C∞-REGULARITY OF THE TANGENTIAL CAUCHY–RIEMANN EQUATIONS ON LEVI-FLAT SUBMANIFOLDS OF Cⁿ

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Abstract. Regularity of solutions of the tangential Cauchy–Riemann (CR) equation is shown on domains in Levi-flat CR manifolds for the class of C∞ forms up to the boundary. These domains are transversal intersections of the CR submanifold with pseudoconvex domains with piecewise smooth boundary.

1. Introduction

Results on global regularity of the tangential Cauchy–Riemann (CR) equation have been given in many articles. Here we want to consider certain domains in Levi-flat submanifolds in Cⁿ. The submanifolds will be of any codimension where the problem makes sense and the differential forms are of any degree. The boundary of the domain is piecewise smooth and weakly pseudoconvex in the sense described below. In order to achieve regularity up to the boundary we only treat the regularity class of C∞ forms. Also, some global geometrical restrictions are needed.

We consider regions in generic CR submanifolds X in Cⁿ which arise by intersecting X with a pseudoconvex domain Ω having piecewise smooth boundary. More precisely we consider the following set-up.

Let U be a non-empty open set in Cⁿ, n ≥ 2, and let ρ₁, ρ₂, ..., ρₖ ∈ C∞(U) be real-valued functions such that

\[ \text{rk}_C(\bar{\partial}\rho_{j_1}, \bar{\partial}\rho_{j_2}, \ldots, \bar{\partial}\rho_{j_l}) = l \]

on \{z ∈ U | ρ_{j_1}(z) = ρ_{j_2}(z) = \cdots = ρ_{j_l}(z) = 0\} for all \( 1 \leq j_1 < j_2 < \cdots < j_l \leq k, 1 \leq l \leq k. \)

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Then, in particular, $X := X_k = \{ z \in U \mid \rho_1(z) = \rho_2(z) = \cdots = \rho_k(z) = 0 \}$ is a generic CR manifold or empty.

Let $\Omega_1, \Omega_2, \ldots, \Omega_N$ be $N$ bounded weakly pseudoconvex domains with $C^\infty$ boundary and let $r_i : U_i \to \mathbb{R}$ denote a defining function of $\Omega_i$ on an open neighborhood of $\Omega_i$. Then $\Omega_i = \{ z \in U_i \mid r_i(z) < 0 \} \subset \subset U_i$ and $dr_i \neq 0$ on $b\Omega_i$. We assume that $\Omega = \bigcap_{i=1}^N \Omega_i \subset \subset U$ and

$$rk_\mathbb{R}(dr_{i_1}, dr_{i_2}, \ldots, dr_{i_m}) = m$$

for all $z \in b\Omega$, $1 \leq i_1 < \cdots < i_m \leq N$, $1 \leq m \leq N$, with $r_{i_1}(z) = r_{i_2}(z) = \cdots = r_{i_m}(z) = 0$.

Moreover, we assume for all $z \in b\Omega$, $1 \leq i_1 < \cdots < i_m \leq N$, $1 \leq m \leq N$, $1 \leq j_1 < j_2 < \cdots < j_l \leq k$, $1 \leq l \leq k$, with $r_{i_1}(z) = r_{i_2}(z) = \cdots = r_{i_m}(z) = r_{j_1}(z) = \cdots = r_{j_l}(z) = 0$, that the following holds:

$$rk_\mathbb{R}(dr_{i_1}, dr_{i_2}, \ldots, dr_{i_m}, dr_{j_1}, dr_{j_2}, \ldots, dr_{j_l}) = m + l.$$

In particular, this means that there is a nested sequence of generic CR manifolds

$$X_l = \{ z \in U \mid \rho_1(z) = \rho_2(z) = \cdots = \rho_l(z) = 0 \}$$

beginning with $X_0 = U$ and ending up with $X = X_k$, such that each $X_l$ intersects $b\Omega$ transversally.

Although we want to show a result for $X \cap \Omega$ these rather restrictive conditions are necessary for an induction over increasing codimension.

Set $M_l = X_l \cap \Omega$, $M = M_k$. Then $M_0 = \Omega$. We call sets of type $M_l$ pseudoconvex regions in a generic CR manifold. $M_l$ is bounded and has piecewise $C^\infty$ boundary.

2. The main result

Our goal is to solve the tangential CR equation $\bar{\partial} b u = f$ on $M$ with $C^\infty$ regularity up to the boundary. To avoid restrictions on the solvability we assume that $X$ is Levi-flat on $\overline{M}$.

