BOUSSINESQ EQUATIONS IN THIN SPHERICAL DOMAINS

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Abstract. The atmosphere is just the incompressible fluid occupying a thin layer around the earth. It is known that the equations describing its motion are the three-dimensional (3D) incompressible Navier–Stokes equations in thin spherical shells and the validity of the two-dimensional (2D) approximation of flows in thin spherical layers is proved. In this paper, we consider the 3D Boussinesq equations in thin spherical domains and prove the convergence of the average of weak solutions of the 3D Boussinesq equations to a 2D problem.

1. Introduction

Let Q_ε be a thin domain defined by

$$Q_\varepsilon = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 1 < |x| < 1 + \varepsilon\}, \quad (1.1)$$

where $0 < \varepsilon < 1/2$. We denote the sphere by S , namely

$$S = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}. \quad (1.2)$$

In this paper we are concerned with the following initial boundary value problem in a thin domain:

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p + 2\Omega \times u = g\theta \quad \text{in } Q_\varepsilon \times (0, T), \quad (1.3)$$

$$\operatorname{div} u = 0 \quad \text{in } Q_\varepsilon \times (0, T), \quad (1.4)$$

$$\partial_t \theta + (u \cdot \nabla)\theta - \kappa \Delta \theta = 0 \quad \text{in } Q_\varepsilon \times (0, T), \quad (1.5)$$

$$u \cdot \mathbf{n} = 0, \operatorname{curl} u \times \mathbf{n} = 0 \text{ and } \nabla \theta \cdot \mathbf{n} = 0 \quad \text{on } \partial Q_\varepsilon \times (0, T), \quad (1.6)$$

$$u(\cdot, 0) = u_0, \theta(\cdot, 0) = \theta_0 \quad \text{in } Q_\varepsilon, \quad (1.7)$$

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where $u = (u_1, u_2, u_3)$ is the fluid velocity, p is the pressure, θ is the temperature, g is the gravitational vector, $2\Omega \times u$ is the Coriolis acceleration, ν (kinematic viscosity) and κ (thermal diffusivity) are positive constants and $\mathbf{n}$ is the unit outer normal vector to the boundary ∂Q_ε . u_0 (respectively θ_0) is a vector (respectively scalar) function defined on Q_ε .

The system of equations (1.3)–(1.5) is called the Boussinesq equations. For not only a three-dimensional (3D) case but an arbitrary dimension, the existence and uniqueness of the weak solutions of the Boussinesq approximation have been studied (see [4, 7]). Moreover, the planetary geostrophic (PG) equations are derived from the Boussinesq equations using standard scale analysis, and the existence of the weak solutions of the PG equations also has been studied (see [12]).

The solutions of the Navier–Stokes equations (NSE) in thin domains (flat, curved and with various boundary conditions) has been extensively studied; see [1, 8, 10, 11]. Azérad and Guillen derived, by making use of different horizontal and vertical eddy viscosities, a 3D limit nonlinear model in [1]. By averaging along the vertical direction and using the uniqueness of solutions of two-dimensional (2D) NSE, Temam and Ziane obtained 2D limit models, together with existence and global regularity results in [10, 11].

Now we use the spherical coordinate system (r, λ, φ) , where $\lambda \in (0, \pi)$ is the colatitude and $\varphi \in (0, 2\pi)$ is the longitude; $\mathbf{e}_r, \mathbf{e}_\lambda, \mathbf{e}_\varphi$ is the local orthonormal basis associated with these coordinates. Using the spherical coordinate system, we can denote that $u = u_r \mathbf{e}_r + u_\lambda \mathbf{e}_\lambda + u_\varphi \mathbf{e}_\varphi = (u_r, u_\lambda, u_\varphi)$, $g = (g_0, 0, 0)$ (where g_0 is the gravitational acceleration constant) and $\Omega = f(\cos \lambda, -\sin \lambda, 0)$ (f a constant).

Our purpose in this paper is to prove that the averages in the vertical direction of the weak solution of the 3D Boussinesq equations converge as the thickness $\varepsilon \rightarrow 0$ to the weak solution of the following 2D initial boundary value problem:

$$\partial_t \tilde{u} + (\tilde{u} \cdot \nabla') \tilde{u} - \nu \Delta' \tilde{u} + \nabla' \tilde{p} + \tilde{b}(\tilde{u}) = 0 \quad \text{in } S \times (0, T), \quad (1.8)$$

$$\operatorname{div}' \tilde{u} = 0 \quad \text{in } S \times (0, T), \quad (1.9)$$

$$\partial_t \tilde{\theta} + (\tilde{u} \cdot \nabla') \tilde{\theta} - \kappa \Delta' \tilde{\theta} = 0 \quad \text{in } S \times (0, T), \quad (1.10)$$

$$\tilde{u}(x', 0) = \tilde{u}_0, \quad \tilde{\theta}(x', 0) = \tilde{\theta}_0 \quad \text{in } S, \quad (1.11)$$

where $\tilde{u} = (0, \tilde{u}_\lambda, \tilde{u}_\varphi)$ and $\tilde{b}(\tilde{u}) = 2f \cos \lambda (0, -\tilde{u}_\varphi, \tilde{u}_\lambda)$. We denote, here and henceforth, the operators in spherical coordinates with a prime, for example

$$\nabla' = \left(0, \partial_\lambda, \frac{1}{\sin \lambda} \partial_\varphi \right).$$

In order to explain $\tilde{u}_0$ and $\tilde{\theta}_0$, we introduce some Hilbert spaces.

We denote by $H^s(Q_\varepsilon)$ and $H^s(S)$, $s \in \mathbb{R}$, the Sobolev space constructed on $L^2(Q_\varepsilon)$ and $L^2(S)$. Moreover, we define the following:

$$\mathbf{H}_\varepsilon = \{v \in L^2(Q_\varepsilon)^3 : \operatorname{div} v = 0 \text{ in } Q_\varepsilon \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \partial Q_\varepsilon\},$$

$$\mathbf{V}_\varepsilon = \{v \in H^1(Q_\varepsilon)^3 : \operatorname{div} v = 0 \text{ in } Q_\varepsilon \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \partial Q_\varepsilon\},$$

$$\mathbf{H}_S = \{v \in L^2(S)^2 : \operatorname{div}' v = 0 \text{ in } S\},$$

$$\mathbf{V}_S = \{v \in H^1(S)^2 : \operatorname{div}' v = 0 \text{ in } S\}.$$

Here and henceforth, $\mathbf{X}^*$ denotes the dual space of $\mathbf{X}$. The scalar product in $L^2(Q_\varepsilon)^d$ is denoted by $(\cdot, \cdot)$ and in $H^1(Q_\varepsilon)^d$ is denoted by $((\cdot, \cdot))$, $d \in \mathbb{N}$, and the associated norms are denoted by $|\cdot|$ and $\|\cdot\|$ respectively.

Now we assume that $\tilde{u}_0$ and $\tilde{\theta}_0$ are the functions such that the following equations are satisfied:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} ru_0(r, \cdot) dr = \tilde{u}_0 \quad \text{weakly in } \mathbf{H}_S,$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \theta_0(r, \cdot) dr = \tilde{\theta}_0 \quad \text{weakly in } L^2(S).$$

Our result is the following theorem.

THEOREM 1.1. *Let T be a finite positive constant and let $\{u, \theta\}$ be the weak solution of Boussinesq equations (1.3)–(1.7). We assume that $u_0 \in \mathbf{V}_\varepsilon$ and $\theta_0 \in H^1(Q_\varepsilon)$. If $(\|u_0\|_{\mathbf{V}_\varepsilon})_{\varepsilon>0}$ and $(\|\theta_0\|_{H^1(Q_\varepsilon)})_{\varepsilon>0}$ are bounded and there exist $\tilde{u}_0 \in \mathbf{H}_S$ and $\tilde{\theta}_0 \in L^2(S)$ such that*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} ru_0(r, \cdot) dr = \tilde{u}_0 \quad \text{weakly in } \mathbf{H}_S, \quad (1.12)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \theta_0(r, \cdot) dr = \tilde{\theta}_0 \quad \text{weakly in } L^2(S), \quad (1.13)$$

then there exist $\tilde{u}$ and $\tilde{\theta}$ such that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} ru(r, \cdot) dr = \tilde{u} \quad \text{strongly in } \mathcal{C}([0, T]; \mathbf{V}_S^*) \cap L^2(0, T; \mathbf{H}_S), \quad (1.14)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \theta(r, \cdot) dr = \tilde{\theta} \quad \text{strongly in } \mathcal{C}([0, T]; (H^1(S))^*) \cap L^2(0, T; L^2(S)), \quad (1.15)$$

where $\{\tilde{u}, \tilde{\theta}\}$ is the weak solution of (1.8)–(1.11).

