if and only if the end 0 is asymptotic to a catenoid or a plane in $\mathbf{E}^3$ [6]. Analogous to this fact, we introduce the simple end for maximal maps (Definition 2.2).

In this paper, we consider a maximal map with simple ends and show the following theorem (Theorem 2.7).

THEOREM. *Let $\Delta := \{z \in \mathbb{C}; |z| < 1\}$ and $\Delta^* := \Delta \setminus \{0\}$. Assume $f: \Delta^* \rightarrow \mathbf{L}^3$ is a maximal map such that the end 0 is simple. Then f is asymptotic to one of the following three types:*

- (I) $(ar^{-1} \cos \theta, ar^{-1} \sin \theta, c \log r),$
- (II) $(ar^{-1} \sin \theta, c \log r, ar^{-1} \sin \theta),$
- (III) $(br^{-1} \cos \theta + c \log r, 0, -br^{-1} \cos \theta - c \log r),$

on a neighborhood of 0, where $z := re^{\sqrt{-1}\theta}$, $a, b \in \mathbb{R} \setminus \{0\}$, and $c \in \mathbb{R}$.

In the final section, we consider the flux vector of maximal maps similarly to the case of minimal surfaces in $\mathbf{E}^3$ (Definition 3.1), and we show the relationship between the simple ends and their flux (Theorem 3.3).

1. Preliminaries

Let $\mathbf{L}^3$ denote the three-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentz metric

$$\langle \cdot, \cdot \rangle = (dx^1)^2 + (dx^2)^2 - (dx^3)^2,$$

where (x^1, x^2, x^3) are the canonical coordinates in $\mathbb{R}^3$. An immersion $f: M \rightarrow \mathbf{L}^3$ of a two-dimensional manifold M is said to be *space-like* if the pull-back ds^2 of $\langle \cdot, \cdot \rangle$ by f is a Riemannian metric on M . Let ν be the unit normal vector field of a space-like immersion f , namely at each point $p \in M$, ν_p is perpendicular to the tangent plane and $\langle \nu_p, \nu_p \rangle = -1$. The second fundamental form is a quadratic form on M defined by

$$h = \langle df, d\nu \rangle.$$

Then the mean curvature is defined as a half the trace of h with respect to ds^2 .

Definition 1.1. A space-like immersion f is said to be *maximal* if the mean curvature of f vanishes identically.

A space-like immersion $f: M \rightarrow \mathbf{L}^3$ induces a conformal structure on M by ds^2 . Hence, if M is orientable, f can be regarded as a conformal immersion of a Riemann surface M .

Let H^2 be a space-like surface in $\mathbf{L}^3$ defined by

$$\begin{aligned} H^2 &= \{(x^1, x^2, x^3) \in \mathbf{L}^3; (x^1)^2 + (x^2)^2 - (x^3)^2 = -1\} \\ &= \{(x^1, x^2, x^3) \in H^2; x^3 > 0\} \cup \{(x^1, x^2, x^3) \in H^2; x^3 < 0\} \\ &= H_+^2 \cup H_-^2. \end{aligned}$$

If $f: M \rightarrow \mathbf{L}^3$ is a space-like immersion, the unit normal vector ν of f determines a point of H^2 at each point of M .

The map

$$\sigma: H^2 \rightarrow \mathbb{C} \cup \{\infty\} \setminus \{z \in \mathbb{C}; |z| = 1\}$$

defined by

$$\sigma(x^1, x^2, x^3) = \frac{1}{1-x^3}(x^1 + \sqrt{-1}x^2) \quad (1.1)$$

is conformal. The map σ is called the *stereographic projection*.

Definition 1.2. For a space-like immersion $f: M \rightarrow \mathbf{L}^3$, the map

$$g: M \rightarrow \mathbb{C} \cup \{\infty\} \setminus \{z \in \mathbb{C}; |z| = 1\}; \quad g(p) = \sigma(\nu_p)$$

is called the *Gauss map* of f , where v_p is the unit normal vector of f at p , and σ is the stereographic projection as in (1.1).

It is well-known that a maximal immersion can be represented in terms of holomorphic data.