In [7] a solution operator on $\overline{\Omega}$ for the $\bar{\partial}$ equation was given. If $f$ is $C^\infty$ up to the boundary the solution $u$ is in the same regularity class. We require that this operator also exists on the domains $\{ z \in \Omega \mid \rho_1(z) > 0 \}$ and $\{ z \in \Omega \mid \rho_1(z) < 0 \}$. Then the hypersurface $\{ z \in \Omega \mid \rho_1(z) = 0 \}$ has to be Levi-flat. Moreover, since up to now such operators can only be constructed if the above hypersurface is in the common boundary of two disjoint smoothly bounded pseudoconvex domains we require the following property.
Definition. Let $1 \leq i \leq k$. We call a hypersurface $H_i = \{z \in U \mid \rho_i(z) = 0\}$ admissible with respect to $\Omega$ if there exist bounded pseudoconvex domains $D_{i,+}$ and $D_{i,-}$, with $C^\infty$ boundary, such that:

(i) $\Omega \subset \overline{D_{i,+}} \cup \overline{D_{i,-}}, \ D_{i,+} \cap D_{i,-} = \emptyset$;
(ii) $\Omega \cap D_{i,+} \subset \{z \in \Omega \mid \rho_i(z) > 0\}, \ \Omega \cap D_{i,-} \subset \{z \in \Omega \mid \rho_i(z) < 0\}$.

Obviously $\Omega \cap H_i$ is an open subset in the Levi-flat part of $H_i$. By using the results of [7], solution operators respecting $C^\infty$ regularity up to the boundary exist on $\Omega \cap D_{i,\pm}$.

When all these hypotheses hold we call $M$ an admissible pseudoconvex region in a Levi-flat generic CR manifold. In this paper we show the following theorem in which $[f]_M$ denotes the equivalence class of a differential form $f$ in the sense of the $\overline{\partial}_b = \overline{\partial}_{b,M}$ complex on $M$. The notation will be explained in the next section.

**Theorem.** Let $M$ be an admissible pseudoconvex region in a Levi-flat generic CR manifold. Let $q \geq 1$ and $[f]_M \in C^\infty_{0,q} (\overline{M})$ be a $\overline{\partial}_{b,M}$ closed form. Then there exist $[u]_M \in C^\infty_{0,q-1} (\overline{M})$ with $\overline{\partial}_{b,M} [u] = [f]$.

3. Notation and proofs

Let $0 \leq q \leq n$. Elements of $C^\infty_{0,q} (\overline{\Omega})$, when restricted to $\overline{M}$, will be called $C^\infty$ forms on $\overline{M}$ of bi-degree $(0, q)$. Two forms $f_1, f_2 \in C^\infty_{0,q} (\overline{\Omega})$ are called equivalent with respect to $\overline{M}$ if there exist $A_1, A_2, \ldots, A_k \in C^\infty_{0,q} (\overline{\Omega})$, $B_1, B_2, \ldots, B_k \in C^\infty_{0,q-1} (\overline{\Omega})$, with

$$f_1 - f_2 = \sum_{i=1}^k \rho_i A_i + \overline{\partial}_b \rho_i \wedge B_i.$$ 

We denote equivalence classes by $[f]_M$ and the space of such classes by $C^\infty_{0,q} (\overline{M})$.

If $[f]_M \in C^\infty_{0,q} (\overline{M})$ we define the tangential CR operator by $\overline{\partial}_{b,M} [f]_M = [\overline{\partial} f]_M$.

The resulting class only depends on the class of $f$. Analogously we define the $\overline{\partial}_b$ operator on $M_l$. Clearly $[f]_{M_0} = [f]$ and $\overline{\partial}_{b,M_0} = \overline{\partial}_b$. $M_0$ will be the starting point of our induction.

The proof of the theorem will be based on the following technical lemma. Here for a fixed $1 \leq l \leq k$, we set $M_{l-1} = S$, $M_l = T$.