This paper is organized as follows. In Section 2 we introduce the average operators M_ε and another N_ε , and study some of their properties. Moreover, we introduce some fundamental inequalities and Theorem 5.1 of [1], which play an important role in the proof of Theorem 1.1. In Section 3 we derive the necessary *a priori* estimates for M_ε and N_ε , and prove the lemma to take advantage of the result of Theorem 5.1 of [1]. Finally, in Section 4, we prove Theorem 1.1.

2. Preliminaries

For a scalar function $\Psi(r, \lambda, \varphi) \in L^2(Q_\varepsilon)$, we define its average in the radial direction as follows:

$$(M_\varepsilon^1 \Psi)(\lambda, \varphi) = \frac{1}{\varepsilon} \int_1^{1+\varepsilon} r \Psi(r, \lambda, \varphi) dr \quad (2.1)$$

and

$$(M_\varepsilon^2 \Psi)(\lambda, \varphi) = \frac{1}{\varepsilon} \int_1^{1+\varepsilon} \Psi(r, \lambda, \varphi) dr \quad (2.2)$$

and set

$$N_\varepsilon^1 \Psi = \Psi - \frac{M_\varepsilon^1 \Psi}{r} \quad (2.3)$$

and

$$N_\varepsilon^2 \Psi = \Psi - M_\varepsilon^2 \Psi. \quad (2.4)$$

We also define the average operator $\tilde{M}_\varepsilon^1$ as

$$\tilde{M}_\varepsilon^1 u = (0, M_\varepsilon^1 u_\lambda, M_\varepsilon^1 u_\varphi) \quad \text{for } u = (u_r, u_\lambda, u_\varphi) \in L^2(Q_\varepsilon)^3 \quad (2.5)$$

and set

$$\tilde{N}_\varepsilon^1 u = u - \frac{\tilde{M}_\varepsilon^1 u}{r}. \quad (2.6)$$

It is clear from the definition of $N_\varepsilon^i, i = 1, 2$, that

$$\int_1^{1+\varepsilon} r N_\varepsilon^1 u_\lambda dr = \int_1^{1+\varepsilon} r N_\varepsilon^1 u_\varphi dr = \int_1^{1+\varepsilon} N_\varepsilon^2 \theta dr = 0. \quad (2.7)$$

For the boundary conditions under consideration, $\tilde{M}_\varepsilon^1 u/r, \tilde{N}_\varepsilon^1 u, M_\varepsilon^2 \theta$ and $N_\varepsilon^2 \theta$ satisfy the conditions

$$\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \mathbf{n} = \tilde{N}_\varepsilon^1 u \cdot \mathbf{n} = \nabla M_\varepsilon^2 \theta \cdot \mathbf{n} = \nabla N_\varepsilon^2 \theta \cdot \mathbf{n} = 0 \quad \text{on } \partial Q_\varepsilon. \quad (2.8)$$

Furthermore, because of $\nabla' M_\varepsilon^1 = M_\varepsilon^1 \nabla'$ (∇' is differentiation in the sense of distribution), we have the following properties:

If $u \in \mathbf{H}_\varepsilon$, then $\operatorname{div} \frac{\tilde{M}_\varepsilon^1 u}{r} = 0$ and $\operatorname{div} \tilde{N}_\varepsilon^1 u = 0$ in Q_ε .

Hence, we obtain

$$\frac{\tilde{M}_\varepsilon^1 u}{r} \in \mathbf{H}_\varepsilon \quad \text{and} \quad \tilde{N}_\varepsilon^1 u \in \mathbf{H}_\varepsilon, \quad \forall u \in \mathbf{H}_\varepsilon. \quad (2.9)$$

In the following lemma, we give the basic properties of the operators M_ε^1 , N_ε^1 , M_ε^2 and N_ε^2 . From the lemma, we can obtain the estimates of $\tilde{M}_\varepsilon^1 u$, $\tilde{N}_\varepsilon^1 u$, $M_\varepsilon^2 \theta$ and $N_\varepsilon^2 \theta$ used in Section 3.

LEMMA 2.1. *For all $u, v \in L^2(Q_\varepsilon)$, we have*

$$\left(\frac{M_\varepsilon^1 u}{r}, N_\varepsilon^1 v \right) = 0, \quad (2.10)$$

which implies

$$|u|^2 = \left| \frac{M_\varepsilon^1 u}{r} \right|^2 + |N_\varepsilon^1 u|^2. \quad (2.11)$$

For all $u, v \in H^1(Q_\varepsilon)$, we obtain

$$(\nabla M_\varepsilon^2 u, \nabla N_\varepsilon^2 v) = 0, \quad (2.12)$$

which implies

$$|\nabla u|^2 = |\nabla M_\varepsilon^2 u|^2 + |\nabla N_\varepsilon^2 u|^2. \quad (2.13)$$

Proof. For the proof of (2.10) and (2.11), see [11, Lemma 1.2].

For (2.12), because $\nabla' M_\varepsilon^2 = M_\varepsilon^2 \nabla'$ (∇' is differentiation in the sense of distribution), we have

$$\begin{aligned} (\nabla M_\varepsilon^2 u, \nabla N_\varepsilon^2 v) &= \int_{Q_\varepsilon} \nabla M_\varepsilon^2 u \cdot \nabla N_\varepsilon^2 v \, dx \\ &= \int_S \int_1^{1+\varepsilon} \nabla' M_\varepsilon^2 u \cdot \nabla' N_\varepsilon^2 u \sin \lambda \, dr \, d\lambda \, d\varphi \\ &= \int_S M_\varepsilon^2(\partial_\lambda u) \left(\int_1^{1+\varepsilon} N_\varepsilon^2(\partial_\lambda u) \, dr \right) \sin \lambda \, d\lambda \, d\varphi \\ &\quad + \int_S M_\varepsilon^2 \left(\frac{1}{\sin \lambda} \partial_\varphi u \right) \left(\int_1^{1+\varepsilon} N_\varepsilon^2 \left(\frac{1}{\sin \lambda} \partial_\varphi u \right) \, dr \right) \sin \lambda \, d\lambda \, d\varphi. \end{aligned} \quad (2.14)$$

Hence, from (2.7), we get the required result (2.12). $\square$

The following lemmas are necessary for estimating $\tilde{N}_\varepsilon^1 u$ and $N_\varepsilon^2 \theta$.

LEMMA 2.2. (Poincaré's inequality in thin spherical shells) For $0 < \varepsilon \leq 1/2$, we have

$$|\tilde{N}_\varepsilon^1 u| \leq 2\varepsilon |\operatorname{curl} \tilde{N}_\varepsilon^1 u| = 2\varepsilon |\nabla \tilde{N}_\varepsilon^1 u|, \quad \forall u \in \mathbf{V}_\varepsilon, \quad (2.15)$$

and

$$|N_\varepsilon^2 \theta| \leq \frac{3}{2}\varepsilon \|N_\varepsilon^2 \theta\|, \quad \forall \theta \in H^1(Q_\varepsilon). \quad (2.16)$$

Proof. For the proof of (2.15), see [11, Lemma 2.1].

Now we prove (2.16). First note that $C^1(\bar{Q}_\varepsilon)$ is dense in $H^1(Q_\varepsilon)$. Thanks to a density argument, we assume that $\theta \in C^1(\bar{Q}_\varepsilon)$. We have for any ζ, η in $[1, 1 + \varepsilon]$ that

$$\theta^2(\zeta, \lambda, \varphi) + \theta^2(\eta, \lambda, \varphi) = 2\theta(\zeta, \lambda, \varphi)\theta(\eta, \lambda, \varphi) + \left(\int_\eta^\zeta \partial_r \theta(s, \lambda, \varphi) ds \right)^2.$$

We fix ζ and integrate with respect to η to obtain

$$\begin{aligned} \varepsilon \theta^2(\zeta, \lambda, \varphi) + \int_1^{1+\varepsilon} \theta^2(\eta, \lambda, \varphi) d\eta \\ = 2\theta(\zeta, \lambda, \varphi) \int_1^{1+\varepsilon} \theta(\eta, \lambda, \varphi) d\eta + \int_1^{1+\varepsilon} \left(\int_\eta^\zeta \partial_r \theta(s, \lambda, \varphi) ds \right)^2 d\eta. \end{aligned} \quad (2.17)$$

