$\mathcal{C}^\infty$ -REGULARITY OF THE TANGENTIAL CAUCHY–RIEMANN EQUATIONS ON LEVI-FLAT SUBMANIFOLDS OF $\mathbb{C}^n$

Joachim MICHEL

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Abstract. Regularity of solutions of the tangential Cauchy–Riemann (CR) equation is shown on domains in Levi-flat CR manifolds for the class of $\mathcal{C}^\infty$ forms up to the boundary. These domains are transversal intersections of the CR submanifold with pseudoconvex domains with piecewise smooth boundary.

1. Introduction

Results on global regularity of the tangential Cauchy–Riemann (CR) equation have been given in many articles. Here we want to consider certain domains in Levi-flat submanifolds in $\mathbb{C}^n$. The submanifolds will be of any codimension where the problem makes sense and the differential forms are of any degree. The boundary of the domain is piecewise smooth and weakly pseudoconvex in the sense described below. In order to achieve regularity up to the boundary we only treat the regularity class of $\mathcal{C}^\infty$ forms. Also, some global geometrical restrictions are needed.

We consider regions in generic CR submanifolds X in $\mathbb{C}^n$ which arise by intersecting X with a pseudoconvex domain Ω having piecewise smooth boundary. More precisely we consider the following set-up.

Let U be a non-empty open set in $\mathbb{C}^n$, $n \geq 2$, and let $\rho_1, \rho_2, \dots, \rho_k \in \mathcal{C}^\infty(U)$ be real-valued functions such that

$$\mathrm{rk}_{\mathbb{C}}(\bar{\partial}\rho_{j_1}, \bar{\partial}\rho_{j_2}, \dots, \bar{\partial}\rho_{j_l}) = l$$

on $\{z \in U \mid \rho_{j_1}(z) = \rho_{j_2}(z) = \dots = \rho_{j_l}(z) = 0\}$ for all $1 \leq j_1 < j_2 < \dots < j_l \leq k$, $1 \leq l \leq k$.

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Then, in particular, $X := X_k = \{z \in U \mid \rho_1(z) = \rho_2(z) = \cdots = \rho_k(z) = 0\}$ is a generic CR manifold or empty.

Let $\Omega_1, \Omega_2, \dots, \Omega_N$ be N bounded weakly pseudoconvex domains with $\mathcal{C}^\infty$ boundary and let $r_i : U_i \rightarrow \mathbb{R}$ denote a defining function of Ω_i on an open neighborhood of $\overline{\Omega_i}$. Then $\Omega_i = \{z \in U_i \mid r_i(z) < 0\} \subset\subset U_i$ and $dr_i \neq 0$ on $b\Omega_i$. We assume that $\Omega = \bigcap_{i=1}^N \Omega_i \subset\subset U$ and

$$\text{rk}_{\mathbb{R}}(dr_{i_1}, dr_{i_2}, \dots, dr_{i_m}) = m$$

for all $z \in b\Omega$, $1 \leq i_1 < \cdots < i_m \leq N$, $1 \leq m \leq N$, with $r_{i_1}(z) = r_{i_2}(z) = \cdots = r_{i_m}(z) = 0$.

Moreover, we assume for all $z \in b\Omega$, $1 \leq i_1 < \cdots < i_m \leq N$, $1 \leq m \leq N$, $1 \leq j_1 < j_2 < \cdots < j_l \leq k$, $1 \leq l \leq k$, with $r_{i_1}(z) = r_{i_2}(z) = \cdots = r_{i_m}(z) = \rho_{j_1}(z) = \rho_{j_2}(z) = \cdots = \rho_{j_l}(z) = 0$, that the following holds:

$$\text{rk}_{\mathbb{R}}(dr_{i_1}, dr_{i_2}, \dots, dr_{i_m}, d\rho_{j_1}, d\rho_{j_2}, \dots, d\rho_{j_l}) = m + l.$$

In particular, this means that there is a nested sequence of generic CR manifolds

$$X_l = \{z \in U \mid \rho_1(z) = \rho_2(z) = \cdots = \rho_l(z) = 0\}$$

beginning with $X_0 = U$ and ending up with $X = X_k$, such that each X_l intersects $b\Omega$ transversally.