**Lemma.** Let $q \geq 1$ and $[f]_T \in C^\infty_{0,q} (\overline{T})$ be a $\overline{\partial}_{b,T}$ closed form. Then there exist differential forms $\tilde{f}, \tilde{f}'_j \in C^\infty_{0,q} (\overline{\Omega}), \ f''_j, C \in C^\infty_{0,q+1} (\overline{\Omega})$ such that $[f]_T = [\tilde{f}]_T, \ C|_S \in C^\infty_{0,q+1} (\overline{\Omega})$. The notation will be explained in the next section.
vanishes of infinite order on \( T \) and
\[
\overline{\partial} \tilde{f} = \sum_{j=1}^{k} (\rho_j \tilde{f}_j^\prime + \overline{\partial} \rho_j \wedge \tilde{f}_j^\prime) + C.
\]

Here \( C|_S \) denotes the pointwise restriction of \( C \) to \( \overline{S} \). The vanishing of \( C|_S \) means that for any finite family of \( C^\infty \) tangential vector fields \( X^1, \ldots, X^P \) on \( S \), \( X^1 X^2 \ldots X^P C|_S \) vanishes on \( \overline{T} \).

**Proof of the theorem.** The theorem holds if \( k = 0 \) as was shown in [7]. We suppose that it is true for all admissible pseudoconvex regions in generic Levi-flat CR manifolds of codimension less than or equal to \( k - 1 \). Let \( M = M_k \) and \( f|_M \in C^\infty_0(\overline{M}) \), \( q \geq 1 \), be \( \overline{\partial}_b,M \) closed. We adopt the notation of the lemma. In particular, \( S = M_{k-1} \) and \( T = M \). Let \( S_+ = \{ z \in S \mid \rho_k(z) > 0 \} \) and \( S_- = \{ z \in S \mid \rho_k(z) < 0 \} \). Then \( S_\pm \) are admissible pseudoconvex regions in generic Levi-flat CR manifolds of codimension \( k - 1 \). By hypothesis we can solve the \( \overline{\partial}_b \) equation with \( C^\infty \) regularity up to the boundary on \( S_\pm \) and \( S \).

Since (\( \ast \)) holds we can define a \( C^\infty_{0,q+1} \) form \( g \) on \( \overline{S} \) by
\[
g := \begin{cases} C & \text{on } \overline{S}_+, \\ 0 & \text{on } \overline{S}_-. \end{cases}
\]

We also denote a \( C^\infty \) extension of \( g \) to \( \overline{\Omega} \) by \( \tilde{g} \). Such extensions are possible since edges in the boundary are locally diffeomorphic to cube-like edges. Then clearly \( [g]_S \in C^\infty_{0,q+1}(\overline{S}) \) and \( \overline{\partial}_b,[g]_S = 0 \). Let \( [h]_S \in C^\infty_0(\overline{S}) \) with \( [g]_S = \overline{\partial}_b,[h]_S = [\tilde{g}]_S \). Therefore there exist forms \( h'_j \in C^\infty_{0,q}(\overline{\Omega}) \) with
\[
g|_S = \overline{\partial} h|_S + \sum_{j=1}^{k-1} \overline{\partial} \rho_j \wedge h'_j|_S.
\]

Set \( \tilde{h} = h + \sum_{j=1}^{k-1} \rho_j h'_j \). Hence \( \tilde{h}|_S = h|_S \) and \( g|_S = \overline{\partial} \tilde{h}|_S \). Since \( \overline{\partial} h|_{S_-} = 0 \) there exist \( k_j \in C^\infty_{0,q-1}(\overline{\Omega}) \) with
\[
\tilde{h}|_{S_-} = \overline{\partial} k|_{S_-} + \sum_{j=1}^{k-1} \overline{\partial} \rho_j \wedge k'_j|_{S_-} = \overline{\partial} k|_{S_-},
\]
with \( k = k + \sum_{j=1}^{k-1} \rho_j k'_j \). Set \( h^* = \tilde{h} - \overline{\partial} k \). Then
\[
\overline{\partial} h^*|_S = \overline{\partial} h|_S = g|_S
\]
and $h^+|s_+ = 0$. On $\overline{S}_+$ we obtain

$$\overline{\rho}(\overline{f} - h^+)|s_+ = (\overline{\rho} \overline{f} - C)|s_+ = \sum_{j=1}^{k-1} \overline{\rho}_{\overline{f}} j \wedge \overline{\rho}^j|s_+.$$ 

Let $[l]|s_+ \in C^\infty_{0,q-1}(\overline{S}_+)$ with $[\overline{f} - h^+]|s_+ = \overline{\partial}_{h,s+}[l]|s_+$. Since $h^+|T = 0$ we have

$$[\overline{f}]_T = [\overline{f} - h^+]_T = \overline{\partial}_{h,T}[l]_T. \quad \Box$$