If $\int_1^{1+\varepsilon} \theta(\eta, \lambda, \varphi) d\eta = 0$, we infer from (2.17) that

$$\begin{aligned} \int_1^{1+\varepsilon} \theta^2(\eta, \lambda, \varphi) d\eta &\leq \int_1^{1+\varepsilon} \left(\int_\eta^\zeta \partial_r \theta(s, \lambda, \varphi) ds \right)^2 d\eta \\ &\leq \int_1^{1+\varepsilon} |\zeta - \eta| d\eta \int_1^{1+\varepsilon} |\partial_r \theta(s, \lambda, \varphi)|^2 ds \\ &\leq \varepsilon^2 \int_1^{1+\varepsilon} |\partial_r \theta(s, \lambda, \varphi)|^2 ds. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_1^{1+\varepsilon} \theta^2(\eta, \lambda, \varphi) \eta^2 d\eta &\leq \frac{9}{4} \int_1^{1+\varepsilon} \theta^2(\eta, \lambda, \varphi) d\eta \\ &\leq \frac{9}{4} \varepsilon^2 \int_1^{1+\varepsilon} |\partial_r \theta(s, \lambda, \varphi)|^2 ds \\ &\leq \frac{9}{4} \varepsilon^2 \int_1^{1+\varepsilon} |\partial_r \theta(s, \lambda, \varphi)|^2 s^2 ds, \end{aligned}$$

because $0 < \varepsilon < 1/2$. Then we integrate with respect to λ and φ to obtain

$$|\theta|_{L^2(Q_\varepsilon)}^2 \leq \frac{9}{4} \varepsilon^2 \int_{Q_\varepsilon} |\partial_r \theta|^2 dx \leq \frac{9}{4} \varepsilon^2 \|\theta\|_{H^1(Q_\varepsilon)}^2$$

for $\theta \in \mathcal{C}^1(\bar{Q}_\varepsilon)$. The proof is complete. $\square$

LEMMA 2.3. *For $0 < \varepsilon$, we have*

$$|\gamma_0 N_\varepsilon^2 \theta|_{L^2(S)}^2 \leq \varepsilon \|N_\varepsilon^2 \theta\|^2, \quad \forall \theta \in H^1(Q_\varepsilon), \quad (2.18)$$

where γ_0 is the trace operator defined by $\gamma_0(\theta) = \theta|_{\partial Q_\varepsilon}$, $\forall \theta \in \mathcal{C}^1(\bar{Q}_\varepsilon)$.

Proof. From (2.17) we have

$$\varepsilon \theta^2(\zeta, \lambda, \varphi) \leq \int_1^{1+\varepsilon} \left(\int_\eta^\zeta \partial_r \theta(s, \lambda, \varphi) ds \right)^2 d\eta \leq \varepsilon^2 \int_1^{1+\varepsilon} |\partial_r \theta(s, \lambda, \varphi)|^2 s^2 ds,$$

$\forall \theta \in \mathcal{C}^1(\bar{Q}_\varepsilon)$ satisfying $\int_1^{1+\varepsilon} \theta(\eta, \lambda, \varphi) d\eta = 0$. Then we integrate with respect to λ and φ to obtain

$$|\theta(\zeta, \lambda, \varphi)|_{L^2(S)}^2 \leq \varepsilon |\partial_r \theta|^2, \quad (2.19)$$

for $1 \leq \zeta \leq 1 + \varepsilon$. Now we take $\zeta = 1$ or $\zeta = 1 + \varepsilon$ in (2.19). By the definition of the trace operator, we have

$$|\gamma_0 \theta|_{L^2(S)}^2 \leq \varepsilon |\partial_r \theta|^2 \leq \varepsilon \|\theta\|^2,$$

$\forall \theta \in \mathcal{C}^1(\bar{Q}_\varepsilon)$ satisfying $\int_1^{1+\varepsilon} \theta(\eta, \lambda, \varphi) d\eta = 0$. Hence, thanks to a density argument and the trace theorem, inequality (2.18) follows. $\square$

LEMMA 2.4. (Ladyzhenskaya's inequality) *There exist constants c_1 and c_2 , independent of ε , such that*

$$|N_\varepsilon^1 u|_{L^6(Q_\varepsilon)}^2 \leq c_1 \|N_\varepsilon^1 u\|^2, \quad \forall u \in \mathbf{V}_\varepsilon, \quad (2.20)$$

and

$$|N_\varepsilon^2 \theta|_{L^6(Q_\varepsilon)}^2 \leq c_2 \|N_\varepsilon^2 \theta\|^2, \quad \forall \theta \in H^1(Q_\varepsilon). \quad (2.21)$$

For the proof, see [11, Lemma 2.3].

By Lemmas 2.2, 2.4 and Hölder's inequality, we get

$$|N_\varepsilon^1 u|_{L^3(Q_\varepsilon)}^2 \leq c_1 \varepsilon \|N_\varepsilon^1 u\|^2, \quad \forall u \in \mathbf{V}_\varepsilon, \quad (2.22)$$

$$|N_\varepsilon^2 \theta|_{L^3(Q_\varepsilon)}^2 \leq c_2 \varepsilon \|N_\varepsilon^2 \theta\|^2, \quad \forall \theta \in H^1(Q_\varepsilon). \quad (2.23)$$

Finally, we quote the following theorem. The theorem plays an important role in proving our main result.

THEOREM 2.1. [1] Let $T > 0$, and let the Banach spaces $\mathbf{X} \xrightarrow{\text{compact}} \mathbf{B} \hookrightarrow \mathbf{Y}$. Let $(f_\varepsilon)_{\varepsilon>0}$ be a family of functions of $L^p(0, T; \mathbf{X})$, $1 \leq p \leq \infty$, with the extra condition that $(f_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}([0, T]; \mathbf{Y})$ if $p = \infty$, such that

(H1) $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(0, T; \mathbf{X})$,

(H2) $|f_\varepsilon(x, t+h) - f_\varepsilon(x, t)|_{L^p(0, T-h; \mathbf{Y})} \leq \varphi(h) + \psi(\varepsilon)$ with

$$\begin{cases} \lim_{h \rightarrow 0} \varphi(h) = 0, \\ \lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = 0. \end{cases}$$

Then the family $(f_\varepsilon)_{\varepsilon>0}$ possesses a cluster point in $L^p(0, T; \mathbf{B})$ and also in $\mathcal{C}([0, T]; \mathbf{B})$ if $p = \infty$, as $\varepsilon \rightarrow 0$.

For the proof, see [1, Theorem 5.1].

3. Weak formulation and *a priori* estimates

In this section we derive some *a priori* estimates for $\tilde{M}_\varepsilon^1 u$, $\tilde{N}_\varepsilon^1 u$, $M_\varepsilon^2 \theta$ and $N_\varepsilon^2 \theta$. Using some standard formulas from vector analysis, the weak formulations of (1.3)–(1.7) are then as follows:

$$\frac{d}{dt}(u, v) + ((u \cdot \nabla)u, v) + v(\operatorname{curl} u, \operatorname{curl} v) + 2(\Omega \times u, v) = (g\theta, v), \quad (3.1)$$

$$\frac{d}{dt}(\theta, w) + ((u \cdot \nabla)\theta, w) + \kappa(\nabla\theta, \nabla w) = 0 \quad (3.2)$$

for all $v = (v_r, v_\lambda, v_\varphi) \in \mathbf{V}_\varepsilon$ and $w \in H^1(Q_\varepsilon)$.

Now we define the weak solution of (1.3)–(1.7).

Definition 3.1. A pair of functions $\{u, \theta\}$ is called a *weak solution* of (1.3)–(1.7) if:

- (1) $\{u, \theta\}$ satisfies (3.1) and (3.2) for any $v \in \mathbf{V}_\varepsilon$ and $w \in H^1(Q_\varepsilon)$;
- (2) $\{u, \theta\}$ also satisfies energy inequalities

$$|\theta(t)|^2 + 2\kappa \int_0^t |\nabla\theta|^2 ds \leq |\theta_0|^2 \quad (3.3)$$

and

$$|u(t)|^2 + \nu \int_0^t |\nabla u|^2 ds \leq |u_0|^2 + \frac{1}{\nu} \int_0^t |g\theta|^2 ds. \quad (3.4)$$

From (3.3), we have

$$\theta \in L^\infty(0, T; L^2(Q_\varepsilon)) \cap L^2(0, T; H^1(Q_\varepsilon)) \quad (3.5)$$

because of the assumption of Theorem 1.1. We also obtain

$$g\theta \in L^\infty(0, T; L^2(Q_\varepsilon)^3) \cap L^2(0, T; H^1(Q_\varepsilon)^3). \quad (3.6)$$

Similarly, we obtain

$$u \in L^\infty(0, T; \mathbf{H}_\varepsilon) \cap L^2(0, T; \mathbf{V}_\varepsilon) \quad (3.7)$$

because of $u_0 \in \mathbf{V}_\varepsilon$ and (3.4).