Although we want to show a result for $X \cap \Omega$ these rather restrictive conditions are necessary for an induction over increasing codimension.

Set $M_l = X_l \cap \Omega$, $M = M_k$. Then $M_0 = \Omega$. We call sets of type M_l pseudoconvex regions in a generic CR manifold. M_l is bounded and has piecewise $\mathcal{C}^\infty$ boundary.

2. The main result

Our goal is to solve the tangential CR equation $\bar{\partial}_b u = f$ on M with $\mathcal{C}^\infty$ regularity up to the boundary. To avoid restrictions on the solvability we assume that X is Levi-flat on $\overline{M}$.

In [7] a solution operator on $\overline{\Omega}$ for the $\bar{\partial}$ equation was given. If f is $\mathcal{C}^\infty$ up to the boundary the solution u is in the same regularity class. We require that this operator also exists on the domains $\{z \in \Omega \mid \rho_i(z) > 0\}$ and $\{z \in \Omega \mid \rho_i(z) < 0\}$. Then the hypersurface $\{z \in \Omega \mid \rho_i(z) = 0\}$ has to be Levi-flat. Moreover, since up to now such operators can only be constructed if the above hypersurface is in the common boundary of two disjoint smoothly bounded pseudoconvex domains we require the following property.

Definition. Let $1 \leq i \leq k$. We call a hypersurface $H_i = \{z \in U \mid \rho_i(z) = 0\}$ *admissible* with respect to Ω if there exist bounded pseudoconvex domains $D_{i,+}$ and $D_{i,-}$, with C^∞ boundary, such that:

- (i) $\overline{\Omega} \subset \overline{D}_{i,+} \cup \overline{D}_{i,-}$, $D_{i,+} \cap D_{i,-} = \emptyset$;
- (ii) $\Omega \cap D_{i,+} \subset \{z \in \Omega \mid \rho_i(z) > 0\}$, $\Omega \cap D_{i,-} \subset \{z \in \Omega \mid \rho_i(z) < 0\}$.

Obviously $\Omega \cap H_i$ is an open subset in the Levi-flat part of H_i . By using the results of [7], solution operators respecting C^∞ regularity up to the boundary exist on $\Omega \cap D_{i,\pm}$.

When all these hypotheses hold we call M an *admissible pseudoconvex region* in a Levi-flat generic CR manifold. In this paper we show the following theorem in which $[f]_M$ denotes the equivalence class of a differential form f in the sense of the $\overline{\partial}_b = \overline{\partial}_{b,M}$ complex on M . The notation will be explained in the next section.

THEOREM. *Let M be an admissible pseudoconvex region in a Levi-flat generic CR manifold. Let $q \geq 1$ and $[f]_M \in \mathcal{C}_{0,q}^\infty(\overline{M})$ be a $\overline{\partial}_{b,M}$ closed form. Then there exist $[u]_M \in \mathcal{C}_{0,q-1}^\infty(\overline{M})$ with $\overline{\partial}_{b,M}[u] = [f]$.*

3. Notation and proofs

Let $0 \leq q \leq n$. Elements of $\mathcal{C}_{0,q}^\infty(\overline{\Omega})$, when restricted to $\overline{M}$, will be called C^∞ forms on $\overline{M}$ of bi-degree $(0, q)$. Two forms $f_1, f_2 \in \mathcal{C}_{0,q}^\infty(\overline{\Omega})$ are called *equivalent* with respect to $\overline{M}$ if there exist $A_1, A_2, \dots, A_k \in \mathcal{C}_{0,q}^\infty(\overline{\Omega})$, $B_1, B_2, \dots, B_k \in \mathcal{C}_{0,q-1}^\infty(\overline{\Omega})$, with