LEMMA 1.3. (Weierstrass-type representation [4]) *Let M be a Riemann surface and $f: M \rightarrow \mathbf{L}^3$ a conformal maximal immersion. Then there exist a meromorphic function $g: M \rightarrow \mathbb{C} \cup \{\infty\} \setminus \{z \in \mathbb{C}; |z| = 1\}$ and a holomorphic 1-form ω on M such that*

$$f(z) = \operatorname{Re} \int_{z_0}^z \alpha, \quad \alpha = (1 + g^2, \sqrt{-1}(1 - g^2), -2g)\omega, \quad (1.2)$$

where z is a complex coordinate on M and $z_0 \in M$ is a base point.

Moreover, g is the Gauss map of f , and the induced metric of f is written as

$$ds^2 = (1 - |g|^2)^2 |\omega|^2. \quad (1.3)$$

Conversely, let $g: M \rightarrow \mathbb{C} \cup \{\infty\} \setminus \{z \in \mathbb{C}; |z| = 1\}$ be a meromorphic function and ω a holomorphic 1-form on M , such that ds^2 in (1.3) is non-degenerate.

If

$$\operatorname{Re} \oint_{\gamma} \alpha = 0$$

for any loops γ on M , then there exists a maximal immersion $f: M \rightarrow \mathbf{L}^3$ such that f is given by (1.2).

We call the pair (g, ω) in Lemma 1.3 the *Weierstrass data* of f .

2. Maximal map

If $g: M \rightarrow \mathbb{C} \cup \{\infty\}$ is a meromorphic function, then the map f given by (1.2) degenerates on

$$\{z \in M; |g(z)| = 1\}.$$

So we consider a notion of maximal surfaces as a surface with degenerate points.

Definition 2.1. Let M be a Riemann surface and g (respectively ω) a meromorphic function (respectively meromorphic 1-form) on M . We denote by P the set of poles of

$$\alpha = (1 + g^2, \sqrt{-1}(1 - g^2), -2g)\omega \quad (2.1)$$

and $S := \{z \in M; |g(z)| = 1\}$. Then a map $f: M^* := M \setminus P \rightarrow \mathbf{L}^3$ given by (1.2) is called a *maximal map* with Weierstrass data (g, ω) if:

- (1) P is a discrete set of M ;
- (2) the restriction $f|_{M^* \setminus S}$ is a conformal maximal immersion;
- (3) the Weierstrass data of $f|_{M^* \setminus S}$ is (g, ω) .

We call a point $q \in P$ an *end* of f .

For a minimal immersion f in $\mathbf{E}^3$ with Weierstrass data (g, ω) , if g is one-to-one on a neighborhood of an end of f and ωdg has a pole of order two at the end of f , then f is asymptotic to a catenoid on the neighborhood of the end of f . So, we introduce the following notion for ends of maximal maps.

Definition 2.2. An end $q \in P$ of maximal map with Weierstrass data (g, ω) is called *simple* if and only if the $\mathbb{C}^3$ -valued 1-form α in (2.1) has a pole of order two at q .

Example 2.3. [5] Let $M^* := \mathbb{C} \cup \{\infty\} \setminus \{0, \infty\}$ and $(g, \omega) = (z, az^{-2}dz)$, where $a \in \mathbb{R} \setminus \{0\}$. Then (g, ω) induces a maximal map f_1 :

$$\begin{aligned} f_1(z) &= \operatorname{Re} \int \left(a \left(\frac{1}{z^2} + 1 \right), \sqrt{-1}a \left(\frac{1}{z^2} - 1 \right), -\frac{2a}{z} \right) dz \\ &= a \operatorname{Re} \left(-\frac{1}{z} + z, -\sqrt{-1} \left(\frac{1}{z} + z \right), -2 \log z \right). \end{aligned}$$

Let $z = re^{\sqrt{-1}\theta}$ ($r > 0, -\pi \leq \theta \leq \pi$). Then

$$f_1 = a \left(\left(r - \frac{1}{r} \right) \cos \theta, \left(r - \frac{1}{r} \right) \sin \theta, -2 \log r \right).$$

Therefore, f_1 is asymptotic to

$$a \left(-\frac{1}{r} \cos \theta, -\frac{1}{r} \sin \theta, -2 \log r \right)$$

on a neighborhood of $z = 0$.