Now we estimate $\tilde{M}_\varepsilon^1 u$, $\tilde{N}_\varepsilon^1 u$, $M_\varepsilon^2 \theta$ and $N_\varepsilon^2 \theta$. From (2.11), (3.7) and Lemma 2.2, taking into account (3.4) and the assumption of Theorem 1.1, we have

$$\begin{aligned} \varepsilon |\tilde{M}_\varepsilon^1 u(t)|_{L^2(S)}^2 &= \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right|^2 \\ &\leq c\varepsilon (|\tilde{M}_\varepsilon^1 u_0|_{L^2(S)}^2 + \|\tilde{N}_\varepsilon^1 u_0\|^2 + \|\tilde{N}_\varepsilon^1 g\theta\|_{L^2(0, T; H^1(Q_\varepsilon))}^2) \\ &\leq C\varepsilon, \end{aligned} \quad (3.8)$$

where C is independent of ε . Then (3.8) allows us to say that

$$\{\tilde{M}_\varepsilon^1 u\}_{\varepsilon>0} \text{ is a bounded sequence in } L^\infty(0, T; \mathbf{H}_S). \quad (3.9)$$

Similarly, from (2.11) and (3.7), we obtain

$$|\tilde{N}_\varepsilon^1 u(t)|^2 \leq C\varepsilon. \quad (3.10)$$

Therefore we see that

$$\{\|\tilde{N}_\varepsilon^1 u\|_{L^\infty(0, T; \mathbf{H}_\varepsilon)}\}_{\varepsilon>0} \text{ is bounded.} \quad (3.11)$$

Next, we consider a weighted L^2 -product on $\mathbf{H}_\varepsilon$:

$$(u, v)_{1, \varepsilon, r} = \int_{Q_\varepsilon} r^2 u \cdot v \, dx.$$

The corresponding norm $|\cdot|_{1, \varepsilon, r}$ is equivalent to $|\cdot|_{L^2(Q_\varepsilon)}$, uniformly with respect to ε for, say, $0 < \varepsilon < 1/2$:

$$|u|_{L^2(Q_\varepsilon)} \leq |u|_{1, \varepsilon, r} \leq \frac{3}{2} |u|_{L^2(Q_\varepsilon)}, \quad \forall u \in \mathbf{H}_\varepsilon,$$

and we have

$$\left(\nabla \frac{\tilde{M}_\varepsilon^1 u}{r}, \nabla \tilde{N}_\varepsilon^1 u \right)_{1, \varepsilon, r} = 0, \quad \forall u \in \mathbf{V}_\varepsilon.$$

Hence we obtain

$$\begin{aligned} |\nabla u|_{L^2(Q_\varepsilon)}^2 &\geq c|\nabla u|_{1,\varepsilon,r}^2 \geq c \left| \nabla \frac{\tilde{M}_\varepsilon^1 u}{r} \right|_{1,\varepsilon,r}^2 \geq c \left| \nabla \frac{\tilde{M}_\varepsilon^1 u}{r} \right|_{L^2(Q_\varepsilon)}^2 \\ &\geq c\varepsilon |\nabla' \tilde{M}_\varepsilon^1 u|_{L^2(S)}^2 \end{aligned} \quad (3.12)$$

and, similarly,

$$|\nabla u|_{L^2(Q_\varepsilon)}^2 \geq c|\nabla \tilde{N}_\varepsilon^1 u|_{L^2(Q_\varepsilon)}^2, \quad \forall u \in \mathbf{V}_\varepsilon. \quad (3.13)$$

Hence, from (3.12) and (3.13), calculating the same as for (3.10), we have

$$\int_0^T |\nabla' \tilde{M}_\varepsilon^1 u|_{L^2(S)}^2 ds \leq C_0, \quad (3.14)$$

and

$$\int_0^T |\nabla \tilde{N}_\varepsilon^1 u|^2 ds \leq C_0 \varepsilon, \quad (3.15)$$

where C_0 is independent of ε . Then (3.14) and (3.15) imply that

$$\{\tilde{M}_\varepsilon^1 u\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; \mathbf{V}_S), \quad (3.16)$$

and

$$\{\|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T;\mathbf{V}_\varepsilon)}\}_{\varepsilon>0} \text{ is bounded.} \quad (3.17)$$

Moreover, we also consider a weighted L^2 -product on $L^2(Q_\varepsilon)$ as

$$(u, v)_{2,\varepsilon,r} = \int_{Q_\varepsilon} \frac{1}{r^2} uv \, dx, \quad u, v \in L^2(Q_\varepsilon).$$

We have

$$\frac{2}{3}|\theta|_{L^2(Q_\varepsilon)} \leq |\theta|_{2,\varepsilon,r} \leq |\theta|_{L^2(Q_\varepsilon)},$$

and

$$(M_\varepsilon^2 \theta, N_\varepsilon^2 \theta)_{2,\varepsilon,r} = 0, \quad \forall \theta \in L^2(Q_\varepsilon).$$

Hence we obtain

$$\begin{aligned} |\theta|_{L^2(Q_\varepsilon)}^2 &\geq |\theta|_{2,\varepsilon,r}^2 \geq |M_\varepsilon^2 \theta|_{2,\varepsilon,r}^2 \geq c|M_\varepsilon^2 \theta|_{L^2(Q_\varepsilon)}^2 \\ &\geq c\varepsilon |M_\varepsilon^2 \theta|_{L^2(S)}^2 \end{aligned} \quad (3.18)$$

and

$$|\theta|_{L^2(Q_\varepsilon)}^2 \geq c|N_\varepsilon^2 \theta|_{L^2(Q_\varepsilon)}^2, \quad \forall \theta \in L^2(Q_\varepsilon). \quad (3.19)$$

From (3.5), (3.19), (3.20) and Lemma 2.2, we have

$$|M_\varepsilon^2 \theta|_{L^2(S)}^2 \leq c(|M_\varepsilon^2 \theta_0|_{L^2(S)}^2 + \|N_\varepsilon^2 \theta_0\|^2),$$

and

$$|N_\varepsilon^2 \theta|^2 \leq c\varepsilon(|M_\varepsilon^2 \theta_0|_{L^2(S)}^2 + \|N_\varepsilon^2 \theta_0\|^2).$$

Therefore, because of the assumption of Theorem 1.1, there exists a positive constant C_1 , independent of ε , such that

$$|M_\varepsilon^2 \theta|_{L^2(S)}^2 \leq C_1, \quad (3.20)$$

and

$$|N_\varepsilon^2 \theta|^2 \leq C_1 \varepsilon. \quad (3.21)$$

Then (3.20) and (3.21) allow us to say that

$$\{M_\varepsilon^2 \theta\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0, T; L^2(S)), \quad (3.22)$$

and

$$\{\|N_\varepsilon^2 \theta\|_{L^\infty(0, T; L^2(Q_\varepsilon))}\}_{\varepsilon>0} \text{ is bounded.} \quad (3.23)$$

Similarly, from (2.13) and (3.5), we can see that

$$\int_0^T |\nabla' M_\varepsilon^2 \theta|_{L^2(S)}^2 ds \leq C_2,$$

and

$$\int_0^T |\nabla N_\varepsilon^2 \theta|^2 ds \leq C_2 \varepsilon, \quad (3.24)$$

where C_2 is independent of ε . Hence, by the assumption of Theorem 1.1, we have

$$\{M_\varepsilon^2 \theta\}_{\varepsilon>0} \text{ is bounded in } L^2(0, T; H^1(S)), \quad (3.25)$$

and

$$\{\|N_\varepsilon^2 \theta\|_{L^2(0, T; H^1(Q_\varepsilon))}\}_{\varepsilon>0} \text{ is bounded.} \quad (3.26)$$

From (3.9) and (3.16), we have

$$\{\tilde{M}_\varepsilon^1 u\}_{\varepsilon>0} \text{ is bounded in } L^4(0, T; L^3(S)^3) \quad (3.27)$$

by interpolation between $L^\infty(0, T; L^2(S))$ and $L^2(0, T; L^6(S))$. Similarly, we obtain that

$$\{M_\varepsilon^2 \theta\}_{\varepsilon>0} \text{ is bounded in } L^4(0, T; L^3(S)) \quad (3.28)$$

from (3.22) and (3.25).

