ON THE FIRST SUBCONSTITUENT OF A QUADRATIC FORMS SCHEME

Seiichi IWAMOTO

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Abstract. As a step to find the eigenvalues of the first subconstituent X_1 of a quadratic forms scheme, we determine the orbits of $GL_n(q)$ acting on $X_1 \times X_1$.

1. Introduction

Let q be a power of an odd prime and let $GF(q)$ denote the finite field of q elements. Let n be a natural number ($n \geq 3$) and let X denote the set of symmetric matrices over $GF(q)$ of size n . For an integer i ($0 \leq i \leq (n+1)/2$), let X_i denote the subset of X that consists of matrices in X of rank $2i-1$ or $2i$. Let R_i ($0 \leq i \leq (n+1)/2$) be a relation on X defined by $(M, M') \in R_i \iff M' - M \in X_i$, i.e. $\text{rank}(M' - M) = 2i-1, 2i$. Then we have a P - and Q -polynomial association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq (n+1)/2})$, which is called the *quadratic forms scheme* [1]. The subset X_i is called the i th subconstituent.

Let Γ denote the graph whose vertex set is X_1 and whose edge set is the restriction of R_1 on $X_1 \times X_1$. In the problem of characterizing the quadratic forms scheme by the parameters, the eigenvalues of the graph Γ on the first subconstituent X_1 are crucially important in the sense that certain subsets of them are in one-to-one correspondence with the isomorphism classes of irreducible thin T -modules of endpoint 1, where T is the Terwilliger algebra of the P - and Q -polynomial association scheme, and that they determine the global structure of the P - and Q -polynomial association scheme to some extent [4]. Note that the eigenvalues of the first subconstituent graph Γ are known for other classical forms schemes, i.e. bilinear, hermitian and alternating forms schemes, whereas those for the quadratic forms scheme are yet to be found [3].

Let G denote the general linear group $GL_n(q)$, the group of non-singular matrices over $GF(q)$ of size n . Then G acts on Γ as a group of automorphisms $M^g = {}^t g M g$ for $g \in G, M \in X_1$. Let G act on $X_1 \times X_1$ by $(M, M')^g = (M^g, M'^g)$ for $g \in G, M, M' \in X_1$. Each of the G -orbits on $X_1 \times X_1$ gives rise to a relation

on $X_1 \times X_1$ and we have a coherent configuration on X_1 with the G -orbits on $X_1 \times X_1$ as the defining relations [2]. Note that the coherent configuration is not an association scheme, since G is not transitive on X_1 . The graph Γ is reconstructed from the coherent configuration by fusing relations, and the eigenvalues of Γ can be obtained in the framework of the coherent configuration by determining the intersection numbers p_{ij}^k and the eigenvalues of the intersection matrix $B_i = (p_{ij}^k)$. In this paper, we determine the G -orbits on $X_1 \times X_1$ as the first step of our approach to the eigenvalue problem of Γ . We plan to determine the eigenvalues of Γ in the framework of the coherent configuration in a forthcoming paper. Throughout this paper, we keep the notation fixed as in this section.

2. The G -orbits on X_i

Let ϵ be a fixed non-square element in $GF(q)$. G acts on X_i by $M^g = {}^t g M g$ ($g \in G, M \in X_i$) and it is well known that G has four orbits on X_i represented by

$$\begin{pmatrix} J_{2i-1}^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} J_{2i-1}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} J_{2i}^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} J_{2i}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

respectively, where

$$\begin{aligned} J_{2i-1}^+ &= \left(\begin{array}{cc|c} \mathbf{0} & I_{i-1} & 0 \\ I_{i-1} & \mathbf{0} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \quad (\text{size } 2i-1), \\ J_{2i-1}^- &= \left(\begin{array}{cc|c} \mathbf{0} & I_{i-1} & 0 \\ I_{i-1} & \mathbf{0} & 0 \\ \hline 0 & 0 & \epsilon \end{array} \right) \quad (\text{size } 2i-1), \\ J_{2i}^+ &= \begin{pmatrix} \mathbf{0} & I_i \\ I_i & \mathbf{0} \end{pmatrix} \quad (\text{size } 2i), \\ J_{2i}^- &= \left(\begin{array}{cc|cc} \mathbf{0} & I_{i-1} & 0 & 0 \\ I_{i-1} & \mathbf{0} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\epsilon \end{array} \right) \quad (\text{size } 2i) \end{aligned}$$

with I_j the identity matrix of size j and $\mathbf{0}$ the zero matrix of the corresponding size.

Let J be one of J_{2i-1}^+ , J_{2i-1}^- , J_{2i}^+ , J_{2i}^- and set

$$O_m(q, J) = \{g \in GL_m(q) \mid {}^t g J g = J\},$$

where m is the matrix size of J . For natural numbers k, l , let $\mathcal{M}_{k,l}(q)$ denote the set of $k \times l$ matrices over $GF(q)$. Then the stabilizer of

$$\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \in X_i \text{ in } G \text{ is}$$

$$G_J = \left\{ g \in G \mid g = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \begin{array}{ll} A \in O_m(q, J), & B \in \mathcal{M}_{n-m,m}(q), \\ C \in GL_{n-m}(q) \end{array} \right\}.$$

Let X_J denote the G -orbit on X_i represented by $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Then $G_J \backslash G$, the set of cosets of G by G_J , is isomorphic to X_J as G -sets by the correspondence

$$G_J \cdot g \mapsto {}^t g \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} g.$$

Let $\mathcal{M}_{k,l}^*(q)$ denote the set of $k \times l$ matrices over $GF(q)$ of full rank, i.e. of rank j , where $j = \min\{k, l\}$. For $g \in G$, write g as

$$g = \begin{pmatrix} M \\ N \end{pmatrix}$$

with $M \in \mathcal{M}_{m,n}^*(q)$, $N \in \mathcal{M}_{n-m,n}^*(q)$. Then we have

$${}^t g \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} g = {}^t M J M.$$

LEMMA 2.1. For $g, g' \in G$, write g, g' as

$$g = \begin{pmatrix} M \\ N \end{pmatrix}, \quad g' = \begin{pmatrix} M' \\ N' \end{pmatrix}$$

with $M, M' \in \mathcal{M}_{m,n}^*(q)$, $N, N' \in \mathcal{M}_{n-m,n}^*(q)$. Then $g' = hg$ for some $h \in G_J$ if and only if $M' = AM$ for some $A \in O_m(q, J)$.

Proof. Suppose $g' = hg$ for some $h \in G_J$. Then

$$h = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

for some $A \in O_m(q, J)$, $B \in \mathcal{M}_{n-m,m}(q)$, $C \in GL_{n-m}(q)$. So,

$$\begin{pmatrix} M' \\ N' \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}$$

and we have $M' = AM$. Conversely, suppose that $M' = AM$ for some $A \in O_m(q, J)$. Then

$${}^t g' \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} g' = {}^t M' J M' = {}^t M J M = {}^t g \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} g,$$

since ${}^t A J A = J$. So g, g' belong to the same G_J -coset. $\square$

The set $\mathcal{M}_{m,n}^*(q)$ admits the action of $O_m(q, J)$ from the left and that of G from the right. Let $O_m(q, J) \backslash \mathcal{M}_{m,n}^*(q)$ denote the set of orbits of $O_m(q, J)$ on $\mathcal{M}_{m,n}^*(q)$. Then Lemma 2.1 states that the mapping $G_J \cdot g \mapsto O_m(q, J)M(g = \begin{pmatrix} M \\ N \end{pmatrix})$ from $G_J \backslash G$ to $O_m(q, J) \backslash \mathcal{M}_{m,n}^*(q)$ is an isomorphism of G -sets.

Thus we have the following lemma.