Proof of the lemma. In the first step we show that for any $m \geq 0$ there exist $C^\infty$ forms on $\overline{\Omega}$, $f_m$, $f_{j,m}$, $f_{j,m}'$, $\Delta_m$, $\Xi_m$, $\Lambda_m$, $C_m$, $\overline{C}_m$ with $[f_m]|T = [f]|T$. $C_m = \rho^m_1 \overline{C}_m$ and

$$\overline{\partial} f_m = \sum_{j=1}^{l-1} (\overline{\rho}_{\overline{f}} f_{j,m}' + \overline{\partial} \overline{\rho}_{\overline{f}} j \wedge f_{j,m}') + C_m$$

such that for all $m \geq 1$

$$f_m - f_{m-1} = \rho^m_1 \Delta_m, \quad f_{j,m}' - f_{j,m-1}' = \rho^m_1 \Xi_m, \quad f_{j,m}' - f_{j,m-1}' = \rho^m_1 \Lambda_m.$$ 

Let $m = 0$. We set $f_0 = f$. Since

$$\overline{\partial} f = \sum_{i=1}^{l} (\overline{\rho} A_i + \overline{\partial} \overline{\rho} \wedge B_i)$$

we set $f_{j,0}' = A_j$, $f_{j,0}' = B_j$, $C_0 = \rho_1 A + \overline{\partial} \overline{\rho} \wedge B_1$. Since $\overline{\partial} \overline{\rho} \wedge B_1 = \overline{\partial}(\overline{\rho} \wedge B_1) + \rho_1 \overline{\partial} B_1$ this also settles the case $m = 1$ when we set $f_1 = f_0 - \rho_1 \wedge B_1$, $C_1 = \overline{\partial} B_1$. 

Let the assertion be true for $m - 1 \geq 1$. We now show it for $m$. When we apply $\overline{\partial}$ to

$$\overline{\partial} f_{m-1} = \sum_{j=1}^{l-1} (\overline{\rho}_{\overline{f}} f_{j,m-1}' + \overline{\partial} \overline{\rho}_{\overline{f}} j \wedge f_{j,m-1}') + \rho^m_1 \overline{C}_{m-1}$$

we obtain

$$0 = \sum_{j=1}^{l-1} \overline{\partial} \overline{\rho}_{\overline{f}} j \wedge (f_{j,m-1}' - \overline{\partial} f_{j,m-1}') + \rho^m_1 \overline{\partial} f_{j,m-1}' + \rho^{m-2}_1 (m - 1) \overline{\partial} \overline{\rho} \wedge \overline{C}_{m-1} + \rho_1 \overline{\partial} \overline{C}_{m-1}.$$ 

Let $\overline{V}$ be a neighborhood of $\overline{T}$ where $\overline{\partial} \overline{\rho} \overline{\partial} \rho_2 \wedge \cdots \wedge \overline{\partial} \overline{\rho} \neq 0$. Set $\omega = \overline{\partial} \overline{\rho} \wedge \overline{\partial} \overline{\rho}_2 \wedge \cdots \wedge \overline{\partial} \overline{\rho}_l$. We multiply the above equation by $\omega$ and on $\overline{V}$ obtain

$$(m - 1) \omega \wedge \overline{\partial} \overline{\rho} \wedge \overline{C}_{m-1} + \rho_1 \omega \wedge \overline{\partial} \overline{C}_{m-1} = 0.$$
This restricted to $T$ gives
\[ \omega \wedge \overline{\partial} \rho_l \wedge \overline{C}_{m-1} = 0. \]

It follows that there exist $C^\infty$ forms $\alpha_{j,m-1}, \beta_{j,m-1}$ on $\overline{\Omega}$ such that
\[ \overline{C}_{m-1} = \sum_{j=1}^{l-1} (\rho_j \beta_{j,m-1} + \overline{\partial} \rho_j \wedge \alpha_{j,m-1}). \]

We put this into (***) and get a decomposition of $\overline{\partial} f_{m-1}$:
\[
\sum_{j=1}^{l-1} \left[ \rho_j (f^m_{j,m-1} + \rho_l^{-1} \beta_{j,m-1}) + \overline{\partial} \rho_j \wedge (f^m_{j,m-1} + \rho_l^{-1} \alpha_{j,m-1}) \right] \\
+ \rho_l^m \beta_{l,m-1} + \rho_l^{-1} \overline{\partial} \rho_l \wedge \alpha_{l,m-1}.
\]