The image of f_1 is

$$f_1(M^*) = \left\{ (x^1, x^2, x^3) \in \mathbf{L}^3; (x^1)^2 + (x^2)^2 - 4a^2 \sinh^2 \left(\frac{x^3}{2a} \right) = 0 \right\}.$$

(See Figure 1.)

FIGURE 1. The surface in Example 2.3 (called the Lorentz catenoid).

Example 2.4. [5] Let $M^* := \mathbb{C} \cup \{\infty\} \setminus \{1, -1\}$ and

$$(g, \omega) = \left(z, \frac{\sqrt{-1}b}{(z-1)^2(z+1)^2} dz \right),$$

where $b \in \mathbb{R} \setminus \{0\}$.

Then (g, ω) induces a maximal map f_2 :

$$f_2(z) = b \operatorname{Re} \left(-\sqrt{-1} \frac{z}{(z+1)(z-1)}, -\frac{1}{2} \log \frac{z+1}{z-1}, \sqrt{-1} \frac{1}{(z+1)(z-1)} \right)$$

Let $w = (z+1)(z-1)^{-1}$. Then

$$f_2(w) = b \operatorname{Re} \left(-\frac{\sqrt{-1}}{4} \frac{w^2 - 1}{w}, -\frac{1}{2} \log w, \frac{\sqrt{-1}}{4} \frac{(w-1)^2}{w} \right).$$

Moreover, let $w = re^{\sqrt{-1}\theta}$ ($r > 0, -\pi \leq \theta \leq \pi$). Then

$$f_2(r, \theta) = b \left(\frac{1}{4} \left(r + \frac{1}{r} \right) \sin \theta, -\frac{1}{2} \log r, -\frac{1}{4} \left(r - \frac{1}{r} \right) \sin \theta \right).$$

FIGURE 2. The surface in Example 2.4.

Therefore, f_2 is asymptotic to

$$\frac{b}{4} \left(\frac{1}{r} \sin \theta, -2 \log r, \frac{1}{r} \sin \theta \right)$$

on a neighborhood of $z = -1$.

The image of f_2 is

$$f_2(M^*) = \left\{ (x^1, x^2, x^3) \in \mathbf{L}^3; x^3 - x^1 \tanh \left(\frac{2x^2}{b} \right) = 0, |x^1| \leq \frac{b}{2} \cosh \frac{2x^2}{b} \right\}.$$

(See Figure 2.)

Example 2.5. Let $M^* := \mathbb{C} \cup \{\infty\} \setminus \{0, 1, -1\}$ and

$$(g, \omega) = \left(\frac{2z}{z^2 + 1}, \frac{c(z^2 + 1)^2}{z^2(z^2 - 1)^2} dz \right),$$

where $c \in \mathbb{R} \setminus \{0\}$. Then (g, ω) induces a maximal map f_3 :

$$f_3(z) = -c \operatorname{Re} \left(\frac{5z^2 - 1}{z(z^2 - 1)} + 2 \log \frac{z - 1}{z + 1}, \frac{\sqrt{-1}}{z}, -\frac{4}{z^2 - 1} + 2 \log \frac{z^2}{z^2 - 1} \right).$$

FIGURE 3. The surface in Example 2.5.

Let $z - 1 = re^{\sqrt{-1}\theta}$ ($r > 0$, $-\pi \leq \theta \leq \pi$). Then it is seen that f_3 is asymptotic to

$$-2c \left(\frac{1}{r} \cos \theta + \log r, 0, -\frac{1}{r} \cos \theta - \log r \right)$$

on a neighborhood of $z = 1$. (See Figure 3.)

Remark 2.6. Even if the order of α at an end is -2 , the order of ωdg at the end is not always equal to -2 . (See Example 2.5.)

In the following theorem, we prove that any simple end is asymptotic to one of the ends shown in Examples 2.3, 2.4 and 2.5 or a space-like plane or a null line.