$$f_1 - f_2 = \sum_{i=1}^k \rho_i A_i + \overline{\partial} \rho_i \wedge B_i.$$

We denote equivalence classes by $[f]_M$ and the space of such classes by $\mathcal{C}_{0,q}^\infty(\overline{M})$. If $[f]_M \in \mathcal{C}_{0,q}^\infty(\overline{M})$ we define the tangential CR operator by $\overline{\partial}_{b,M}[f]_M = [\overline{\partial} f]_M$. The resulting class only depends on the class of f . Analogously we define the $\overline{\partial}_b$ operator on M_l . Clearly $[f]_{M_0} = \{f\}$ and $\overline{\partial}_{b,M_0} = \overline{\partial}$. M_0 will be the starting point of our induction.

The proof of the theorem will be based on the following technical lemma. Here for a fixed $1 \leq l \leq k$, we set $M_{l-1} = S$, $M_l = T$.

LEMMA. *Let $q \geq 1$ and $[f]_T \in \mathcal{C}_{0,q}^\infty(\overline{T})$ be a $\overline{\partial}_{b,T}$ closed form. Then there exist differential forms $\tilde{f}, \tilde{f}'_j \in \mathcal{C}_{0,q}^\infty(\overline{\Omega})$, $\tilde{f}''_j, C \in \mathcal{C}_{0,q+1}^\infty(\overline{\Omega})$ such that $[f]_T = [\tilde{f}]_T$, $C|_S$*

vanishes of infinite order on $\overline{T}$ and

$$\bar{\partial} \tilde{f} = \sum_{j=1}^k (\rho_j \tilde{f}_j'' + \bar{\partial} \rho_j \wedge \tilde{f}_j') + C. \quad (*)$$

Here $C|_S$ denotes the pointwise restriction of C to $\overline{S}$. The vanishing of $C|_S$ means that for any finite family of $\mathcal{C}^\infty$ tangential vector fields $X^1, \dots, X^P$ on $\overline{S}$, $X^1 X^2 \dots X^P C|_S$ vanishes on $\overline{T}$.

Proof of the theorem. The theorem holds if $k = 0$ as was shown in [7]. We suppose that it is true for all admissible pseudoconvex regions in generic Levi-flat CR manifolds of codimension less than or equal to $k - 1$. Let $M = M_k$ and $[f]_M \in \mathcal{C}_{0,q}^\infty(\overline{M})$, $q \geq 1$, be $\bar{\partial}_{b,M}$ closed. We adopt the notation of the lemma. In particular, $S = M_{k-1}$ and $T = M$. Let $S_+ = \{z \in S \mid \rho_k(z) > 0\}$ and $S_- = \{z \in S \mid \rho_k(z) < 0\}$. Then $S_\pm$ are admissible pseudoconvex regions in generic Levi-flat CR manifolds of codimension $k - 1$. By hypothesis we can solve the $\bar{\partial}_b$ equation with $\mathcal{C}^\infty$ regularity up to the boundary on $S_\pm$ and S .

Since (*) holds we can define a $\mathcal{C}_{0,q+1}^\infty$ form g on $\overline{S}$ by

$$g := \begin{cases} C & \text{on } \overline{S}_+, \\ 0 & \text{on } \overline{S}_-. \end{cases}$$

We also denote a $\mathcal{C}^\infty$ extension of g to $\overline{\Omega}$ by g . Such extensions are possible since edges in the boundary are locally diffeomorphic to cube-like edges. Then clearly $[g]_S \in \mathcal{C}_{0,q+1}^\infty(\overline{S})$ and $\bar{\partial}_{b,S}[g]_S = 0$. Let $[h]_S \in \mathcal{C}_{0,q}^\infty(\overline{S})$ with $[g]_S = \bar{\partial}_{b,S}[h]_S = [\bar{\partial}h]_S$. Therefore there exist forms $h'_j \in \mathcal{C}_{0,q}^\infty(\overline{\Omega})$ with

$$g|_S = \bar{\partial}h|_S + \sum_{j=1}^{k-1} \bar{\partial}\rho_j \wedge h'_j|_S.$$