LEMMA 2.2. $O_m(q, J) \backslash \mathcal{M}_{m,n}^*(q)$ is isomorphic to X_J as G -sets by the correspondence

$$O_m(q, J)M \mapsto {}^t M J M.$$

For G -orbits $X_J, X_{J'}$, G acts on $X_J \times X_{J'}$ by $(M, M')^g = (M^g, M'^g)$. The G -orbits on $X_J \times X_{J'}$ are in one-to-one correspondence with the G_J -orbits on $X_{J'}$, since the stabilizer of

$$\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \in X_J$$

in G is G_J . By Lemma 2.2, $X_{J'}$ is isomorphic to $O_{m'}(q, J') \backslash \mathcal{M}_{m',n}^*(q)$, where m' is the matrix size of J' . Let $O_{m'}(q, J') \backslash \mathcal{M}_{m',n}^*(q) / G_J$ denote the G_J -orbits on $O_{m'}(q, J') \backslash \mathcal{M}_{m',n}^*(q)$, i.e. the $(O_{m'}(q, J') \times G_J)$ -orbits on $\mathcal{M}_{m',n}^*(q)$. Then we have the following.

COROLLARY 2.3. The G -orbits on $X_J \times X_{J'}$ are in one-to-one correspondence with $O_{m'}(q, J') \backslash \mathcal{M}_{m',n}^*(q) / G_J$.

Let us set

$$H_J = \left\{ g \in G \mid g = \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}, A \in O_m(q, J) \right\},$$

$$K_m(q) = \left\{ g \in G \mid g = \begin{pmatrix} I_m & 0 \\ 0 & C \end{pmatrix}, C \in GL_{n-m}(q) \right\},$$

$$U_m(q) = \left\{ g \in G \mid g = \begin{pmatrix} I_m & 0 \\ B & I_{n-m} \end{pmatrix}, B \in \mathcal{M}_{n-m,m}(q) \right\}.$$

Then $G_J = U_m(q) \rtimes (H_J \times K_m(q))$, the semi-direct product of $U_m(q)$ by $H_J \times K_m(q)$. We frequently use this decomposition later.

In the next section, we specialize $i = 1$ so that J is one of

$$J_1^+ = 1, \quad J_1^- = \epsilon, \quad J_2^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2^- = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

For $J = J_1^\pm$, we have $O_1(q, J) = \{\pm 1\}$. For $J = J_2^+$, we have

$$O_2(q, J) = O_2^+(q) = \left\langle \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid x \in GF(q)^\times \right\rangle,$$

where $GF(q)^\times$ is the set of non-zero elements of $GF(q)$. For $J = J_2^-$, we have

$$O_2(q, J) = O_2^-(q) = \left\langle \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mid x, y \in GF(q), x^2 - \epsilon y^2 = 1 \right\rangle.$$

Let F denote the extension field of $GF(q)$ by $\sqrt{\epsilon}$, a square root of ϵ . Since F is a finite field of q^2 elements, we may use $GF(q^2)$ in place of F . Set

$$P = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{\epsilon} & -1/\sqrt{\epsilon} \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{\epsilon} \\ 1 & -\sqrt{\epsilon} \end{pmatrix}.$$

Then P is an element of $GL_2(q^2)$, the set of non-singular matrices over $GF(q^2)$ of size 2. Observe that J_2^+, J_2^- are in the same orbit of $GL_2(q^2)$; in fact, it holds that ${}^tPJ_2^-P = 2J_2^+$ and 2 is a square in $GF(q^2)^\times$, since the mapping $\xi \mapsto \xi^{q+1}$ from $GF(q^2)^\times$ to $GF(q)^\times$ is surjective. Thus we have

$$P^{-1}O_2(q, J_2^-)P \subset O_2(q^2, J_2^+).$$

Let σ denote the generator of the Galois group of $GF(q^2)$ over $GF(q)$, i.e. $\sigma : \xi \mapsto \xi^q$. We use the following lemma frequently in the next section.

LEMMA 2.4.

(i)

$$\left\{ P^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in GF(q) \right\} = \left\{ \begin{pmatrix} \alpha \\ \alpha^\sigma \end{pmatrix} \mid \alpha \in GF(q^2) \right\}.$$

(ii)

$$\{(a, b)P \mid a, b \in GF(q)\} = \{(\alpha, \alpha^\sigma) \mid \alpha \in GF(q^2)\}.$$

(iii)

$$\left\{ P^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \mid a, b, c, d \in GF(q) \right\} = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^\sigma & \alpha^\sigma \end{pmatrix} \mid \alpha, \beta \in GF(q^2) \right\}.$$

Proof.(i) For $\alpha, \beta \in GF(q^2)$, set

$$\begin{pmatrix} a \\ b \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Then a, b belong to $GF(q)$, i.e. $a^\sigma = a, b^\sigma = b$ if and only if

$$P^\sigma \begin{pmatrix} \alpha^\sigma \\ \beta^\sigma \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Since $\sqrt{\epsilon}^\sigma = -\sqrt{\epsilon}$, we have

$$P^{-1} P^\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the desired result.

(ii) The same proof as in (i) is valid.

(iii) For $\alpha, \beta, \gamma, \delta \in GF(q^2)$, set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = P \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P^{-1}.$$

Then a, b, c, d belong to $GF(q)$ if and only if

$$P^\sigma \begin{pmatrix} \alpha^\sigma & \beta^\sigma \\ \gamma^\sigma & \delta^\sigma \end{pmatrix} (P^{-1})^\sigma = P \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P^{-1}.$$

Since

$$P^{-1} P^\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (P^{-1})^\sigma P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain the desired result. □

Since

$$O_2(q^2, J_2^+) = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid \xi \in GF(q^2)^\times \right\rangle,$$

we have, by Lemma 2.4(iii),

$$P^{-1} O_2(q, J_2^-) P = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid \xi \in GF(q^2)^\times, \xi^\sigma = \xi^{-1} \right\rangle.$$

3. The G -orbits on $X_1 \times X_1$

We keep the same notation as in Sections 1 and 2. As we saw in Section 2, G has four orbits on X_1 . They are X_J for $J \in \{J_1^+, J_1^-, J_2^+, J_2^-\}$. Recall that

$$J_1^+ = 1, \quad J_1^- = \epsilon, \quad J_2^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2^- = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

Recall that X_J is the G -orbit containing $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$. Thus we have

$$X_1 \times X_1 = \bigcup_{J, J'} X_J \times X_{J'} \quad (\text{disjoint union}),$$

where J, J' run through $\{J_1^\pm, J_2^\pm\}$. In order to determine the G -orbits on $X_1 \times X_1$, it suffices to determine the G -orbits on each $X_J \times X_{J'}$. Note that the G -orbits on $X_J \times X_{J'}$ and those on $X_{J'} \times X_J$ are in one-to-one correspondence by the mapping $(M, M') \mapsto (M', M)$. Since $X_{J_1^-} = \epsilon X_{J_1^+}$, the G -orbits on $X_J \times X_{J_1^+}$ and those on $X_J \times X_{J_1^-}$ are in one-to-one correspondence by the mapping $(M, M') \mapsto (M, \epsilon M')$. Thus among the 16 $X_J \times X_{J'}$, we only need to treat $X_J \times X_{J'}$ for $(J, J') \in \{(J_1^+, J_1^+), (J_2^\pm, J_1^+), (J_2^\pm, J_2^\pm), (J_2^-, J_2^-)\}$.

By Corollary 2.3, we determine the $(O_{m'}(q, J') \times G_J)$ -orbits on $\mathcal{M}_{m',n}^*(q)$ instead of the G -orbits on $X_J \times X_{J'}$ ($m' = 1, 2$). For $M \in \mathcal{M}_{m',n}^*(q)$, write M as $M = (M_0, M_1)$ with $M_0 \in \mathcal{M}_{m',m'}(q)$, $M_1 \in \mathcal{M}_{m',n-m'}(q)$. We call M of type I, II, III, respectively (for $m' = 1$, only type I and II appear), as rank $M_1 = m', m' - 1, m' - 2$. Let $\Omega_I, \Omega_{II}, \Omega_{III}$ denote the set of matrices in $\mathcal{M}_{m',n}^*(q)$ of type I, II, III, respectively. Apparently, each of $\Omega_I, \Omega_{II}, \Omega_{III}$ is fixed as a set by the action of $(O_{m'}(q, J') \times G_J)$.