THEOREM 2.7. *Let $\Delta := \{z \in \mathbb{C}; |z| < 1\}$ and $\Delta^* := \Delta \setminus \{0\}$. Assume $f: \Delta^* \rightarrow \mathbf{L}^3$ is a maximal map such that the end 0 is simple. Then f is asymptotic to one of the following three types:*

- (I) $(ar^{-1} \cos \theta, ar^{-1} \sin \theta, c \log r)$,
 - (II) $(ar^{-1} \sin \theta, c \log r, ar^{-1} \sin \theta)$,
 - (III) $(br^{-1} \cos \theta + c \log r, 0, -br^{-1} \cos \theta - c \log r)$,
- on a neighborhood of 0, where $z := re^{\sqrt{-1}\theta}$, $a, b \in \mathbb{R} \setminus \{0\}$, and $c \in \mathbb{R}$.

Proof. By the definition of a maximal map, f can be written as

$$f = \operatorname{Re} \int \alpha,$$

where α is as in (2.1). Since 0 is a simple end, the order of α at 0 is -2 . Hence α can be expanded as the following:

$$\alpha = \left(\frac{1}{z^2} \mathbf{a}_{-2} + \frac{1}{z} \mathbf{a}_{-1} + \cdots \right) dz,$$

where $\mathbf{a}_{-2}, \mathbf{a}_{-1}, \dots \in \mathbb{C}^3$, and $\mathbf{a}_{-2} \neq \mathbf{0}$.

By (2.1), $\langle \alpha, \alpha \rangle_{\mathbb{C}} = 0$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the complexification of the Lorentz metric $\langle \cdot, \cdot \rangle$. Then we have

$$\frac{1}{z^4} \langle \mathbf{a}_{-2}, \mathbf{a}_{-2} \rangle_{\mathbb{C}} + \frac{2}{z^3} \langle \mathbf{a}_{-2}, \mathbf{a}_{-1} \rangle_{\mathbb{C}} + \cdots = 0,$$

that is,

$$\begin{cases} \langle \mathbf{a}_{-2}, \mathbf{a}_{-2} \rangle_{\mathbb{C}} = 0, \\ \langle \mathbf{a}_{-2}, \mathbf{a}_{-1} \rangle_{\mathbb{C}} = 0. \end{cases} \quad (2.2)$$

Since

$$\operatorname{Re} \oint_{\gamma} \alpha = 0$$

holds for any loop γ surrounding 0, $\mathbf{a}_{-1}$ is real vector.

Therefore, if we set $\mathbf{a} + \sqrt{-1}\mathbf{b} := \mathbf{a}_{-2}$, $\mathbf{c} := \mathbf{a}_{-1}$, ($\mathbf{a}, \mathbf{b}, \mathbf{c}$ are real vectors) such that $(\mathbf{a}, \mathbf{b}) \neq (\mathbf{0}, \mathbf{0})$, then (2.2) is equivalent to the following conditions:

$$\begin{cases} \langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle, \\ \langle \mathbf{a}, \mathbf{b} \rangle = 0, \\ \langle \mathbf{a}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{c} \rangle = 0. \end{cases} \quad (2.3)$$

Then the triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is one of the cases in Table 1.

In the case of (I), we may set, without loss of generality, that

$$\mathbf{a} = -a(1, 0, 0), \quad \mathbf{b} = -a(0, 1, 0), \quad \mathbf{c} = c(0, 0, 1),$$

where $a, c \in \mathbb{R}$, $a \neq 0$. Hence

$$\begin{aligned} \alpha &= \left(-\frac{a}{z^2} (1, \sqrt{-1}, 0) + \frac{c}{z} (0, 0, 1) + \cdots \right) dz \\ \int \alpha &= \frac{a}{z} (1, \sqrt{-1}, 0) + c \log z (0, 0, 1) + o(1). \end{aligned}$$

Set $z = re^{\sqrt{-1}\theta}$ ($r > 0$, $-\pi \leq \theta \leq \pi$), then

$$f = \operatorname{Re} \int \alpha = (ar^{-1} \cos \theta, ar^{-1} \sin \theta, c \log r) + o(1).$$

Similarly, in the cases of (II-i)–(III-iii), f is asymptotic to the following:

TABLE 1. Types of triple (a, b, c) .