Remark. Because of (3.21) and (3.24), we can see that the estimate of the norm of $L^\infty(0, T; L^2(Q_\varepsilon))$ and $L^2(0, T; H^1(Q_\varepsilon))$ for the difference between the temperature in the thin spherical domain Q_ε and the average temperature in the vertical direction is less than $c\varepsilon^{1/2}$, where c is a constant depending only on the initial value. For the velocity, we can obtain a similar estimate from (3.10) and (3.15).

Let

$$\mathbf{W}_S = \{v \in H^2(S)^2 : \operatorname{div}' v = 0 \text{ in } S\}.$$

Then, from the Sobolev–Rellich embeddings, one deduces easily that

$$\mathbf{W}_S \hookrightarrow \mathbf{V}_S \hookrightarrow \mathbf{H}_S \equiv \mathbf{H}_S^* \hookrightarrow \mathbf{V}_S^* \hookrightarrow \mathbf{W}_S^*, \quad (3.29)$$

where all are dense and compact embeddings.

Now we state the important lemma. Because of the lemma, we can apply the result of Theorem 2.1 to $\tilde{M}_\varepsilon^1 u$ and $M_\varepsilon^2 \theta$. Here and henceforth, $\tau_h f(t)$ denotes $f(t+h)$.

LEMMA 3.1. *For $0 < \forall h < T$, there exist positive constants c_0 and c_1 , independent of h and ε , such that*

$$\|\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u\|_{L^\infty(0, T-h; \tilde{\mathbf{W}}^*)} \leq c_0(h + \varepsilon^{2/3}), \quad (3.30)$$

and

$$\|\tau_h M_\varepsilon^2 \theta - M_\varepsilon^2 \theta\|_{L^\infty(0, T-h; (H^2(S))^*)} \leq c_1(h^{1/2} + \varepsilon^{2/3}). \quad (3.31)$$

Proof. First, we prove (3.30). From (2.3) and (3.1), we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) + \frac{d}{dt} \left(\tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \\ &= - \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) - \left(\left(\tilde{N}_\varepsilon^1 u \cdot \nabla \right) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \\ & \quad - \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) - \left(\left(\tilde{N}_\varepsilon^1 u \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \\ & \quad - v \left(\operatorname{curl} \frac{\tilde{M}_\varepsilon^1 u}{r}, \operatorname{curl} \frac{v}{r} \right) - v \left(\operatorname{curl} \tilde{N}_\varepsilon^1 u, \operatorname{curl} \frac{v}{r} \right) \\ & \quad - 2 \left(\Omega \times \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) - 2 \left(\Omega \times \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) + \left(\tilde{N}_\varepsilon^1 g \theta, \frac{v}{r} \right), \quad \forall v \in \mathbf{W}_S. \end{aligned} \quad (3.32)$$

The estimates of each term in (3.32) are as follows:

$$\begin{aligned}
 \left| \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right| &= \left| \left(\frac{\tilde{M}_\varepsilon^1 u}{r}, \left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \frac{v}{r} \right) \right| \\
 &\leq \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right|_{L^3(Q_\varepsilon)} \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right| \left| \nabla \frac{v}{r} \right|_{L^6(Q_\varepsilon)} \\
 &\leq c\varepsilon^{4/3} |\tilde{M}_\varepsilon^1 u|_{L^3(S)} |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\left(\tilde{N}_\varepsilon^1 u \cdot \nabla \right) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right| &\leq |\tilde{N}_\varepsilon^1 u|_{L^3(Q_\varepsilon)} \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right| \left| \nabla \frac{v}{r} \right|_{L^6(Q_\varepsilon)} \\
 &\leq c\varepsilon^{3/2} |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|\tilde{N}_\varepsilon^1 u\| \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \right| &\leq c\varepsilon^{3/2} |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|\tilde{N}_\varepsilon^1 u\| \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\left(\tilde{N}_\varepsilon^1 u \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \right| &\leq |\tilde{N}_\varepsilon^1 u| |\tilde{N}_\varepsilon^1 u|_{L^3(Q_\varepsilon)} \left| \nabla \frac{v}{r} \right|_{L^6(Q_\varepsilon)} \\
 &\leq c\varepsilon |\tilde{N}_\varepsilon^1 u| \|\tilde{N}_\varepsilon^1 u\| \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\operatorname{curl} \frac{\tilde{M}_\varepsilon^1 u}{r}, \operatorname{curl} \frac{v}{r} \right) \right| &\leq \left| \left(\frac{\tilde{M}_\varepsilon^1 u}{r}, \Delta \frac{v}{r} \right) \right| \\
 &\leq \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right| \left| \Delta \frac{v}{r} \right| \\
 &\leq \varepsilon |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\operatorname{curl} \tilde{N}_\varepsilon^1 u, \operatorname{curl} \frac{v}{r} \right) \right| &\leq |\tilde{N}_\varepsilon^1 u| \left| \Delta \frac{v}{r} \right| \\
 &\leq \varepsilon^{3/2} \|\tilde{N}_\varepsilon^1 u\| \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\Omega \times \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right| &\leq |\Omega|_{\mathcal{C}(S)} \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right| \left| \frac{v}{r} \right| \\
 &\leq c\varepsilon |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|v\|_{\mathbf{W}_S}, \\
 \left| \left(\Omega \times \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \right| &\leq |\Omega|_{\mathcal{C}(S)} |\tilde{N}_\varepsilon^1 u| \left| \frac{v}{r} \right| \\
 &\leq c\varepsilon^{3/2} \|\tilde{N}_\varepsilon^1 u\| \|v\|_{\mathbf{W}_S},
 \end{aligned}$$

and

$$\left(\tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) = \left(\tilde{N}_\varepsilon^1 g\theta, \frac{v}{r} \right) = 0,$$

because of (2.7), (2.15), (2.22) and the fact that $|\nabla(v/r)|_{L^6(Q_\varepsilon)} \leq c\|\nabla(v/r)\| \leq c\|v\|_{\mathbf{W}_S}$.

From the above-mentioned estimate, we integrate (2.23) from t to $t + h$, $t \in [0, T - h]$, to obtain

$$\begin{aligned} & \left| \left(\frac{\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right| \\ & \leq \int_t^{t+h} c\varepsilon \left(\varepsilon^{1/3} |\tilde{M}_\varepsilon^1 u|_{L^3(S)} |\tilde{M}_\varepsilon^1 u|_{L^2(S)} + 2\varepsilon^{1/2} |\tilde{M}_\varepsilon^1 u|_{L^2(S)} \|\tilde{N}_\varepsilon^1 u\| \right. \\ & \quad \left. + \varepsilon \|\tilde{N}_\varepsilon^1 u\|^2 + |\tilde{M}_\varepsilon^1 u|_{L^2(S)} + \varepsilon^{1/2} \|\tilde{N}_\varepsilon^1 u\| + \varepsilon^{1/2} \|u\| \right) ds \cdot \|v\|_{\mathbf{W}_S} \\ & \leq (\text{with Hölder's inequality}) \\ & \leq c\varepsilon \left(\varepsilon^{1/3} h^{3/4} \|\tilde{M}_\varepsilon^1 u\|_{L^4(0,T; L^3(S))} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; L^2(S))} \right. \\ & \quad + 2\varepsilon^{1/2} h^{1/2} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; L^2(S))} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; H^1(Q_\varepsilon))} \\ & \quad + \varepsilon \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; H^1(Q_\varepsilon))}^2 + h \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; L^2(S))} \\ & \quad + \varepsilon^{1/2} h^{1/2} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; H^1(Q_\varepsilon))} \\ & \quad \left. + h \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; \mathbf{H}_S)} + h^{1/2} \varepsilon^{1/2} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; \mathbf{V}_\varepsilon)} \right) \|v\|_{\mathbf{W}_S}, \end{aligned}$$

and, taking into account $0 < h < T$ and $0 < \varepsilon < 1/2$, according to (3.9), (3.17) and (3.27), we obtain

$$\begin{aligned} \left| \left(\frac{\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right| & \leq c\varepsilon (\varepsilon^{1/3} h^{3/4} + 4\varepsilon^{1/2} h^{1/2} + \varepsilon + 2h) \|v\|_{\mathbf{W}_S} \\ & \leq c_0 \varepsilon (h + \varepsilon^{2/3}). \end{aligned}$$

Therefore, since $((\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u)/r, v/r) = \varepsilon(\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u, v)_{L^2(S)}$, we obtain

$$|(\tau_h \tilde{M}_\varepsilon^1 u - \tilde{M}_\varepsilon^1 u, v)_{L^2(S)}| \leq c_0 (h + \varepsilon^{2/3}) \|v\|_{\mathbf{W}_S}, \quad \forall v \in \mathbf{W}_S. \quad (3.33)$$

The proof of (3.31) is finished.