3.1. The case $(J, J') = (J_1^+, J_1^+)$

We apply Corollary 2.3 with $J = J' = J_1^+$, $m' = 1$, $O_{m'}(q, J') = \{\pm 1\}$, $G_J = U_1(q) \rtimes (H_J \times K_1(q))$, where

$$H_J = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \right\rangle.$$

Apparently, we have the following theorem.

THEOREM 3.1. $O_1(q, J_1^+) \backslash \mathcal{M}_{1,n}^*(q) / G_{J_1^+}$ consists of:

- (i) $\{(a, \mathbf{b}) \mid a \in GF(q), \mathbf{b} \in \mathcal{M}_{1,n-1}^*(q)\}$;
- (ii) $\{(D, 0, \dots, 0), (-D, 0, \dots, 0)\}$ for $D \in GF(q)^\times / \sim$, where $GF(q)^\times = GF(q) - \{0\}$, and $D' \sim D$ is defined by $D' = \pm D$.

There are $1 + (q - 1)/2$ orbits.

3.2. The case $(J, J') = (J_2^+, J_1^+)$

We apply Corollary 2.3 with

$$J = J_2^+, \quad J' = J_1^+, \quad m' = 1, \quad O_{m'}(q, J') = \{\pm 1\},$$

$$G_J = U_2(q) \rtimes (H_J \times K_2(q)), \quad H_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid A \in O_2^+(q) \right\}.$$

THEOREM 3.2. $O_1(q, J_1^+) \backslash \mathcal{M}_{1,n}^*(q) / G_{J_2^+}$ consists of:

- (i) $\Omega_1^0 = \{(a, b, \mathbf{c}) \mid a, b \in GF(q), \mathbf{c} \in \mathcal{M}_{1,n-2}^*(q)\};$
- (ii) $\Omega_{\Pi}^+(D) = \{(a, b, 0, \dots, 0) \mid a, b \in GF(q), ab = D, (a, b) \neq (0, 0)\}$ for $D \in GF(q).$

There are $1 + q$ orbits.

Proof. Apparently, Ω_1^0 and $\Omega_{\Pi}^+(D)$ are fixed as a set by the action of $O_1(q, J_1^+) = \{\pm 1\}$ and that of $G_J = U_2(q) \rtimes (H_J \times K_2(q)).$

Every element of Ω_1^0 is sent to $(a, b, 1, 0, \dots, 0)$ for some $a, b \in GF(q)$ by the action of $K_2(q)$. Any element $(a, b, 1, 0, \dots, 0)$ is sent to $(0, 0, 1, 0, \dots, 0)$ by $U_2(q)$. Thus Ω_1^0 is an orbit.

Every element of $\Omega_{\Pi}^+(D)$ is sent to $(1, D, 0, \dots, 0)$ by the action of H_J , since $O_2^+(q)$ is generated by

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} (x \in GF(q)^\times) \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So $\Omega_{\Pi}^+(D)$ is an orbit. □

3.3. The case $(J, J') = (J_2^-, J_1^+)$

We apply Corollary 2.3 with

$$J = J_2^-, \quad J' = J_1^+, \quad m' = 1, \quad O_{m'}(q, J') = \{\pm 1\},$$

$$G_J = U_2(q) \rtimes (H_J \times K_2(q)), \quad H_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid A \in O_2^-(q) \right\}.$$

Recall that

$$P = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{\epsilon} & -1/\sqrt{\epsilon} \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{\epsilon} \\ 1 & -\sqrt{\epsilon} \end{pmatrix}, \quad \langle \sigma \rangle = \text{Gal}(GF(q^2)/GF(q)),$$

and the Galois group of $GF(q^2)$ over $GF(q)$. Set

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & I_{n-2} \end{pmatrix} \in GL_n(q^2).$$

THEOREM 3.3. $O_1(q, J_1^+) \backslash \mathcal{M}_{1,n}^*(q) / G_{J_2^-}$ consists of:

- (i) $\Omega_1^0 = \{(a, b, \mathbf{c}) \mid a, b \in GF(q), \mathbf{c} \in \mathcal{M}_{1,n-2}^*(q)\};$
- (ii) $\Omega_{\Pi}^-(D) = \Lambda_{\Pi}^-(D) \tilde{P}^{-1}$ for $D \in GF(q)^\times$, where

$$\Lambda_{\Pi}^-(D) = \{(\alpha, \alpha^\sigma, 0, \dots, 0) \mid \alpha \in GF(q^2)^\times, \alpha\alpha^\sigma = D\}.$$

There are $1 + (q - 1)$ orbits.

Proof. The argument in the proof of Theorem 3.2 is valid for Ω_1^0 . So Ω_1^0 is an orbit.

We have $\mathcal{M}_{1,n}^*(q) - \Omega_1^0 = \{(a, b, 0, \dots, 0) \mid a, b \in GF(q), (a, b) \neq (0, 0)\}$, and $K_2(q)$, $U_2(q)$ act on it trivially. The action of H_J on $\mathcal{M}_{1,n}^*(q) - \Omega_1^0$ is isomorphic to that of $O_2^-(q)$ on $\mathcal{M}_{1,2}^*(q)$. The action of $O_2^-(q)$ on $\mathcal{M}_{1,2}^*(q)$ is isomorphic to that of $P^{-1} O_2^-(q) P$ on $\mathcal{M}_{1,2}^*(q) P$ by the correspondence $(M, A) \mapsto (MP, P^{-1}AP)$ from $\mathcal{M}_{1,2}^* \times O_2^-(q)$ to $\mathcal{M}_{1,2}^*(q) P \times P^{-1} O_2^-(q) P$. By Lemma 2.4(ii), we have $\mathcal{M}_{1,2}^* P = \{(\alpha, \alpha^\sigma) \mid \alpha \in GF(q^2)^\times\}$. As we saw in Section 2,

$$P^{-1} O_2^-(q) P = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid \xi \in GF(q^2)^\times, \xi^\sigma = \xi^{-1} \right\}.$$

So the orbits of $P^{-1} O_2^-(q) P$ on $\mathcal{M}_{1,2}^* P$ are

$$\Lambda(D) = \{(\alpha, \alpha^\sigma) \mid \alpha \in GF(q^2), \alpha\alpha^\sigma = D\} \quad \text{for } D \in GF(q)^\times.$$

Here we have used the fact that the mapping $\xi \mapsto \xi\xi^\sigma$ is a surjection from $GF(q^2)^\times$ to $GF(q)^\times$. Thus $\Lambda(D)P^{-1}$ ($D \in GF(q)^\times$) are the orbits of $O_2^-(q)$ on $\mathcal{M}_{1,2}^*(q)$ and the theorem follows. $\square$

3.4. The case $(J, J') = (J_2^+, J_2^+)$

We apply Corollary 2.3 with

$$J = J' = J_2^+, \quad m' = 2, \quad O_{m'}(q, J') = O_2^+(q),$$

$$G_J = U_2(q) \rtimes (H_J \times K_2(q)), \quad H_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid A \in O_2^+(q) \right\}.$$

Let χ denote the quadratic residue character of $GF(q)$, i.e. $\chi(x) = 0, 1$ or -1 , respectively, as x is 0, a square of $GF(q)^\times$ or a non-square of $GF(q)^\times$.