Set $\tilde{h} = h + \sum_{j=1}^{k-1} \rho_j h'_j$. Hence $\tilde{h}|_S = h|_S$ and $g|_S = \bar{\partial}\tilde{h}|_S$. Since $\bar{\partial}\tilde{h}|_{S_-} = 0$ there exist $k, k'_j \in \mathcal{C}_{0,q-1}^\infty(\overline{\Omega})$ with

$$\tilde{h}|_{S_-} = \bar{\partial}k|_{S_-} + \sum_{j=1}^{k-1} \bar{\partial}\rho_j \wedge k'_j|_{S_-} = \bar{\partial}\tilde{k}|_{S_-},$$

with $\tilde{k} = k + \sum_{j=1}^{k-1} \rho_j k'_j$. Set $h^* = \tilde{h} - \bar{\partial}\tilde{k}$. Then

$$\bar{\partial}h^*|_S = \bar{\partial}\tilde{h}|_S = g|_S$$

and $h^*|_{S_-} = 0$. On $\bar{S}_+$ we obtain

$$\bar{\partial}(\tilde{f} - h^*)|_{S_+} = (\bar{\partial}\tilde{f} - C)|_{S_+} = \sum_{j=1}^{k-1} \bar{\partial}\rho_j \wedge \tilde{f}'_j|_{S_+}.$$

Let $[l]_{S_+} \in C_{0,q-1}^\infty(\bar{S}_+)$ with $[\tilde{f} - h^*]_{S_+} = \bar{\partial}_{b,S_+}[l]_{S_+}$. Since $h^*|_T = 0$ we have

$$[f]_T = [\tilde{f} - h^*]_T = \bar{\partial}_{b,T}[l]_T. \quad \square$$

Proof of the lemma. In the first step we show that for any $m \geq 0$ there exist $\mathcal{C}^\infty$ forms on $\bar{\Omega}$, $f_m, f'_{j,m}, f''_{j,m}, \Delta_m, \Xi_m, \Lambda_m, C_m, \tilde{C}_m$ with $[f_m]_T = [f]_T$, $C_m = \rho_l^m \tilde{C}_m$ and

$$\bar{\partial}f_m = \sum_{j=1}^{l-1} (\rho_j f''_{j,m} + \bar{\partial}\rho_j \wedge f'_{j,m}) + C_m$$

such that for all $m \geq 1$

$$f_m - f_{m-1} = \rho_l^m \Delta_m, \quad f'_{j,m} - f'_{j,m-1} = \rho_l^{m-1} \Xi_m, \quad f''_{j,m} - f''_{j,m-1} = \rho_l^{m-1} \Lambda_m.$$

Let $m = 0$. We set $f_0 = f$. Since

$$\bar{\partial}f = \sum_{i=1}^l (\rho_i A_i + \bar{\partial}\rho_i \wedge B_i)$$

we set $f'_{j,0} = A_j$, $f''_{j,0} = B_j$, $C_0 = \rho_l A_l + \bar{\partial}\rho_l \wedge B_l$. Since $\bar{\partial}\rho_l \wedge B_l = \bar{\partial}(\rho_l \wedge B_l) - \rho_l \bar{\partial}B_l$ this also settles the case $m = 1$ when we set $f_1 = f_0 - \rho_l \wedge B_l$, $\tilde{C}_1 = -\bar{\partial}B_l$.