Type	c	a	b
(I)	Time-like or $\mathbf{0}$	Space-like	Space-like
(II-i)	Space-like or $\mathbf{0}$	$\mathbf{0}$	Null
(II-ii)	Space-like or $\mathbf{0}$	Null	$\mathbf{0}$
(II-iii)	Space-like or $\mathbf{0}$	Null	Null
(III-i)	Null or $\mathbf{0}$	$\mathbf{0}$	Null
(III-ii)	Null or $\mathbf{0}$	Null	$\mathbf{0}$
(III-iii)	Null or $\mathbf{0}$	Null	Null

$$(II-i) \quad (ar^{-1} \sin \theta, c \log r, ar^{-1} \sin \theta);$$

$$(II-ii) \quad (ar^{-1} \cos \theta, c \log r, ar^{-1} \cos \theta);$$

$$(II-iii) \quad (r^{-1}(a \cos \theta + b \sin \theta), c \log r, r^{-1}(a \cos \theta + b \sin \theta));$$

$$(III-i) \quad (br^{-1} \sin \theta + c \log r, 0, -br^{-1} \sin \theta - c \log r);$$

$$(III-ii) \quad (br^{-1} \cos \theta + c \log r, 0, -br^{-1} \cos \theta - c \log r);$$

$$(III-iii) \quad (r^{-1}(a \cos \theta + b \sin \theta) + c \log r, 0, -r^{-1}(a \cos \theta + b \sin \theta) - c \log r).$$

By the addition formula of the trigonometric function, we have $(II-i) = (II-ii) = (II-iii)$, $(III-i) = (III-ii) = (III-iii)$. $\square$

Remark 2.8. In (I) in Theorem 2.7, if $c = 0$, then f is asymptotic to a space-like plane. Similarly, in the cases of (II) and (III), f is asymptotic to a null line.

Remark 2.9. For a minimal immersion $f: M \rightarrow \mathbf{E}^3$, it is well-known that the Weierstrass representation is

$$f(z) = \operatorname{Re} \int_{z_0}^z \tilde{\alpha}, \quad \tilde{\alpha} = (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\omega,$$

and that, if f is a complete minimal immersion with embedded ends, then the order of $\tilde{\alpha}$ at the end is equal to -2 . However, in our case, we can consider the case that the order of α at the end is equal to -1 .

If the order α at the end 0 is -1 , then

$$\alpha = \left(\frac{1}{z} \mathbf{a}_{-1} + \cdots \right) dz,$$

where $\mathbf{a}_{-1}$ is non-zero real vector.

Since $\langle \alpha, \alpha \rangle_{\mathbb{C}} = 0$, $\langle \mathbf{a}_{-1}, \mathbf{a}_{-1} \rangle = 0$. That is, $\mathbf{a}_{-1}$ is the null vector. We may set, without loss of generality, that $\mathbf{a}_{-1} = a(1, 0, 1)$, $a \in \mathbb{R} \setminus \{0\}$. Hence

$$\alpha = \left(\frac{1}{z} (1, 0, 1) + \cdots \right) dz,$$

$$\int \alpha = a(1, 0, 1) \log z + o(1).$$

Therefore,

$$f = \operatorname{Re} \int \alpha = (a \log r, 0, a \log r) + o(1),$$

where $z = r e^{\sqrt{-1}\theta}$ ($r > 0$, $-\pi \leq \theta \leq \pi$).

For example, $M^* := \mathbb{C} \cup \{\infty\} \setminus \{0, \infty\}$ and $(g, \omega) = (z + 1, z^{-1} dz)$.

3. Flux

In this section, we consider the relationship between the flux and the simple ends.

Definition 3.1. Let $M, (g, \omega)$ and P be as in Definition 2.1. Then, for a simple end $q \in P$ of a maximal map $f: M^* = M \setminus P \rightarrow \mathbf{L}^3$, the *flux vector* of q is defined by

$$\varphi_q := \operatorname{Im} \oint_{\gamma_q} \alpha,$$

where γ_q is a positively-oriented curve surrounding q .

PROPOSITION 3.2. *For the flux vector φ_q of the simple end $q \in P$ of a maximal map f , if f is asymptotic to (I) ($c \neq 0$) in Theorem 2.7 on a neighborhood of q , then*

$$\varphi_q = \oint_{\gamma_q} \vec{n} ds,$$

where $\vec{n}$ is the conormal vector along γ_q that is $(\gamma'_q, \vec{n})$ is positively oriented.