THEOREM 3.4. $O_2(q, J_2^+) \backslash \mathcal{M}_{2,n}^*(q) / G_{J_2^+}$ consists of:

- (i) $\Omega_I = \{(M_0, M_1) \mid M_0 \in \mathcal{M}_{2,2}(q), M_1 \in \mathcal{M}_{2,n-2}^*(q)\};$
- (ii) $\Omega_{II}^{++}(D_1, D_2) = \Lambda_{II}^{++}(D_1, D_2)K_2(q)$ for $(D_1, D_2) \in GF(q) \times GF(q)/\sim$,
where

$$\Lambda_{II}^{++}(D_1, D_2) = \left\{ (\mathbf{a}, \mathbf{b}, \mathbf{c}, 0, \dots, 0) \mid (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathcal{M}_{2,3}^*(q), \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \right. \\ \left. (\det(\mathbf{a}, \mathbf{c}) \det(\mathbf{b}, \mathbf{c}), c_1 c_2) \sim (D_1, D_2) \right\},$$

and $(D_1, D_2) \sim (D'_1, D'_2) \Leftrightarrow D'_1 = x^2 D_1, D'_2 = x^2 D_2$ for some $x \in GF(q)^\times$;

- (iii)

$$\Omega_{III}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\}) = \left\{ \begin{pmatrix} a & c & \mathbf{0} \\ b & d & \mathbf{0} \end{pmatrix} \in \mathcal{M}_{2,n}^*(q) \mid \{ad, bc\} = \{D_1, D_2\}, \right. \\ \left. \{\chi(ab), \chi(cd)\} = \{\kappa_1, \kappa_2\} \text{ as multisets} \right\}$$

for $D_1, D_2 \in GF(q)$ ($D_1 \neq D_2$), $\kappa_1, \kappa_2 \in \{0, 1, -1\}$ such that $\kappa_1 \kappa_2 = \chi(D_1 D_2)$.

There are $1 + (2q + 3) + \frac{1}{4}(q - 1)(3q + 5)$ orbits.

Proof. It is easily checked that the action of $O_2^+(q)$ fixes each of $\Omega_I, \Omega_{II}^{++}(D_1, D_2), \Omega_{III}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ as a set. Also $U_2(q), K_2(q), H_J$ fix each of $\Omega_I, \Omega_{II}^{++}(D_1, D_2), \Omega_{III}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ as a set, and so G_J fixes each of them as a set.

Every element of Ω_I is sent to $(M_0, I, \mathbf{0})$ for some $M_0 \in \mathcal{M}_{2,2}(q)$ by $K_2(q)$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any element $(M_0, I, \mathbf{0})$ is sent to $(0, I, \mathbf{0})$ by $U_2(q)$. So, Ω_I is an orbit.

Every element of $\Omega_{II}^{++}(D_1, D_2)$ is, by the action of $K_2(q)$, sent to $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0})$ such that

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $c_1 c_2 = D_2$. By the action of $O_2^+(q)$, we may assume that $c_1 = D_2, c_2 = 1$. Any such element $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0})$ is sent to $\begin{pmatrix} a_1 & b_1 & D_2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{0}$ for some $a_1, b_1 \in GF(q)$ by the action of $U_2(q)$. Since it belongs to $\Omega_{II}^{++}(D_1, D_2)$, we have $a_1 b_1 = D_1$. The action of H_J sends $\begin{pmatrix} a_1 & b_1 & D_2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{0}$ to $\begin{pmatrix} 1 & D_1 & D_2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{0}$. Thus $\Omega_{II}^{++}(D_1, D_2)$ is an orbit.

For

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Omega_{\text{III}}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\}),$$

we may assume that $ad = D_1, bc = D_2, \chi(ab) = \kappa_1, \chi(cd) = \kappa_2$ by applying

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in O_2^+(q), \quad \left(\begin{array}{cc|c} 0 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-2} \end{array} \right) \in H_J.$$

Given $\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}$ such that $ad = a'd' = D_1, bc = b'c' = D_2, \chi(ab) = \chi(a'b') = \kappa_1, \chi(cd) = \chi(c'd') = \kappa_2$, we claim that there exist $x, y \in GF(q)^\times$ such that

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix},$$

i.e. $a' = xya, b' = x^{-1}yb, c' = xy^{-1}c, d' = x^{-1}y^{-1}d$. Suppose that $\kappa_1 \neq 0$. Then a, b, a', b' are non-zero by $\chi(ab) = \chi(a'b') = \kappa_1$. Let $\sqrt{a}, \sqrt{b}, \sqrt{a'}, \sqrt{b'} \in GF(q^2)^\times$ be square roots of a, b, a', b' and set $x = \sqrt{a'}\sqrt{b}/\sqrt{a}\sqrt{b'}, y = \sqrt{a'}\sqrt{b'}/\sqrt{a}\sqrt{b}$. Then $x^2 = a'b/ab', y^2 = a'b'/ab$ are squares in $GF(q)^\times$ by $\chi(ab) = \chi(a'b')$. So x, y are contained in $GF(q)^\times$. It holds that $xya = a', x^{-1}yb = b', xy^{-1}c = bc/b' = c', x^{-1}y^{-1}d = ad/a' = d'$, since $bc = b'c' = D_2, ad = a'd' = D_1$. So x, y are desired. Suppose that $\kappa_2 \neq 0$. Then c, d, c', d' are non-zero and $x = \sqrt{c'}\sqrt{d}/\sqrt{c}\sqrt{d'}, y = \sqrt{c}\sqrt{d}/\sqrt{c'}\sqrt{d'}$ are desired. Suppose that $\kappa_1 = \kappa_2 = 0$. If $D_1 \neq 0$, then $ad = a'd' = D_1 \neq 0, b = c = 0, b' = c' = 0$, and so $x = a'/a = d/d', y = 1$ meet the requirement. If $D_2 \neq 0$, then $bc = b'c' = D_2 \neq 0, a = d = 0, a' = d' = 0$, and so $x = b/b' = c'/c, y = 1$ meet the requirement. The case $D_1 = D_2 = 0$ does not occur, since $D_1 \neq D_2$ due to the non-singularity of the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Thus the claim is proved and $\Omega_{\text{III}}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ is an orbit.

Apparently, each element of $\mathcal{M}_{2,n}^*(q)$ belongs to exactly one of $\Omega_{\text{I}}, \Omega_{\text{II}}^{++}(D_1, D_2), \Omega_{\text{III}}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$. Thus they are the orbits of $O_2^+(q) \times G_{J_2^+}$ on $\mathcal{M}_{2,n}^*(q)$.

For the orbits $\Omega_{\text{II}}^{++}(D_1, D_2)$, the parameter (D_1, D_2) ranges over $GF(q) \times GF(q)/\sim$, where $(D_1, D_2) \sim (D'_1, D'_2) \Leftrightarrow D'_1 = x^2D_1, D'_2 = x^2D_2$ for some $x \in GF(q)^\times$. For $(D_1, D_2) \neq (0, 0)$, the equivalence class to which (D_1, D_2) belongs contains $(q-1)/2$ elements, while $(0, 0)$ is itself an equivalence class. Since $1 + 2(q^2 - 1)/(q - 1) = 2q + 3$, there are $2q + 3$ equivalence classes, i.e. $2q + 3$ orbits $\Omega_{\text{II}}^{++}(D_1, D_2)$.