Next, we take w/r^2 as a test function in (3.2) and integrate over $(t, t+h)$. Then, from (2.4), we obtain

$$\begin{aligned}
 & \left(\tau_h M_\varepsilon^2 \theta(t) - M_\varepsilon^2 \theta(t), \frac{w}{r^2} \right) + \left(\tau_h N_\varepsilon^2 \theta(t) - N_\varepsilon^2 \theta(t), \frac{w}{r^2} \right) \\
 &= - \int_t^{t+h} \left[\left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) \right. \\
 & \quad + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) \\
 & \quad \left. + \kappa \left(\nabla M_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) + \kappa \left(\nabla N_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) \right] ds, \quad \forall w \in H^2(S). \quad (3.34)
 \end{aligned}$$

In a similar way to the case of $\tilde{M}_\varepsilon^1 u$, we have the following estimates:

$$\begin{aligned}
 & \left| \int_t^{t+h} \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 & \leq ch^{3/4} \varepsilon^{4/3} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; L^2(S))} \|\tilde{M}_\varepsilon^2 \theta\|_{L^4(0,T; L^3(S))} \|w\|_{H^2(S)}, \\
 & \left| \int_t^{t+h} \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 & \leq ch^{1/2} \varepsilon^{3/2} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T; L^2(S))} \|N_\varepsilon^2 \theta\|_{L^2(0,T; H^1(Q_\varepsilon))} \|w\|_{H^2(S)}, \\
 & \left| \int_t^{t+h} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 & \leq ch^{1/2} \varepsilon^{3/2} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; H^1(Q_\varepsilon))} \|M_\varepsilon^2 \theta\|_{L^\infty(0,T; L^2(S))} \|w\|_{H^2(S)}, \\
 & \left| \int_t^{t+h} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 & \leq c\varepsilon^2 \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T; H^1(Q_\varepsilon))} \|N_\varepsilon^2 \theta\|_{L^2(0,T; H^1(Q_\varepsilon))} \|w\|_{H^2(S)}, \\
 & \left| \int_t^{t+h} \left(\nabla M_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) ds \right| \\
 & \leq ch^{1/2} \varepsilon \|M_\varepsilon^2 \theta\|_{L^2(0,T; H^1(S))} \|w\|_{H^2(S)}.
 \end{aligned}$$

For the last term of the right-hand side in (3.34), according to the trace theorem, Lemma 2.2, Lemma 2.3 and (3.26), we obtain

$$\begin{aligned}
& \left| \int_t^{t+h} \left(\nabla N_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) ds \right| \\
& \leq \int_t^{t+h} \left[\left| \left(\gamma_0 N_\varepsilon^2 \theta, \gamma_1 \frac{w}{r^2} \right)_{L^2(S)} \right| + \left| \left(N_\varepsilon^2 \theta, \Delta \frac{w}{r^2} \right) \right| \right] ds \quad (\text{by Green's formula}) \\
& \leq \int_t^{t+h} \left(|\gamma_0 N_\varepsilon^2 \theta|_{L^2(S)} \left| \gamma_1 \frac{w}{r^2} \right|_{L^2(S)} + |N_\varepsilon^2 \theta| \left| \Delta \frac{w}{r^2} \right| \right) ds \\
& \leq ch^{1/2} (\varepsilon^{1/2} + 2\varepsilon) \|N_\varepsilon^2 \theta\|_{L^2(0,T; H^1(Q_\varepsilon))} \left\| \frac{w}{r^2} \right\|_{H^2(Q_\varepsilon)} \\
& \leq ch^{1/2} \varepsilon^{3/2} \|w\|_{H^2(S)}, \tag{3.35}
\end{aligned}$$

where γ_1 is the trace operator defined by $\gamma_1(w) = (\mathbf{n} \cdot \nabla)w|_{\partial Q_\varepsilon}$, $\forall w \in C^2(\bar{Q}_\varepsilon)$. On the other hand, from (2.7), we have

$$\left(\tau_h N_\varepsilon^2 \theta - N_\varepsilon^2 \theta, \frac{w}{r^2} \right) = \int_S \left(\int_1^{1+\varepsilon} N_\varepsilon^2 (\tau_h \theta - \theta) dr \right) w d\sigma = 0.$$

Hence, from the above, we obtain that

$$\begin{aligned}
& \left| \left(\tau_h M_\varepsilon^2 \theta - M_\varepsilon^2 \theta, \frac{w}{r^2} \right) \right| \\
& \leq c\varepsilon(h^{3/4}\varepsilon^{1/3} + 3h^{1/2}\varepsilon^{1/2} + \varepsilon + h^{1/2}) \|w\|_{H^2(S)} \\
& \leq c_1\varepsilon(h^{1/2} + \varepsilon^{2/3}) \|w\|_{H^2(S)},
\end{aligned}$$

because of (3.9), (3.17), (3.22), (3.25), (3.26) and (3.28). Therefore, by the fact that $(\tau_h M_\varepsilon^2 \theta - M_\varepsilon^2 \theta, w/r^2) = \varepsilon(\tau_h M_\varepsilon^2 \theta - M_\varepsilon^2 \theta, w)_{L^2(S)}$, we have,

$$|(\tau_h M_\varepsilon^2 \theta - M_\varepsilon^2 \theta, w)_{L^2(S)}| \leq c_1(h^{1/2} + \varepsilon^{2/3}) \|w\|_{H^2(S)}, \quad \forall w \in H^2(S).$$

The proof is complete. $\square$

4. Proof of the theorem

Let

$$\text{curl}' u = \frac{1}{\sin \lambda} \partial_\lambda (u_\varphi \sin \lambda) - \frac{1}{\sin \lambda} \partial_\varphi u_\lambda, \quad \forall u = (0, u_\lambda, u_\varphi) \in H^1(S)^3.$$

Then the space-time weak formulations of (1.8)–(1.11) are the following:

$$(\tilde{u}, v) = (\tilde{u}_0, v) - \int_0^t [((\tilde{u} \cdot \nabla') \tilde{u}, v) + \nu(\operatorname{curl}' \tilde{u}, \operatorname{curl}' v) + (\tilde{b}(\tilde{u}), v)] ds, \quad (4.1)$$

$$(\tilde{\theta}, w) = (\tilde{\theta}_0, w) - \int_0^t [((\tilde{u} \cdot \nabla') \tilde{\theta}, w) + \kappa(\nabla' \tilde{\theta}, \nabla' w)] ds, \quad (4.2)$$

for all $v \in \mathbf{V}_S$ and $w \in H^1(S)$.

Definition 4.1. A pair of functions $\{\tilde{u}, \tilde{\theta}\}$ is called a *weak solution* of (1.8)–(1.11) if $\{\tilde{u}, \tilde{\theta}\}$ satisfies (4.1) and (4.2) for any $v \in \mathbf{V}_S$ and $w \in H^1(S)$.

The purpose of the following is to show that $\tilde{M}_\varepsilon^1 u$ and $M_\varepsilon^2 \theta$ converge, as $\varepsilon \rightarrow 0$, to the weak solution $\{\tilde{u}, \tilde{\theta}\}$ in $\mathcal{C}([0, T]; \mathbf{V}_S^*) \cap L^2(0, T; \mathbf{H}_S)$ and $\mathcal{C}([0, T]; (H^1(S))^*) \cap L^2(0, T; L^2(S))$ respectively.

First, we obtain the convergent subsequences of $\tilde{M}_\varepsilon^1 u$ and $M_\varepsilon^2 \theta$.