Proof. Rewrite $\varphi := \varphi_q$ and let $z(t)$ be a parameterization of γ_q . Put $z(t) := u(t) + \sqrt{-1}v(t)$ and define a positively-oriented orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \nu\}$ by

$$\mathbf{e}_1 := e^{-\lambda} \frac{\partial f}{\partial u}, \quad \mathbf{e}_2 := e^{-\lambda} \frac{\partial f}{\partial v},$$

ν : the limit normal vector at q , where $ds^2 = e^{2\lambda} dz d\bar{z}$.

Then

$$\begin{aligned}
 \frac{df}{dt} &= \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial \bar{z}} \frac{d\bar{z}}{dt} \\
 &= \frac{1}{2} \left(\frac{\partial f}{\partial u} - \sqrt{-1} \frac{\partial f}{\partial v} \right) \left(\frac{du}{dt} + \sqrt{-1} \frac{dv}{dt} \right) \\
 &\quad + \frac{1}{2} \left(\frac{\partial f}{\partial u} + \sqrt{-1} \frac{\partial f}{\partial v} \right) \left(\frac{du}{dt} - \sqrt{-1} \frac{dv}{dt} \right) \\
 &= \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt} \\
 &= e^\lambda \left(\mathbf{e}_1 \frac{du}{dt} + \mathbf{e}_2 \frac{dv}{dt} \right).
 \end{aligned}$$

Hence,

$$\frac{df}{dt} \times v = e^\lambda \left(-\mathbf{e}_2 \frac{du}{dt} + \mathbf{e}_1 \frac{dv}{dt} \right)$$

and

$$\left| \frac{df}{dt} \right| = e^\lambda \sqrt{\left(\frac{du}{dt} \right)^2 + \left(\frac{dv}{dt} \right)^2} = e^\lambda \left| \frac{dz}{dt} \right|.$$

Therefore,

$$\begin{aligned}
 \vec{n} &= \left| \frac{df}{dt} \right|^{-1} \frac{df}{dt} \times v \\
 &= \left| \frac{dz}{dt} \right|^{-1} \left(\mathbf{e}_1 \frac{dv}{dt} - \mathbf{e}_2 \frac{du}{dt} \right).
 \end{aligned}$$

Moreover,

$$ds = \left| \frac{df}{dt} \right| dt = e^\lambda \left| \frac{dz}{dt} \right| dt.$$

Therefore,

$$\begin{aligned}
 \vec{n} ds &= e^\lambda (\mathbf{e}_1 dv - \mathbf{e}_2 du) \\
 &= \frac{\partial f}{\partial u} dv - \frac{\partial f}{\partial v} du \\
 &= \frac{-\sqrt{-1}}{2} \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \right) (dz - d\bar{z}) - \frac{\sqrt{-1}}{2} \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) (dz + d\bar{z}) \\
 &= -\sqrt{-1} \left(\frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial \bar{z}} d\bar{z} \right) \\
 &= \operatorname{Im} \frac{\partial f}{\partial z} dz.
 \end{aligned}$$

□

By definition, the flux vector of q is written as the residue of α at $z = q$:

$$\varphi_q = 2\pi \operatorname{Res}(\alpha(z); q).$$

By the property of the residue, if M is compact, then $\sum_{q \in P} \varphi_q = \mathbf{0}$ (balancing formula).

Furthermore, in the proof of Theorem 2.7, α is written as

$$\alpha = \left(\frac{1}{(z-q)^2} (\mathbf{a} + \sqrt{-1}\mathbf{b}) + \frac{1}{z-q} \mathbf{c} + \cdots \right) dz.$$

Therefore,

$$\varphi_q = 2\pi \mathbf{c}.$$

Hence we can rewrite the three types of Theorem 2.7 as in the following theorem. (See Table 1.)

THEOREM 3.3. *Let $\Delta := \{z \in \mathbb{C}; |z| < 1\}$ and $\Delta^* := \Delta \setminus \{0\}$. Assume $f: \Delta^* \rightarrow \mathbf{L}^3$ is a maximal map such that the end 0 is simple. Then f is asymptotic to (I) (respectively (II), (III)) ($\mathbf{c} \neq \mathbf{0}$) of Theorem 2.7 if and only if the flux vector φ_0 is time-like (respectively space-like, null).*

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