Now
\[ \rho_l^{m-1} \overline{\partial} \rho_l \wedge \alpha_{l,m-1} = \overline{\partial} \left[ \frac{\rho_l^m}{m} \alpha_{l,m-1} \right] - \frac{\rho_l^m}{m} \overline{\partial} \alpha_{l,m-1}. \]

We set
\[
fm = f_{m-1} - \frac{\rho_l^m}{m} \alpha_{l,m-1}, \quad f_j^m = f_{j,m-1} + \rho_l^{-1} \alpha_{j,m-1}, \\
f^m_{j,m} = f_{j,m-1} + \rho_l^{-1} \beta_{j,m-1}, \quad \overline{C}_m = \beta_{l,m-1} - \frac{1}{m} \overline{\partial} \alpha_{l,m-1}.
\]

In order to construct $\tilde{f}$ we apply a standard method. We first note that for $m \geq 1$,
\[ f_m = f_0 + (f_1 - f_0) + \cdots + (f_m - f_{m-1}). \]

Let $\eta$ be a positive-valued $C^\infty$ cut-off function on $\mathbb{R}$, with $\eta(x) = 1$ if $|x| \leq 1/2$, and $\eta(x) = 0$ if $|x| \geq 1$. For $\delta > 0$ we set $\eta_\delta(x) = \eta(x/\delta)$. Then $\eta_\delta(x) = 1$ if $|x| \leq \delta/2$ and $\eta_\delta(x) = 0$ if $|x| \geq \delta$.

Now we want to estimate derivatives near $S$ of terms of type
\[ X^k_m(z) = \eta_\delta(\rho_l(z)) \rho_l^m(z) \Delta_m(z). \]

It suffices to assume that $|\rho_l(z)| \leq \delta$. Let $D^k$ be a differentiation of order $k \leq m/4$. If we apply $D^k$ to $X^k_m(z)$ we obtain a sum of terms
\[ D^{k_1} \eta_\delta(\rho_l(z)) D^{k_2} \rho_l^m(z) D^{k_3} \Delta_m(z), \]
with $k_1 + k_2 + k_3 = k$. Then it is obvious that we can majorize its absolute value by
\[
\text{const.} \frac{1}{\delta^k} |\rho_l(z)|^{m-k} \leq \text{const.} |\rho_l(z)|^{m-k-k_1} \leq \text{const.} |\rho_l(z)|^{3m/4},
\]
where the constant does not depend on \( z \). Therefore we can choose \( \delta = \delta_m \) so small that for all such differentiations

\[
|D^k X_m^\delta(z)| \leq \text{const. } 2^{-m} |\rho_l(z)|^{m/2},
\]

where the constant does not depend on \( m \) and \( z \). By shrinking \( \delta_m \) again we can achieve that the same holds true for

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

where the constant does not depend on \( m \) and \( z \). By shrinking \( \delta_m \) again we can achieve that the same holds true for

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

\[
\eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma}_{j,m-1}(z), \quad \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1}(z),
\]

Now set

\[
F(z) = \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \Delta_m(z)
\]

and \( \tilde{f} = f + F \). Then \( F \) is of class \( C^\infty \) up to the boundary and \( F \) is only a finite sum on any compact set on which \( \rho_l \) does not vanish. The same holds true for the following forms:

\[
\tilde{f}^j = \tilde{f}^j_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \beta_{j,m-1},
\]

\[
\tilde{f}''_j = \tilde{f}''_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \alpha_{j,m-1},
\]

\[
\Phi = \Phi_0 + \sum_{m=1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \tilde{\gamma} \rho_l^{-1}(z) \rho_l \Delta_m(z),
\]

\[
\Gamma = \sum_{m=1}^{\infty} \eta_{\delta_m}'(\rho_l(z)) \rho_l^{-1}(z) \rho_l \Delta_m(z).
\]

Let \( z_0 \) be fixed with \( \rho_l(z_0) = 0 \) and let \( m_0 \) be a large positive integer. We choose a neighborhood \( V \) of \( z_0 \) such that

\[
\sup_{V} |\rho_l| < \frac{1}{2} \min_{1 \leq i \leq m_0} \delta_i.
\]