For the orbits $\Omega_{\text{III}}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$, the parameter $(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ ranges over $D_1, D_2 \in GF(q)$ ($D_1 \neq D_2$), $\kappa_1, \kappa_2 \in \{0, 1, -1\}$ ($\kappa_1 \kappa_2 = \chi(D_1 D_2)$). The number of $\{D_1, D_2\}$ such that D_1, D_2 are distinct squares in $GF(q)^\times$ is $\frac{1}{2}((q-1)/2)((q-1)/2 - 1)$. For each of such $\{D_1, D_2\}$, the choice of $\{\kappa_1, \kappa_2\}$ such that $\kappa_1 \kappa_2 = \chi(D_1 D_2)$ is either $\{1, 1\}$ or $\{-1, -1\}$. Thus there are $(q-1)/2((q-1)/2 - 1)$ parameters of this type. Similarly there are $(q-1)/2((q-1)/2 - 1)$ parameters $(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ such that D_1, D_2 are distinct non-squares in $GF(q)^\times$. The number of $\{D_1, D_2\}$ such that one of D_1, D_2 is a square in $GF(q)^\times$ and the other is a non-square in $GF(q)^\times$ is $(q-1)/2((q-1)/2)$. For each of such $\{D_1, D_2\}$, the choice of $\{\kappa_1, \kappa_2\}$ such that $\kappa_1 \kappa_2 = \chi(D_1 D_2)$ is only $\{1, -1\}$. Thus there are $(q-1)/2((q-1)/2)$ parameters of this type. The number of $\{D_1, D_2\}$ such that one of D_1, D_2 is zero and the other is non-zero is $q-1$. For each of such $\{D_1, D_2\}$, the choice of $\{\kappa_1, \kappa_2\}$ such that $\kappa_1 \kappa_2 = \chi(D_1 D_2)$ is $\{0, 0\}$, $\{0, 1\}$ or $\{0, -1\}$. Thus there are $3(q-1)$ parameters of this type. Therefore the number of orbits $\Omega_{\text{III}}^{++}(\{D_1, D_2\}, \{\kappa_1, \kappa_2\})$ is

$$\begin{aligned} & \frac{q-1}{2} \left(\frac{q-1}{2} - 1 \right) + \frac{q-1}{2} \left(\frac{q-1}{2} - 1 \right) + \frac{q-1}{2} \frac{q-1}{2} + 3(q-1) \\ &= \frac{1}{4}(q-1)(3q+5). \end{aligned} \quad \square$$

3.5. The case $(J, J') = (J_2^+, J_2^-)$

We apply Corollary 2.3 with

$$\begin{aligned} J &= J_2^+, \quad J' = J_2^-, \quad m' = 2, \quad O_{m'}(q, J') = O_2^-(q), \\ G_J &= U_2(q) \rtimes (H_J \times K_2(q)), \quad H_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid A \in O_2^+(q) \right\}. \end{aligned}$$

Let χ denote the quadratic residue character of $GF(q)$.

Recall that

$$P = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{\epsilon} & -1/\sqrt{\epsilon} \end{pmatrix}$$

and $\langle \sigma \rangle = \text{Gal}(GF(q^2)/GF(q))$, the Galois group of $GF(q^2)$ over $GF(q)$.

THEOREM 3.5. $O_2(q, J_2^-) \backslash \mathcal{M}_{2,n}^*(q) / G_{J_2^+}$ consists of:

- (i) $\Omega_1 = \{(M_0, M_1) \mid M_0 \in \mathcal{M}_{2,2}(q), M_1 \in \mathcal{M}_{2,n-2}^*(q)\};$

(ii) $\Omega_{\Pi}^{+-}(D, \kappa) = P\Lambda_{\Pi}^{+-}(D, \kappa)K_2(q)$; for $(D, \kappa) \in GF(q) \times \{\pm 1\}$, where

$$\Lambda_{\Pi}^{+-}(D, \kappa) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \mathbf{0} \\ \alpha^\sigma & \beta^\sigma & \gamma^\sigma & \mathbf{0} \end{pmatrix} \in \mathcal{M}_{2,n}^*(q^2) \mid \chi(\gamma\gamma^\sigma) = \kappa, \right. \\ \left. \frac{(\alpha\gamma^\sigma - \alpha^\sigma\gamma)(\beta\gamma^\sigma - \beta^\sigma\gamma)}{\gamma\gamma^\sigma} = D \right\};$$

(iii) $\Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\}) = P\Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ for $D \in GF(q^2)^\times - GF(q)^\times$, $\kappa_1, \kappa_2 \in \{\pm 1\}$ such that $\chi(DD^\sigma) = \kappa_1\kappa_2$, where

$$\Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\}) = \left\{ \begin{pmatrix} \alpha & \beta & \mathbf{0} \\ \alpha^\sigma & \beta^\sigma & \mathbf{0} \end{pmatrix} \in \mathcal{M}_{2,n}^*(q^2) \mid \{\alpha\beta^\sigma, \alpha^\sigma\beta\} \right. \\ = \{D, D^\sigma\}, \{\chi(\alpha\alpha^\sigma), \chi(\beta\beta^\sigma)\} \\ \left. = \{\kappa_1, \kappa_2\} \text{ as multisets} \right\}.$$

There are $1 + 2q + \frac{1}{4}(q-1)(3q-1)$ orbits.

Proof. For $M \in \mathcal{M}_{2,n}^*(q)$, write M as $M = (M_0, M_1)$ with $M_0 \in \mathcal{M}_{2,2}(q)$, $M_1 \in \mathcal{M}_{2,n-2}(q)$. Recall that we call M of type I, II, III, respectively, as $\text{rank } M_1 = 2, 1, 0$. Let $\Omega_{\text{I}}, \Omega_{\text{II}}, \Omega_{\text{III}}$ denote the set of matrices in $\mathcal{M}_{2,n}^*(q)$ of type I, II, III, respectively. Apparently, each of $\Omega_{\text{I}}, \Omega_{\text{II}}, \Omega_{\text{III}}$ is fixed as a set by $O_2^-(q), G_J$.

The argument in the proof of Theorem 3.4 is valid for Ω_{I} . So Ω_{I} is an orbit of G_J , and hence of $O_2^-(q) \times G_J$.

Each of $\Omega_{\text{II}}^{+-}(D, \kappa), \Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ is fixed as a set by each of $U_2(q), K_2(q), H_J$ and hence by G_J . Also, each of them is fixed as a set by $O_2^-(q)$; to see this fact, note that the action of $O_2^-(q)$ on $\mathcal{M}_{2,n}^*(q)$ is isomorphic to that of $P^{-1}O_2^-(q)P$ on $P^{-1}\mathcal{M}_{2,n}^*(q)$ by the correspondence $(A, M) \mapsto (P^{-1}AP, P^{-1}M)$ from $O_2^-(q) \times \mathcal{M}_{2,n}^*(q)$ to $P^{-1}O_2^-(q)P \times P^{-1}\mathcal{M}_{2,n}^*(q)$, and check each of $\Lambda_{\text{II}}^{+-}(D, \kappa), \Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ is fixed as a set by $P^{-1}O_2^-(q)P$, using

$$P^{-1}O_2^-(q)P = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid \xi \in GF(q^2)^\times, \xi^\sigma = \xi^{-1} \right\rangle.$$

To show that $\Omega_{\text{II}}^{+-}(D, \kappa)$ is an orbit of $O_2^-(q) \times G_J$, it suffices to show that $P^{-1}O_2^-(q)P \times U_2(q)H_J$ is transitive on $\Lambda_{\text{II}}^{+-}(D, \kappa)K_2(q)$. Note that the mapping $\xi \mapsto \xi\xi^\sigma$ from $GF(q^2)^\times$ to $GF(q)^\times$ is surjective and has the kernel $\{\xi \in GF(q^2)^\times \mid \xi^\sigma = \xi^{-1}\}$. Fix $\gamma_0 \in GF(q^2)^\times$ such that $\chi(\gamma_0\gamma_0^\sigma) = \kappa$. Then, since any $\gamma \in GF(q^2)^\times$ such that $\chi(\gamma\gamma^\sigma) = \kappa$ can be written as $\gamma = \xi t\gamma_0$ for some