Due to Lemma 3.1 and (3.22), we can apply Theorem 2.1 for $p = \infty$ and the spaces $L^2(S) \xrightarrow{\text{compact}} (H^1(S))^* \hookrightarrow (H^2(S))^*$. Therefore, there exists a subsequence, still denoted by $M_\varepsilon^2 \theta$, and a function $\hat{\theta}$ such that

$$M_\varepsilon^2 \theta \rightarrow \hat{\theta} \quad \text{strongly in } \mathcal{C}([0, T]; (H^1(S))^*). \quad (4.3)$$

Similarly, combining Lemma 3.1 and (3.9), we can apply Theorem 2.1 for $p = \infty$ and the spaces $\mathbf{H}_S \xrightarrow{\text{compact}} \mathbf{V}_S^* \hookrightarrow \mathbf{W}_S^*$. Therefore, there exists a subsequence, still denoted by $\tilde{M}_\varepsilon^1 u$, and a function $\hat{u}$ such that

$$\tilde{M}_\varepsilon^1 u \rightarrow \hat{u} \quad \text{strongly in } \mathcal{C}([0, T]; \mathbf{V}_S^*). \quad (4.4)$$

Moreover, we also apply Theorem 2.1 for $p = 2$ and the spaces $H_0^1(S) \xrightarrow{\text{compact}} L^2(S) \hookrightarrow (H^2(S))^*$. Then, because of Lemma 3.1, (3.25) and the fact that $L^\infty(0, T - h; (H^2(S)^*)) \subset L^2(0, T - h; (H^2(S))^*)$, there exists a subsequence, still denoted by $M_\varepsilon^2 \theta$, such that

$$M_\varepsilon^2 \theta \rightarrow \hat{\theta} \quad \text{strongly in } L^2(0, T; L^2(S)). \quad (4.5)$$

Similarly, we apply Theorem 2.1 for $p = 2$ and the spaces $\mathbf{V}_S \xrightarrow{\text{compact}} \mathbf{H}_S \hookrightarrow \mathbf{W}_S^*$. By Lemma 3.1, (3.16) and the fact that $L^\infty(0, T - h; \mathbf{W}_S^*) \subset L^2(0, T - h; \mathbf{W}_S^*)$, there exists a subsequence, still denoted by $\tilde{M}_\varepsilon^1 u$, such that

$$\tilde{M}_\varepsilon^1 u \rightarrow \hat{u} \quad \text{strongly in } L^2(0, T; \mathbf{H}_S). \quad (4.6)$$

Now, we will prove that $\{\hat{u}, \hat{\theta}\}$ is the weak solution of (1.8)–(1.11). First, because of (3.9) and (3.22), there exists the subsequences, still denote by $\tilde{M}_\varepsilon^1 u$ and $M_\varepsilon^2 \theta$, such that

$$\tilde{M}_\varepsilon^1 u \rightarrow \hat{u} \quad \text{weakly in } L^\infty(0, T; \mathbf{H}_S), \quad (4.7)$$

and

$$M_\varepsilon^2 \theta \rightarrow \hat{\theta} \quad \text{weakly in } L^\infty(0, T; L^2(S)). \quad (4.8)$$

Moreover, by the assumption of Theorem 1.1, we have

$$\hat{u}(\lambda, \varphi, 0) = \tilde{u}_0 \quad \text{and} \quad \hat{\theta}(\lambda, \varphi, 0) = \tilde{\theta}_0.$$

The following is the proof that $\hat{\theta}$ satisfies (4.2) for any $w \in H^1(S)$.

For all $w \in C^\infty(S)$, we take w/r^2 as a test function in (3.2) and integrate between 0 and t . Then we have

$$\begin{aligned} \left(M_\varepsilon^2 \theta, \frac{w}{r^2} \right) &= \left(M_\varepsilon^2 \theta_0, \frac{w}{r^2} \right) \\ &\quad - \int_0^t \left[\left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \kappa \left(\nabla M_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) \right. \\ &\quad \left. + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) \right. \\ &\quad \left. + \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \kappa \left(\nabla N_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) \right] ds, \end{aligned} \quad (4.9)$$

for $t \in [0, T]$. For the left-hand side of (4.9), we have

$$\left(M_\varepsilon^2 \theta, \frac{w}{r^2} \right) = \varepsilon (M_\varepsilon^2 \theta, w)_{L^2(S)}. \quad (4.10)$$

For each term of the right-hand side, we obtain the following:

$$\begin{aligned} \left(M_\varepsilon^2 \theta_0, \frac{w}{r^2} \right) &= \varepsilon (M_\varepsilon^2 \theta_0, w)_{L^2(S)}, \\ \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) &= \frac{\varepsilon}{1 + \varepsilon} ((\tilde{M}_\varepsilon^1 u \cdot \nabla') M_\varepsilon^2 \theta, w)_{L^2(S)}, \\ \left(\nabla M_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) &= \frac{\varepsilon}{1 + \varepsilon} (\nabla' M_\varepsilon^2 \theta, \nabla' w)_{L^2(S)}. \end{aligned}$$

Hence, (4.9) is equivalent to

$$\begin{aligned}
 (M_\varepsilon^2 \theta, w)_{L^2(S)} &= (M_\varepsilon^2 \theta_0, w)_{L^2(S)} - \int_0^t \left[\frac{1}{1+\varepsilon} ((\tilde{M}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, w)_{L^2(S)} \right. \\
 &\quad + \frac{1}{\varepsilon} \left\{ \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) \right. \\
 &\quad \left. \left. + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) \right\} \right. \\
 &\quad \left. + \frac{\kappa}{1+\varepsilon} (\nabla' M_\varepsilon^2 \theta, \nabla' w)_{L^2(S)} + \frac{\kappa}{\varepsilon} \left(\nabla N_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) \right] ds, \quad (4.11)
 \end{aligned}$$

for all $w \in C^\infty(S)$ and $t \in [0, T]$. For the nonlinear terms in (4.11), we obtain

$$\begin{aligned}
 &\left| \int_0^t \frac{1}{\varepsilon} \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 &= \left| \int_0^t \frac{1}{\varepsilon} \left(N_\varepsilon^2 \theta, \left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \frac{w}{r^2} \right) ds \right| \\
 &\leq \frac{1}{\varepsilon} \int_0^T |N_\varepsilon^2 \theta| \left| \frac{\tilde{M}_\varepsilon^1 u}{r} \right| ds \cdot \left| \nabla \frac{w}{r^2} \right|_{C(Q_\varepsilon)} \\
 &\leq c\varepsilon^{1/2} \|N_\varepsilon^2 \theta\|_{L^2(0,T;H^1(S))} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0,T;L^2(S))} \left| \nabla \frac{w}{r^2} \right|_{C(Q_\varepsilon)}, \\
 &\left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 &\leq c\varepsilon^{1/2} \|M_\varepsilon^2 \theta\|_{L^\infty(0,T;L^2(S))} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T;\mathbf{V}_S)} \left| \nabla \frac{w}{r^2} \right|_{C(Q_\varepsilon)}, \\
 &\left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| \\
 &\leq c\varepsilon \|\tilde{N}_\varepsilon^1 u\|_{L^2(0,T;\mathbf{V}_S)} \|N_\varepsilon^2 \theta\|_{L^2(0,T;H^1(S))} \left| \nabla \frac{w}{r^2} \right|_{C(Q_\varepsilon)},
 \end{aligned}$$

by Lemma 2.2. Therefore, due to (3.9), (3.17), (3.22) and (3.26), we have

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \frac{1}{\varepsilon} \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| = 0, \quad (4.12)$$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) M_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| = 0, \quad (4.13)$$

and

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) N_\varepsilon^2 \theta, \frac{w}{r^2} \right) ds \right| = 0. \quad (4.14)$$

Moreover, we obtain

$$\begin{aligned} & \left| \int_0^t ((\tilde{M}_\varepsilon^1 u \cdot \nabla') M_\varepsilon^2 \theta, w)_{L^2(S)} ds - \int_0^t ((\hat{u} \cdot \nabla') \hat{\theta}, w)_{L^2(S)} ds \right| \\ & \leq \int_0^T |((\tilde{M}_\varepsilon^1 u - \hat{u}) \cdot \nabla' M_\varepsilon^2 \theta, w)_{L^2(S)}| ds \\ & \quad + \int_0^T |((\hat{u} \cdot \nabla')(M_\varepsilon^2 \theta - \hat{\theta}), w)_{L^2(S)}| ds \\ & \leq \|\tilde{M}_\varepsilon^1 u - \hat{u}\|_{L^2(0,T; \mathbf{H}_S)} \|M_\varepsilon^2 \theta\|_{L^2(0,T; H^1(S))} |w|_{\mathcal{C}(S)} \\ & \quad + \|M_\varepsilon^2 \theta - \hat{\theta}\|_{L^2(0,T; L^2(S))} \|\hat{u}\|_{L^2(0,T; \mathbf{V}_S)} |w|_{\mathcal{C}(S)} \\ & \rightarrow 0, \end{aligned} \quad (4.15)$$

as $\varepsilon \rightarrow 0$, because of (3.25), (4.5) and (4.6). For the remaining terms, by (4.8), we have

$$\begin{aligned} \int_0^t (\nabla' M_\varepsilon^2 \theta, \nabla' w)_{L^2(S)} ds &= - \int_0^t (M_\varepsilon^2 \theta, \Delta' w)_{L^2(S)} ds \\ &\rightarrow - \int_0^t (\hat{\theta}, \Delta' w)_{L^2(S)} ds \\ &= \int_0^t (\nabla' \hat{\theta}, \nabla' w)_{L^2(S)} ds, \end{aligned} \quad (4.16)$$