Let the assertion be true for $m - 1 \geq 1$. We now show it for m . When we apply $\bar{\partial}$ to

$$\bar{\partial}f_{m-1} = \sum_{j=1}^{l-1} (\rho_j f''_{j,m-1} + \bar{\partial}\rho_j \wedge f'_{j,m-1}) + \rho_l^{m-1} \tilde{C}_{m-1} \quad (**)$$

we obtain

$$0 = \sum_{j=1}^{l-1} \bar{\partial}\rho_j \wedge (f''_{j,m-1} - \bar{\partial}f'_{j,m-1}) + \rho_l^{m-2} ((m-1)\bar{\partial}\rho_l \wedge \tilde{C}_{m-1} + \rho_l \bar{\partial}\tilde{C}_{m-1}).$$

Let $\tilde{V}$ be a neighborhood of $\bar{T}$ where $\bar{\partial}\rho_1 \wedge \bar{\partial}\rho_2 \wedge \cdots \wedge \bar{\partial}\rho_l \neq 0$. Set $\omega = \bar{\partial}\rho_1 \wedge \bar{\partial}\rho_2 \wedge \cdots \wedge \bar{\partial}\rho_{l-1}$. We multiply the above equation by ω and on $\tilde{V}$ obtain

$$(m-1)\omega \wedge \bar{\partial}\rho_l \wedge \tilde{C}_{m-1} + \rho_l \omega \wedge \bar{\partial}\tilde{C}_{m-1} = 0.$$

This restricted to $\overline{T}$ gives

$$\omega \wedge \overline{\partial} \rho_l \wedge \tilde{C}_{m-1} = 0.$$

It follows that there exist C^∞ forms $\alpha_{j,m-1}, \beta_{j,m-1}$ on $\overline{\Omega}$ such that

$$\tilde{C}_{m-1} = \sum_{j=1}^l (\rho_j \beta_{j,m-1} + \overline{\partial} \rho_j \wedge \alpha_{j,m-1}).$$

We put this into (**) and get a decomposition of $\overline{\partial} f_{m-1}$:

$$\begin{aligned} & \sum_{j=1}^{l-1} [\rho_j (f''_{j,m-1} + \rho_l^{m-1} \beta_{j,m-1}) + \overline{\partial} \rho_j \wedge (f'_{j,m-1} + \rho_l^{m-1} \alpha_{j,m-1})] \\ & + \rho_l^m \beta_{l,m-1} + \rho_l^{m-1} \overline{\partial} \rho_l \wedge \alpha_{l,m-1}. \end{aligned}$$

Now

$$\rho_l^{m-1} \overline{\partial} \rho_l \wedge \alpha_{l,m-1} = \overline{\partial} \left[\frac{\rho_l^m}{m} \alpha_{l,m-1} \right] - \frac{\rho_l^m}{m} \overline{\partial} \alpha_{l,m-1}.$$

We set

$$\begin{aligned} f_m &= f_{m-1} - \frac{\rho_l^m}{m} \alpha_{l,m-1}, & f'_{j,m} &= f'_{j,m-1} + \rho_l^{m-1} \alpha_{j,m-1}, \\ f''_{j,m} &= f''_{j,m-1} + \rho_l^{m-1} \beta_{j,m-1}, & \tilde{C}_m &= \beta_{l,m-1} - \frac{1}{m} \overline{\partial} \alpha_{l,m-1}. \end{aligned}$$

In order to construct $\tilde{f}$ we apply a standard method. We first note that for $m \geq 1$,

$$f_m = f_0 + (f_1 - f_0) + \cdots + (f_m - f_{m-1}).$$

Let η be a positive-valued C^∞ cut-off function on $\mathbb{R}$, with $\eta(x) = 1$ if $|x| \leq 1/2$, and $\eta(x) = 0$ if $|x| \geq 1$. For $\delta > 0$ we set $\eta_\delta(x) = \eta(x/\delta)$. Then $\eta_\delta(x) = 1$ if $|x| \leq \delta/2$ and $\eta_\delta(x) = 0$ if $|x| \geq \delta$.