$t \in GF(q)^\times, \xi \in GF(q^2)^\times$ with $\xi^\sigma = \xi^{-1}$, every element of $\Lambda_{\Pi}^{+-}(D, \kappa)$ is sent to

$$\begin{pmatrix} \alpha & \beta & \gamma_0 & \mathbf{0} \\ \alpha^\sigma & \beta^\sigma & \gamma_0^\sigma & \mathbf{0} \end{pmatrix} \in \Lambda_{\Pi}^{+-}(D, \kappa)$$

for some $\alpha, \beta \in GF(q^2)$ by the action of $P^{-1}O_2^-(q)P$ and $K_2(q)$. Then we have

$$\frac{\alpha}{\gamma_0} - \left(\frac{\alpha}{\gamma_0}\right)^\sigma = a\sqrt{\epsilon}, \quad \frac{\beta}{\gamma_0} - \left(\frac{\beta}{\gamma_0}\right)^\sigma = b\sqrt{\epsilon}$$

for some $a, b \in GF(q)$, since any element $\xi \in GF(q^2)$ such that $\xi^\sigma = -\xi$ can be written as $\xi = c\sqrt{\epsilon}$ for some $c \in GF(q)$. By the action of H_J , we may assume that $a = 1, b = D/\epsilon\gamma_0\gamma_0^\sigma$, since

$$\left(\frac{\alpha}{\gamma_0} - \frac{\alpha^\sigma}{\gamma_0^\sigma}\right)\left(\frac{\beta}{\gamma_0} - \frac{\beta^\sigma}{\gamma_0^\sigma}\right) = \frac{D}{\gamma_0\gamma_0^\sigma} \quad \text{and} \quad a = b = 0$$

cannot occur due to the condition that $\begin{pmatrix} \alpha & \beta & \gamma_0 \\ \alpha^\sigma & \beta^\sigma & \gamma_0^\sigma \end{pmatrix}$ is of rank 2. Since the mapping $\xi \mapsto (\xi - \xi^\sigma)/\sqrt{\epsilon}$ from $GF(q^2)$ to $GF(q)$, i.e. the mapping $x + y\sqrt{\epsilon} \mapsto 2y$ ($x, y \in GF(q)$), is surjective and has the kernel $GF(q)$ as an additive group, the equation $\xi - \xi^\sigma = a\sqrt{\epsilon}$ has a unique solution $\xi_0 \bmod GF(q)$, i.e. $\alpha = \alpha_0 + s\gamma_0$ ($\xi = \alpha/\gamma_0, \xi_0 = \alpha_0/\gamma_0, s \in GF(q)$). Also, the equation $\xi - \xi^\sigma = b\sqrt{\epsilon}$ has a unique solution $\xi_1 \bmod GF(q)$, i.e. $\beta = \beta_0 + t\gamma_0$ ($\xi = \beta/\gamma_0, \xi_1 = \beta_0/\gamma_0, t \in GF(q)$). By the action of $U_2(q)$, $\begin{pmatrix} \alpha & \beta & \gamma_0 \\ \alpha^\sigma & \beta^\sigma & \gamma_0^\sigma \end{pmatrix}$ is sent to $\begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_0^\sigma & \beta_0^\sigma & \gamma_0^\sigma \end{pmatrix}$. Thus $P^{-1}O_2^-(q)P \times U_2(q)H_J$ is transitive on $\Lambda_{\Pi}^{+-}(D, \kappa)K_2(q)$.

To show that $\Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ is an orbit of $O_2^-(q) \times G_J$, it suffices to show that $P^{-1}O_2^-(q)P \times H_J$ is transitive on $\Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$. For

$$\begin{pmatrix} \alpha & \beta & \mathbf{0} \\ \alpha^\sigma & \beta^\sigma & \mathbf{0} \end{pmatrix} \in \Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\}),$$

we may assume that $\alpha\beta^\sigma = D, \alpha^\sigma\beta = D^\sigma, \chi(\alpha\alpha^\sigma) = \kappa_1, \chi(\beta\beta^\sigma) = \kappa_2$ by applying

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in P^{-1}O_2^-(q)P, \quad \left(\begin{array}{cc|c} 0 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-2} \end{array} \right) \in H_J.$$

For

$$\begin{pmatrix} \alpha & \beta & \mathbf{0} \\ \alpha^\sigma & \beta^\sigma & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' & \mathbf{0} \\ \alpha'^\sigma & \beta'^\sigma & \mathbf{0} \end{pmatrix} \in \Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$$

with $\alpha\beta^\sigma = \alpha'\beta'^\sigma = D$, $\chi(\alpha\alpha^\sigma) = \chi(\alpha'\alpha'^\sigma) = \kappa_1$, $\chi(\beta\beta^\sigma) = \chi(\beta'\beta'^\sigma) = \kappa_2$, we claim that there exist $\xi \in GF(q^2)^\times$ ($\xi^\sigma = \xi^{-1}$), $z \in GF(q)^\times$ such that

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \alpha^\sigma & \beta^\sigma \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \alpha'^\sigma & \beta'^\sigma \end{pmatrix},$$

i.e. $\alpha' = \xi z \alpha$, $\beta' = \xi z^{-1} \beta$. Since $\chi(\alpha\alpha^\sigma) = \chi(\alpha'\alpha'^\sigma) \neq 0$, there exists $z \in GF(q)^\times$ such that $\alpha'\alpha'^\sigma/\alpha\alpha^\sigma = z^2$. Set $\xi = \alpha'/\alpha z$. Then $\xi\xi^\sigma = \alpha'\alpha'^\sigma/\alpha\alpha^\sigma z^2 = 1$ and $\xi z^{-1} \beta = \alpha' \beta / \alpha z^2 = \alpha^\sigma \beta / \alpha'^\sigma = \beta'$ by $\alpha^\sigma \beta = \alpha'^\sigma \beta' = D^\sigma$. Thus these z, ξ meet the requirement. So the claim is proved and hence $P^{-1}O_2^-(q)P \times H_J$ is transitive on $\Lambda_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$.

Apparently, each matrix of type II belongs to exactly one of $\Omega_{\text{II}}^{+-}(D, \kappa)$ and each matrix of type III belongs to exactly one of $\Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$. Thus the theorem is proved except for the number of orbits.

We count the number of orbits $\Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$. Let φ denote the mapping $\xi \mapsto \xi\xi^\sigma$ from $GF(q^2)^\times$ to $GF(q)^\times$. φ is surjective and $\ker \varphi$ is of order $q+1$. $\text{Im } \varphi|_{GF(q)^\times}$ is the set of squares of $GF(q)^\times$ and $\ker \varphi|_{GF(q)^\times}$ is of order 2. So for a square s of $GF(q)^\times$, there are $(q+1)-2$ elements $D \in GF(q^2)^\times - GF(q)^\times$ such that $DD^\sigma = s$, i.e. $(q-1)/2$ such pairs $\{D, D^\sigma\}$. For such a pair $\{D, D^\sigma\}$, the choice of $\{\kappa_1, \kappa_2\}$ such that $\chi(DD^\sigma) = \kappa_1\kappa_2$ is either $\{1, 1\}$ or $\{-1, -1\}$. Since there are $(q-1)/2$ squares of $GF(q)^\times$, the number of $(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ such that $\chi(DD^\sigma) = \kappa_1\kappa_2 = 1$ is $(q-1)/2((q-1)/2) \times 2$. For a non-square $t \in GF(q)^\times$, there are $q+1$ elements $D \in GF(q^2)^\times - GF(q)^\times$ such that $DD^\sigma = t$, i.e. $(q+1)/2$ such pairs $\{D, D^\sigma\}$. For such a pair $\{D, D^\sigma\}$, the choice of $\{\kappa_1, \kappa_2\}$ such that $\chi(DD^\sigma) = \kappa_1\kappa_2$ is only $\{1, -1\}$. Since there are $(q-1)/2$ non-squares of $GF(q)^\times$, the number of $(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ such that $\chi(DD^\sigma) = \kappa_1\kappa_2 = -1$ is $(q-1)/2((q+1)/2)$. Thus the number of orbits $\Omega_{\text{III}}^{+-}(\{D, D^\sigma\}, \{\kappa_1, \kappa_2\})$ is

$$\frac{q-1}{2} \frac{q-1}{2} \times 2 + \frac{q-1}{2} \frac{q+1}{2} = \frac{1}{4}(q-1)(3q-1).$$

Apparently, the number of orbits $\Omega_{\text{I}}, \Omega_{\text{II}}^{+-}(D, \kappa)$ is 1, $2q$, respectively. $\square$