as $\varepsilon \rightarrow 0$. We also calculate as for (3.35) to obtain

$$\begin{aligned} & \left| \int_0^t \frac{1}{\varepsilon} \left(\nabla N_\varepsilon^2 \theta, \nabla \frac{w}{r^2} \right) ds \right| \\ & \leq \int_0^T \frac{1}{\varepsilon} \left[\left| \left(\gamma_0 N_\varepsilon^2 \theta, \gamma_1 \frac{w}{r^2} \right)_{L^2(S)} \right| + \left| \left(N_\varepsilon^2 \theta, \Delta \frac{w}{r^2} \right) \right| \right] ds \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^T \frac{1}{\varepsilon} \left(|\gamma_0 N_\varepsilon^2 \theta|_{L^2(S)} \left| \gamma_1 \frac{w}{r^2} \right|_{L^2(S)} + |N_\varepsilon^2 \theta| \left| \Delta \frac{w}{r^2} \right| \right) ds \\
 &\leq \frac{c}{\varepsilon} (\varepsilon^{1/2} + 2\varepsilon) \|N_\varepsilon^2 \theta\|_{L^2(0,T; H^1(Q_\varepsilon))} \left\| \frac{w}{r^2} \right\|_{H^2(Q_\varepsilon)} \\
 &\leq c\varepsilon^{1/2} \|w\|_{H^2(S)} \\
 &\rightarrow 0,
 \end{aligned} \tag{4.17}$$

as $\varepsilon \rightarrow 0$, because of the trace theorem, Lemma 2.2, Lemma 2.3 and (3.26).

Hence, from (4.8) and (4.12)–(4.17), the right-hand side in (4.11) converges to

$$(\tilde{\theta}_0, w)_{L^2(S)} - \int_0^t [(\hat{u} \cdot \nabla') \hat{\theta}, w]_{L^2(S)} + \kappa(\nabla' \hat{\theta}, \nabla' w)_{L^2(S)}] ds,$$

as $\varepsilon \rightarrow 0$ (t is fixed) which is equal to $(\hat{\theta}(t), w)_{L^2(S)}$. Hence, since $H^1(S)$ is the completion of $C^\infty(S)$ under the $H^1(S)$ norm, we obtain

$$(\hat{\theta}, w) = (\tilde{\theta}_0, w) - \int_0^t [((\hat{u} \cdot \nabla') \hat{\theta}, w) + \kappa(\nabla' \hat{\theta}, \nabla' w)] ds, \quad \forall w \in H^1(S). \tag{4.18}$$

From (4.18), we can see that $\hat{\theta}$ is the weak solution of (4.2).

Next, we prove that $\hat{u}$ satisfies (4.1) for any $v \in \mathbf{V}_S$. From (2.23), we have

$$\begin{aligned}
 (\tilde{M}_\varepsilon^1 u, v)_{L^2(S)} &= (\tilde{M}_\varepsilon^1 u_0, v)_{L^2(S)} - \int_0^t \left[\frac{1}{1+\varepsilon} ((\tilde{M}_\varepsilon^1 u \cdot \nabla') \tilde{M}_\varepsilon u, v)_{L^2(S)} \right. \\
 &\quad + \frac{1}{\varepsilon} \left\{ \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) \right. \\
 &\quad \left. \left. + \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) \right\} \right. \\
 &\quad + \frac{v}{1+\varepsilon} (\operatorname{curl}' \tilde{M}_\varepsilon u, \operatorname{curl}' v)_{L^2(S)} + \frac{v}{\varepsilon} \left(\operatorname{curl} \tilde{N}_\varepsilon^1 u, \operatorname{curl} \frac{v}{r} \right) \\
 &\quad \left. + 2(\Omega \times \tilde{M}_\varepsilon^1 u, v)_{L^2(S)} \right] ds,
 \end{aligned} \tag{4.19}$$

where $v \in \{\varphi \in C^\infty(S)^2 : \operatorname{div}' \varphi = 0 \text{ in } S\}$ and $t \in [0, T]$. We calculate as for the case of $M_\varepsilon^2 \theta$, and obtain

$$\begin{aligned}
& \int_0^t ((\tilde{M}_\varepsilon^1 u \cdot \nabla') \tilde{M}_\varepsilon^1 u, v) ds \rightarrow \int_0^t ((\hat{u} \cdot \nabla') \hat{u}, v) ds, \\
& \int_0^t (\operatorname{curl}' \tilde{M}_\varepsilon^1 u, \operatorname{curl}' v) ds \rightarrow \int_0^t (\operatorname{curl}' \hat{u}, \operatorname{curl}' v) ds, \\
& \int_0^t 2(\Omega \times \tilde{M}_\varepsilon^1 u, v) ds \rightarrow \int_0^t (\tilde{b}(\hat{u}), v) ds, \\
& \left| \int_0^t \frac{1}{\varepsilon} \left(\left(\frac{\tilde{M}_\varepsilon^1 u}{r} \cdot \nabla \right) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) ds \right| \\
& \quad \leq c\varepsilon^{1/2} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0, T; \mathbf{H}_S)} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0, T; \mathbf{V}_\varepsilon)} \left| \nabla \frac{v}{r} \right|_{\mathcal{C}(Q_\varepsilon)} \rightarrow 0, \\
& \left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) \frac{\tilde{M}_\varepsilon^1 u}{r}, \frac{v}{r} \right) ds \right| \\
& \quad \leq c\varepsilon^{1/2} \|\tilde{M}_\varepsilon^1 u\|_{L^\infty(0, T; \mathbf{H}_S)} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0, T; \mathbf{V}_\varepsilon)} \left| \nabla \frac{v}{r} \right|_{\mathcal{C}(Q_\varepsilon)} \rightarrow 0, \\
& \left| \int_0^t \frac{1}{\varepsilon} \left((\tilde{N}_\varepsilon^1 u \cdot \nabla) \tilde{N}_\varepsilon^1 u, \frac{v}{r} \right) ds \right| \leq c\varepsilon \|\tilde{N}_\varepsilon^1 u\|_{L^2(0, T; \mathbf{V}_\varepsilon)}^2 \left| \nabla \frac{v}{r} \right|_{\mathcal{C}(Q_\varepsilon)} \rightarrow 0, \\
& \left| \int_0^t \frac{1}{\varepsilon} \left(\operatorname{curl} \tilde{N}_\varepsilon^1 u, \operatorname{curl} \frac{v}{r} \right) ds \right| \\
& \quad \leq c\varepsilon^{1/2} \|\tilde{N}_\varepsilon^1 u\|_{L^2(0, T; \mathbf{V}_\varepsilon)} (|v|_{\mathcal{C}(S)} + |\Delta' v|_{\mathcal{C}(Q_\varepsilon)}) \rightarrow 0,
\end{aligned}$$

as $\varepsilon \rightarrow 0$. Then the right-hand side in (4.19) converges to

$$(\tilde{u}_0, v)_{L^2(S)} - \int_0^t [((\hat{u} \cdot \nabla') \hat{u}, v)_{L^2(S)} + v(\operatorname{curl}' \hat{u}, \operatorname{curl}' v)_{L^2(S)} + (\tilde{b}(\hat{u}), v)_{L^2(S)}] ds,$$

which is equal to $(\hat{u}, v)_{L^2(S)}$, as $\varepsilon \rightarrow 0$ (t is fixed). Hence we obtain

$$(\hat{u}, v) = (\tilde{u}_0, v) - \int_0^t [((\hat{u} \cdot \nabla') \hat{u}, v) + v(\operatorname{curl}' \hat{u}, \operatorname{curl}' v) + (\tilde{b}(\hat{u}), v)] ds, \quad (4.20)$$

for all $v \in \mathbf{V}_S$, because $\mathbf{V}_S$ is the completion of the space $\{\varphi \in C^\infty(S)^2 : \operatorname{div}' \varphi = 0 \text{ in } S\}$ under the $H^1(S)^2$ norm. From (4.20), we can see that $\hat{u}$ is the weak solution of (4.1).

Finally, because of (4.3) and (4.4), we obtain the weak time-continuity $\hat{\theta} \in \mathcal{C}(0, T; (H^1(S))^*)$ and $\hat{u} \in \mathcal{C}(0, T; \mathbf{V}_S^*)$, so that the initial conditions makes sense. The proof of Theorem 1.1 is finished.

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