Now we want to estimate derivatives near $\overline{S}$ of terms of type

$$X_m^\delta(z) = \eta_\delta(\rho_l(z)) \rho_l^m(z) \Delta_m(z).$$

It suffices to assume that $|\rho_l(z)| \leq \delta$. Let D^k be a differentiation of order $k \leq m/4$. If we apply D^k to $X_m^\delta(z)$ we obtain a sum of terms

$$D^{k_1} \eta_\delta(\rho_l(z)) D^{k_2} \rho_l^m(z) D^{k_3} \Delta_m(z),$$

with $k_1 + k_2 + k_3 = k$. Then it is obvious that we can majorize its absolute value by

$$\text{const.} \frac{1}{\delta^{k_1}} |\rho_l(z)|^{m-k_2} \leq \text{const.} |\rho_l(z)|^{m-k_2-k_1} \leq \text{const.} |\rho_l(z)|^{3m/4},$$

where the constant does not depend on z . Therefore we can choose $\delta = \delta_m$ so small that for all such differentiations

$$|D^k X_m^\delta(z)| \leq \text{const. } 2^{-m} |\rho_l(z)|^{m/2},$$

where the constant does not depend on m and z . By shrinking δ_m again we can achieve that the same holds true for

$$\begin{aligned} & \eta_{\delta_m}(\rho_l(z)) \rho_l^{m-1}(z) \beta_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{m-1}(z) \alpha_{j,m-1}(z), \\ & \eta_{\delta_m}(\rho_l(z)) \rho_l^{m-1}(z) (\rho_l(z) \tilde{C}_m - \tilde{C}_{m-1}), \quad \eta'_{\delta_m}(\rho_l(z)) \frac{\bar{\partial} \rho_l(z)}{\delta_m} \wedge \rho_l^m(z) \Delta_m(z). \end{aligned}$$

Now set

$$F(z) = \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^m(z) \Delta_m(z)$$

and $\tilde{f} = f + F$. Then F is of class C^∞ up to the boundary and F is only a finite sum on any compact set on which ρ_l does not vanish. The same holds true for the following forms:

$$\begin{aligned} \tilde{f}'_j &= \tilde{f}'_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l) \rho_l^{m-1} \beta_{j,m-1}, \\ \tilde{f}''_j &= \tilde{f}''_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l) \rho_l^{m-1} \alpha_{j,m-1}, \\ \Phi &= C_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l) \rho_l^{m-1} (\rho_l \tilde{C}_m - \tilde{C}_{m-1}), \\ \Gamma &= \sum_{m=1}^{\infty} \eta'_{\delta_m}(\rho_l) \frac{\bar{\partial} \rho_l}{\delta_m} \wedge \rho_l^m \Delta_m. \end{aligned}$$

Let z_0 be fixed with $\rho_l(z_0) = 0$ and let m_0 be a large positive integer. We choose a neighborhood V of z_0 such that

$$\sup_V |\rho_l| < \frac{1}{2} \min_{1 \leq i \leq m_0} \delta_i.$$

Then on $\bar{V}$ we have the following decomposition:

$$\begin{aligned} \tilde{f}(z) &= f_{m_0}(z) + \rho_l(z)^{[m_0/2]} \sum_{m=m_0+1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l(z)^{m-[m_0/2]} \Delta_m(z) \\ &= f_{m_0}(z) + \rho_l(z)^{[m_0/2]} \gamma_{m_0}(z), \end{aligned}$$

with a $\mathcal{C}^\infty$ form γ_{m_0} . In particular, $[\tilde{f}]_T = [f]_T$. Now we consider $\bar{\partial}\tilde{f}$. Since

$$\bar{\partial}\tilde{f} = \bar{\partial}f + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l) \bar{\partial}(f_m - f_{m-1}) + \sum_{m=1}^{\infty} \eta'_{\delta_m}(\rho_l) \frac{\bar{\partial}\rho_l}{\delta_m} \wedge \rho_l^m \Delta_m$$

and

$$\bar{\partial}(f_m - f_{m-1}) = \sum_{j=1}^{l-1} (\rho_j \rho_l^{m-1} \beta_{j,m-1} + \bar{\partial}\rho_j \wedge \rho_l^{m-1} \alpha_{j,m-1}) + \rho_l^m \tilde{C}_m - \rho_l^{m-1} \tilde{C}_{m-1}$$

we obtain

$$\bar{\partial}\tilde{f} = \sum_{j=1}^{l-1} (\rho_j \tilde{f}_j'' + \bar{\partial}\rho_j \wedge \tilde{f}_j') + \Phi + \Gamma. \quad (***)$$