3.6. The case $(J, J') = (J_2^-, J_2^-)$

We apply Corollary 2.3 with

$$J = J' = J_2^-, \quad m' = 2, \quad O_{m'}(q, J') = O_2^-(q),$$

$$G_J = U_2(q) \rtimes (H_J \times K_2(q)), \quad H_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \mid A \in O_2^-(q) \right\}.$$

Recall that

$$P = \begin{pmatrix} 1 & 1 \\ 1/\sqrt{\epsilon} & -1/\sqrt{\epsilon} \end{pmatrix}, \quad \langle \sigma \rangle = \text{Gal}(GF(q^2)/GF(q)),$$

and χ is the quadratic residue character of $GF(q)$. Set

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & I_{n-2} \end{pmatrix} \in GL_n(q^2).$$

Set

$$\begin{aligned} GF(q)^\sim &= \{0\} \cup GF(q)^\times \cup \sqrt{\epsilon} GF(q)^\times = \{\xi \in GF(q^2) \mid \xi^2 \in GF(q)\} \\ &= \{\xi^{(q+1)/2} \mid \xi \in GF(q^2)\}. \end{aligned}$$

Then the scalar -1 acts on $GF(q)^\sim \times GF(q)^\sim$ by $(D_1, D_2) \mapsto (-D_1, -D_2)$ and σ acts on $GF(q)^\sim \times GF(q)^\sim$ by $(D_1, D_2) \mapsto (D_1^\sigma, D_2^\sigma)$. Let τ denote the mapping $(D_1, D_2) \mapsto (D_2, D_1)$ from $GF(q)^\sim \times GF(q)^\sim$ onto itself. Let $GF(q)^\sim \times GF(q)^\sim / \sim$ denote the set of orbits of the group $\langle -1 \rangle \times \langle \sigma \rangle \times \langle \tau \rangle$ on $GF(q)^\sim \times GF(q)^\sim$.

THEOREM 3.6. $O_2(q, J_2^-) \backslash \mathcal{M}_{2,n}^*(q) / G_{J_2^-}$ consists of:

- (i) $\Omega_I = \{(M_0, M_1) \mid M_0 \in \mathcal{M}_{2,2}(q), M_1 \in \mathcal{M}_{2,n-2}^*(q)\};$
- (ii) $\Omega_{II}^-(D, \kappa) = P \Lambda_{II}^-(D, \kappa) \tilde{P}^{-1} K_2(q)$ for $(D, \kappa) \in GF(q)^\times \times \{\pm 1\}$, where

$$\begin{aligned} \Lambda_{II}^-(D, \kappa) &= \left\{ \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ \beta^\sigma & \alpha^\sigma & \gamma^\sigma & 0 \end{pmatrix} \in \mathcal{M}_{2,n}^*(q^2) \mid \chi(\gamma \gamma^\sigma) = \kappa, \right. \\ &\quad \left. \frac{(\alpha \gamma^\sigma - \beta^\sigma \gamma)(\alpha^\sigma \gamma - \beta \gamma^\sigma)}{\gamma \gamma^\sigma} = D \right\}; \end{aligned}$$

- (iii) $\Omega_{III}^-(D_1, D_2) = P \Lambda_{III}^-(D_1, D_2) \tilde{P}^{-1}$ for $(D_1, D_2) \in GF(q)^\sim \times GF(q)^\sim / \sim (D_1^2 \neq D_2^2)$, where

$$\begin{aligned} \Lambda_{III}^-(D_1, D_2) &= \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \beta^\sigma & \alpha^\sigma & 0 \end{pmatrix} \in \mathcal{M}_{2,n}^*(q^2) \mid (\alpha^{(q+1)/2}, \beta^{(q+1)/2}) \sim (D_1, D_2) \right\}. \end{aligned}$$

There are $1 + 2(q-1) + \frac{3}{4}(q-1)^2$ orbits.

Proof. Let $\Omega_I, \Omega_{II}, \Omega_{III}$ be the set of matrices of type I, II, III, respectively, as defined in the proof of Theorem 3.5. They are fixed as a set by the action of $O_2^-(q)$, G_J ($J = J_2^-$), and Ω_I is an orbit of $O_2^-(q) \times G_J$ as before.

The action of $O_2^-(q) \times G_J$ on $\mathcal{M}_{2,n}^*(q)$ is isomorphic to that of $(P^{-1}O_2^-(q)P) \times (\tilde{P}^{-1}G_J\tilde{P})$ on $P^{-1}\mathcal{M}_{2,n}^*(q)\tilde{P}$ by the correspondence $(A \times B, M) \mapsto (P^{-1}AP \times \tilde{P}^{-1}B\tilde{P}, P^{-1}M\tilde{P})$ from $(O_2^-(q) \times G_J) \times \mathcal{M}_{2,n}^*(q)$ to $((P^{-1}O_2^-(q)P) \times (\tilde{P}^{-1}G_J\tilde{P})) \times P^{-1}\mathcal{M}_{2,n}^*(q)\tilde{P}$. Note that $\tilde{P}^{-1}G_J\tilde{P} = U'_2(q) \rtimes (H'_J \times K_2(q))$, where, by Lemma 2.4 (ii),

$$U'_2(q) = \left\{ \left(\begin{pmatrix} I_2 & 0 \\ B & I_{n-2} \end{pmatrix} \middle| B = \begin{pmatrix} \lambda_1 & \lambda_1^\sigma \\ \vdots & \vdots \\ \lambda_{n-2} & \lambda_{n-2}^\sigma \end{pmatrix} (\lambda_i \in GF(q^2)) \right\},$$

$$H'_J = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} \middle| A \in P^{-1}O_2^-(q)P \right\},$$

$$P^{-1}O_2^-(q)P = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \middle| \xi \in GF(q^2)^\times, \xi^\sigma = \xi^{-1} \right\}.$$

Also, note that $\tilde{P}$ commutes with each element of $K_2(q)$. It can be easily checked that each of $\Lambda_{II}^{--}(D, \kappa)K_2(q)$, $\Lambda_{III}^{--}(D_1, D_2)$ is fixed as a set by the action of $P^{-1}O_2^-(q)P$, $\tilde{P}^{-1}G_J\tilde{P} = U'_2(q) \rtimes (H'_J \times K_2(q))$. So each of $\Omega_{II}^{--}(D, \kappa)$, $\Omega_{III}^{--}(D_1, D_2)$ is fixed as a set by the action of $O_2^-(q)$, G_J . So to show $\Omega_{II}^{--}(D, \kappa)$ (respectively $\Omega_{III}^{--}(D_1, D_2)$) is an orbit of $O_2^-(q) \times G_J$, it suffices to show that $(P^{-1}O_2^-(q)P) \times (U'_2(q)H'_J)$ is transitive on $\Lambda_{II}^{--}(D, \kappa)$ (respectively $\Lambda_{III}^{--}(D_1, D_2)$).

As in the proof of Theorem 3.5, fix $\gamma_0 \in GF(q^2)^\times$ such that $\chi(\gamma_0\gamma_0^\sigma) = \kappa$. Then every element of $\Lambda_{II}^{--}(D, \kappa)$ is sent to $\begin{pmatrix} \alpha & \beta & \gamma_0 \\ \beta^\sigma & \alpha^\sigma & \gamma_0^\sigma \end{pmatrix} \mathbf{0}$ for some $\alpha, \beta \in GF(q^2)^\times$ by the action of $P^{-1}O_2^-(q)P$ and $K_2(q)$. Fix $\zeta_0 \in GF(q^2)^\times$ such that $\zeta_0\zeta_0^\sigma = D/\gamma_0\gamma_0^\sigma$. Then we have

$$\frac{\alpha}{\gamma_0} - \left(\frac{\beta}{\gamma_0} \right)^\sigma = \xi\zeta_0$$

for some $\xi \in GF(q^2)^\times$ such that $\xi^\sigma = \xi^{-1}$, since

$$\left(\frac{\alpha}{\gamma_0} - \left(\frac{\beta}{\gamma_0} \right)^\sigma \right) \left(\left(\frac{\alpha}{\gamma_0} \right)^\sigma - \frac{\beta}{\gamma_0} \right) = D/\gamma_0\gamma_0^\sigma$$

and the mapping $\zeta \mapsto \zeta\zeta^\sigma$ from $GF(q^2)^\times$ to $GF(q)^\times$ is surjective and has the kernel $\{\xi \in GF(q)^\times \mid \xi^\sigma = \xi^{-1}\}$. By the action of H'_J , we may assume that $\xi = 1$.