Then on \( \bar{V} \) we have the following decomposition:

\[
\tilde{f}(z) = f_{m_0}(z) + \rho_l(z)^{m_0/2} \sum_{m=m_0+1}^{\infty} \eta_{\delta_m}(\rho_l(z)) \rho_l^{-1}(z) \rho_l(z)^{m-m_0/2} \Delta_m(z)
\]

\[
= f_{m_0}(z) + \rho_l(z)^{m_0/2} \gamma_{m_0}(z),
\]
with a $C^\infty$ form $\gamma_{m_0}$. In particular, $[\tilde{f}]_T = [f]_T$. Now we consider $\overline{\partial} \tilde{f}$. Since

$$\overline{\partial} \tilde{f} = \overline{\partial} f + \sum_{m=1}^\infty \eta_{m_0}(\rho_l) \overline{\partial} (f_m - f_{m-1}) + \sum_{m=1}^\infty \eta'_{m_0}(\rho_l) \overline{\partial} \rho_l^m \Delta_m$$

and

$$\overline{\partial} (f_m - f_{m-1}) = \sum_{j=1}^{l-1} (\rho_j \rho_l^{m-1} - \beta_{j,m-1} + \overline{\partial} \rho_l \wedge \rho_l^{m-1} \alpha_{j,m-1}) + \rho_l^m \tilde{C}_m - \rho_l^{m-1} \tilde{C}_{m-1}$$

we obtain

$$\overline{\partial} \tilde{f} = \sum_{j=1}^{l-1} (\rho_j \tilde{f}_j' + \overline{\partial} \rho_l \wedge \tilde{f}_j) + \Phi + \Gamma.$$  (###)

It remains to show that $\Phi + \Gamma$ can be decomposed into two terms where one can be absorbed by the sum and the other vanishes of infinite order on $T$ along $\overline{S}$. Let $m_0$ and $V$ be as before.

On $\overline{S} \cap V$ it follows that

$$\omega \wedge (\Phi + \Gamma) = \omega \wedge \overline{\partial} \tilde{f}$$

$$= \omega \wedge \left( \overline{\partial} f_{m_0} + \rho_l^{[m_0/2]} \overline{\partial} \gamma_{m_0} + \frac{\rho_l^{[m_0/2]}}{2} \overline{\partial} \rho_l \wedge \gamma_{m_0} \right)$$

$$= \rho_l^{[m_0/2]-1} \Pi_{m_0},$$

with a $C^\infty$ form $\Pi_{m_0}$. This yields that $\omega \wedge (\Phi + \Gamma)$ vanishes along $\overline{S}$ of infinite order in $T$.

Now let $\{\psi_1, \psi_2, \ldots, \psi_M\}$ be a partition of unity associated with the open covering $\{U_1, U_2, \ldots, U_M\}$ of a neighborhood of $\overline{S}$. Let the covering be so fine such that there exist on $U_i$, $C^\infty(0, 1)$-forms $e_1, e_2, \ldots, e_n-\ell+1$ with

$$\omega \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_{n-\ell+1} \neq 0.$$ 

Then there exist $C^\infty$ forms $R_{i,j}$ on $U_i \cap \overline{S}$ with

$$\Phi + \Gamma = \sum_{j=1}^{l-1} \overline{\partial} \rho_j \wedge R_{i,j} + R_{i,l},$$

where $R_{i,l}$ does not contain any $\overline{\partial} \rho_j$, $j < l$. Consequently $R_{i,l}$ vanishes of infinite order on $T$ along $\overline{S}$. Since on a neighborhood of $\overline{S}$ we have

$$\Phi + \Gamma = \sum_{j=1}^{l-1} \overline{\partial} \rho_j \wedge \sum_{i=1}^M \psi_i R_{i,j} + \sum_{i=1}^M \psi_i R_{i,l}$$
there exist $C^\infty$ forms $A_j, B_j$ on $\overline{\Omega}$ with

$$\Phi + \Gamma = \sum_{j=1}^{l-1} (\rho_j B_j + \overline{\partial} \rho_j \wedge A_j) + \sum_{i=1}^{M} \psi_i R_i,.$$ 

Inserting this into (***), gives

$$\overline{\partial} \tilde{f} = \sum_{j=1}^{l-1} \left( \rho_j \left( \tilde{f}_j'' + B_j \right) + \overline{\partial} \rho_j \wedge (\tilde{f}_j + A_j) \right) + C,$$

with $C = \sum_{i=1}^{M} \psi_i R_i$. By renaming $\tilde{f}_j'' + B_j$ and $\tilde{f}_j + A_j$ the lemma is proven. □

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