It remains to show that $\Phi + \Gamma$ can be decomposed into two terms where one can be absorbed by the sum and the other vanishes of infinite order on $\bar{T}$ along $\bar{S}$. Let m_0 and V be as before.

On $\bar{S} \cap V$ it follows that

$$\begin{aligned} \omega \wedge (\Phi + \Gamma) &= \omega \wedge \bar{\partial}\tilde{f} \\ &= \omega \wedge \left(\bar{\partial}f_{m_0} + \rho_l^{[m_0/2]} \bar{\partial}\gamma_{m_0} + \left[\frac{m_0}{2} \right] \rho_l^{[m_0/2]-1} \bar{\partial}\rho_l \wedge \gamma_{m_0} \right) \\ &= \rho_l^{[m_0/2]-1} \Pi_{m_0}, \end{aligned}$$

with a $\mathcal{C}^\infty$ form Π_{m_0} . This yields that $\omega \wedge (\Phi + \Gamma)$ vanishes along $\bar{S}$ of infinite order in $\bar{T}$.

Now let $\{\varphi_1, \varphi_2, \dots, \varphi_M\}$ be a partition of unity associated with the open covering $\{U_1, U_2, \dots, U_M\}$ of a neighborhood of $\bar{S}$. Let the covering be so fine such that there exist on U_i , $\mathcal{C}^\infty(0, 1)$ -forms $e_1, e_2, \dots, e_{n-l+1}$ with

$$\omega \wedge e_1 \wedge e_2 \wedge \dots \wedge e_{n-l+1} \neq 0.$$

Then there exist $\mathcal{C}^\infty$ forms $R_{i,j}$ on $U_i \cap \bar{S}$ with

$$\Phi + \Gamma = \sum_{j=1}^{l-1} \bar{\partial}\rho_j \wedge R_{i,j} + R_{i,l},$$

where $R_{i,l}$ does not contain any $\bar{\partial}\rho_j$, $j < l$. Consequently $R_{i,l}$ vanishes of infinite order on $\bar{T}$ along $\bar{S}$. Since on a neighborhood of $\bar{S}$ we have

$$\Phi + \Gamma = \sum_{j=1}^{l-1} \bar{\partial}\rho_j \wedge \sum_{i=1}^M \varphi_i R_{i,j} + \sum_{i=1}^M \varphi_i R_{i,l}$$

there exist C^∞ forms A_j, B_j on $\overline{\Omega}$ with

$$\Phi + \Gamma = \sum_{j=1}^{l-1} (\rho_j B_j + \bar{\partial} \rho_j \wedge A_j) + \sum_{i=1}^M \varphi_i R_{i,l}.$$

Inserting this into (***) gives

$$\bar{\partial} \tilde{f} = \sum_{j=1}^{l-1} (\rho_j (\tilde{f}_j'' + B_j) + \bar{\partial} \rho_j \wedge (\tilde{f}_j' + A_j)) + C,$$

with $C = \sum_{i=1}^M \varphi_i R_{i,l}$. By renaming $\tilde{f}_j'' + B_j$ and $\tilde{f}_j' + A_j$ the lemma is proven. $\square$

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Joachim Michel
Université du Littoral Côte d'Opale
Centre Universitaire de la Mi-Voix
50, rue Ferdinand Buisson
F-62228 Calais
France
(E-mail: michel@lmpa.univ-littoral.fr)