Fix $\alpha_0, \beta_0 \in GF(q^2)$ such that

$$\frac{\alpha_0}{\gamma_0} - \left(\frac{\beta_0}{\gamma_0}\right)^\sigma = \zeta_0.$$

Given $\alpha, \beta \in GF(q^2)$ such that

$$\frac{\alpha}{\gamma_0} - \left(\frac{\beta}{\gamma_0}\right)^\sigma = \zeta_0,$$

set

$$\lambda = \left(\frac{\beta}{\gamma_0}\right)^\sigma - \left(\frac{\beta_0}{\gamma_0}\right)^\sigma.$$

Then $\alpha = \alpha_0 + \lambda\gamma_0, \beta = \beta_0 + \lambda^\sigma\gamma_0$, i.e.

$$\begin{pmatrix} \alpha & \beta & \gamma_0 \\ \beta^\sigma & \alpha^\sigma & \gamma_0^\sigma \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \beta_0^\sigma & \alpha_0^\sigma & \gamma_0^\sigma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \lambda^\sigma & 1 \end{pmatrix}.$$

This means that $\begin{pmatrix} \alpha & \beta & \gamma_0 \\ \beta^\sigma & \alpha^\sigma & \gamma_0^\sigma \end{pmatrix}$ is sent to $\begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \beta_0^\sigma & \alpha_0^\sigma & \gamma_0^\sigma \end{pmatrix}$ by $U'_2(q)$. Therefore $(P^{-1}O_2^-(q)P) \times (U'_2(q)H'_J)$ is transitive on $\Lambda_{\text{II}}^{--}(D, \kappa)K_2(q)$.

Next, we want to show that $(P^{-1}O_2^-(q)P) \times H'_J$ is transitive on $\Lambda_{\text{III}}^{--}(D_1, D_2)$.

For

$$\begin{pmatrix} \alpha & \beta \\ \beta^\sigma & \alpha^\sigma \end{pmatrix} \in \Lambda_{\text{III}}^{--}(D_1, D_2),$$

we may assume that $\alpha^{(q+1)/2} = D_1, \beta^{(q+1)/2} = D_2$ by the action of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in P^{-1}O_2^-(q)P \ (\xi^{(q+1)/2} = -1), \quad \left(\begin{array}{cc|c} 0 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & I_{n-2} \end{array} \right) \in H'_J.$$

For

$$\begin{pmatrix} \alpha & \beta \\ \beta^\sigma & \alpha^\sigma \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \beta'^\sigma & \alpha'^\sigma \end{pmatrix} \in \Lambda_{\text{III}}^{--}(D_1, D_2)$$

with $\alpha^{(q+1)/2} = \alpha'^{(q+1)/2} = D_1, \beta^{(q+1)/2} = \beta'^{(q+1)/2} = D_2$, we claim that there exist $\xi_1, \xi_2 \in GF(q^2)^\times$ ($\xi_1^\sigma = \xi_1^{-1}, \xi_2^\sigma = \xi_2^{-1}$) such that

$$\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^\sigma & \alpha^\sigma \end{pmatrix} \begin{pmatrix} \xi_2 & 0 \\ 0 & \xi_2^{-1} \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \beta'^\sigma & \alpha'^\sigma \end{pmatrix},$$

i.e. $\alpha' = \xi_1\xi_2\alpha, \beta' = \xi_1\xi_2^{-1}\beta$.

Suppose that $D_1 \neq 0, D_2 \neq 0$. Since $(\alpha'/\alpha)^{(q+1)/2} = 1$, α'/α is contained in the cyclic subgroup of order $(q+1)/2$ of $GF(q^2)^\times$ and so there exists $\lambda \in GF(q^2)^\times$ such that $\lambda^{q+1} = 1, \lambda^2 = \alpha'/\alpha$. Since $(\beta'/\beta)^{(q+1)/2} = 1$, there exists $\mu \in GF(q^2)^\times$ such that $\mu^{q+1} = 1, \mu^2 = \beta'/\beta$. Set $\xi_1 = \lambda\mu$ and $\xi_2 = \lambda/\mu$. Then $\xi_1\xi_2\alpha = \lambda^2\alpha = \alpha', \xi_1\xi_2^{-1}\beta = \mu^2\beta = \beta'$ and so ξ_1, ξ_2 meet the requirement.

Suppose that $D_1 = 0$. Then $D_2 \neq 0$, since $D_1^2 \neq D_2^2$. Set $\xi_1 = \beta'/\beta, \xi_2 = 1$. Then $\xi_1^{q+1} = 1$ and so ξ_1, ξ_2 meet the requirement.

Suppose that $D_2 = 0$. Then $D_1 \neq 0$ and $\xi_1 = \alpha'/\alpha, \xi_2 = 1$ meet the requirement. Thus the claim is proved and hence $P^{-1}O_2^-(q)P \times H'_j$ is transitive on $\Lambda_{\text{III}}^-(D_1, D_2)$.

By Lemma 2.4, each matrix of type II belongs to exactly one of $\Omega_{\text{II}}^-(D, \kappa)$ and each matrix of type III belongs to exactly one of $\Omega_{\text{III}}^-(D_1, D_2)$. Thus the theorem is proved except for the number of orbits.

We count the number of orbits $\Omega_{\text{III}}^-(D_1, D_2)$. The length of the orbit of $\langle -1 \rangle \times \langle \sigma \rangle \times \langle \tau \rangle$ containing (D_1, D_2) with $D_1^2 \neq D_2^2$ is four if $(D_1, D_2) \in (GF(q)^\times \times GF(q)^\times) \cup (\sqrt{\epsilon}GF(q)^\times \times \sqrt{\epsilon}GF(q)^\times)$, eight if $(D_1, D_2) \in (GF(q)^\times \times \sqrt{\epsilon}GF(q)^\times) \cup (\sqrt{\epsilon}GF(q)^\times \times GF(q)^\times)$ and four if $D_1 = 0$ or $D_2 = 0$. There are $(q-1)(q-3) + (q-1)(q-3)$ many (D_1, D_2) in $(GF(q)^\times \times GF(q)^\times) \cup (\sqrt{\epsilon}GF(q)^\times \times \sqrt{\epsilon}GF(q)^\times)$ such that $D_1^2 \neq D_2^2$. There are $(q-1)^2 + (q-1)^2$ many (D_1, D_2) in $(GF(q)^\times \times \sqrt{\epsilon}GF(q)^\times) \cup (\sqrt{\epsilon}GF(q)^\times \times GF(q)^\times)$ such that $D_1^2 \neq D_2^2$. There are $2(q-1) + 2(q-1)$ many (D_1, D_2) such that $D_1^2 \neq D_2^2$ and $D_1D_2 = 0$. Thus the number of orbits $\Omega_{\text{III}}^-(D_1, D_2)$ is $\frac{1}{2}(q-1)(q-3) + \frac{1}{4}(q-1)^2 + (q-1) = \frac{3}{4}(q-1)^2$. Apparently, the number of orbits $\Omega_I, \Omega_{\text{II}}^-(D, \kappa)$ is 1, $2(q-1)$, respectively. $\square$

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Seiichi Iwamoto
Graduate School of Natural Science and Technology
Kanazawa University
Kakuma-machi
Kanazawa 920-1192
Japan
(E-mail: iwamoto@kappa.s.kanazawa-u.ac.jp)