ON THE CHARACTERISTIC NUMBERS OF THE MANIFOLDS IMMERSED IN THE MULTIPLE POINT SET OF A SELF-TRANSVERSE IMMERSION

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Abstract. We study characteristic numbers of the manifold immersed in the multiple point set of a self-transverse immersion of a closed manifold in the sphere. The signature of the manifold immersed in the multiple point set is described as the characteristic number with L-classes.

1. Introduction

Let $f : V^n \longrightarrow W^{n+k}$ be a self-transverse immersion. We have the manifold N_d that is immersed in the d -multiple point set in W^{n+k} (cf. [2]). Herbert investigated properties of the manifold N_d in [2] to study the multiple point set and Ronga obtained the important pull-back diagram of immersions including f and the natural d -fold covering $p'' : \widetilde{N}_d \longrightarrow N_d$ in [6]. The definition of the manifolds N_d and $\widetilde{N}_d$ are given in Section 3 and here we call the manifold N_d the d -multiple point manifold. This paper studies the characteristic numbers of the multiple point manifold of a self-transverse immersion of a closed smooth manifold in the sphere. Section 2 is devoted to constructing natural transformations from the cobordism cohomology to the ordinary cohomology that is used to describe the characteristic numbers of the multiple point manifold for a self-transverse immersion in the sphere. Section 3 is

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concerned with a self-transverse immersion of an oriented manifold V^n in the $(n+2k)$ -dimensional sphere S^{n+2k} . We describe the characteristic numbers of the covering space $\widetilde{N}_d$ for the d -multiple point manifold N_d as follows:

$$\langle p_{\mathbf{t}}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = (-1)^{d-1} \langle \{e_H(v_f)\}^{d-1} \{p_{\mathbf{t}}(\tau(V^n))\}^d, [V^n] \rangle,$$

where $p_{\mathbf{t}}(\xi)$ denotes the generalized Pontrjagin class of a vector bundle ξ given in Section 2 and $e_H(v_f)$ is the Euler class in the ordinary cohomology theory. Furthermore, we discuss the unoriented case and we get the same formula as the oriented case for a self-transverse immersion V^n in S^{n+k} as follows:

$$\langle w_{\mathbf{t}}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = \langle \{e_H(v_f)\}^{d-1} \{w_{\mathbf{t}}(\tau(V^n))\}^d, [V^n] \rangle.$$

By virtue of the above formulas we see that if V^n is the boundary of an oriented manifold and d is odd then the d -multiple point manifold N_d is the boundary of an oriented manifold. Section 4 is concerned with the rational oriented cobordism class $[N_d]$ for a self-transverse immersion of an oriented manifold V^n in S^{n+2k} , which is described by the cobordism classes of the complex projective spaces. In Section 5 we generalize the formula of the Euler characteristic modulo 2 of the multiple point manifolds that is given in [1]. In Section 6 we study the signature of N_d for a self-transverse immersion of an oriented manifold V^{4n} in S^{4n+2k} , which is described as

$$\sigma(N_d) = \frac{(-1)^{d-1}}{d} \langle \{e_H(v_f)\}^{d-1} \{L(\tau(V^{4n}))\}^d, [V^{4n}] \rangle,$$

where $L(\xi)$ denotes the L -class of a vector bundle ξ .

2. The natural transformation from the cobordism cohomology to ordinary cohomology $H^*(\cdot)[[t_1, t_2, \dots, t_i, \dots]]$

Let ξ be an n -dimensional vector bundle over X . We consider the characteristic class with respect to a formal power series:

$$f(x) = 1 + xt_1 + x^2t_2 + \dots + x^it_i + \dots$$

in the ring $H^*(X; \mathcal{Q})[[t_1, t_2, \dots, t_i, \dots]]$ consisting of formal power series of t_i with the coefficients in the rational cohomology group $H^*(X; \mathcal{Q})$. Let ξ be an n -dimensional real vector bundle. Here we consider formal indeterminate variables

$\alpha_1, \alpha_2, \dots, \alpha_n$ and the product of $f(\alpha_i)$:

$$\begin{aligned} f(\alpha_1)f(\alpha_2)\cdots f(\alpha_n) &= \prod_{i=1}^n (1 + \alpha_k t_1 + \alpha_k^2 t_2 + \cdots + \alpha_k^i t_i + \cdots) \\ &= \sum s_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\alpha) t_1^{i_1}, \dots, t_\ell^{i_\ell}. \end{aligned}$$

$s_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\alpha)$ is the symmetric polynomial of α_i , which is described by the elementary symmetric polynomials $\sigma_i(\alpha)$ of α_i :

$$s_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\alpha) = p_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\sigma_1(\alpha), \dots, \sigma_n(\alpha)).$$

For the vector bundle ξ we set

$$s_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\xi) = p_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(p_1(\xi), \dots, p_n(\xi)).$$

Then the characteristic class $p_t(\xi)$, called the generalized Pontrjagin class, can be defined as follows:

$$p_t(\xi) = \sum s_{(\underbrace{1, \dots, 1}_{i_1}, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})}(\xi) t_1^{i_1}, \dots, t_\ell^{i_\ell}.$$

The class $p_t(\xi)$ satisfies the following properties.

- (1) (Naturality) $p_t(f^*\xi) = f^*(p_t(\xi))$, for $f : X \longrightarrow Y$.
- (2) (Multiplicativity) $p_t(\xi \oplus \eta) = p_t(\xi)p_t(\eta)$.
- (3) For a complex line bundle ℓ

$$p_t(\ell) = 1 + e_H(\ell)^2 t_1 + \{e_H(\ell)\}^4 t_2 + \cdots + \{e_H(\ell)\}^{2n} t_n + \cdots$$

where $e_H(\ell)$ is the Euler class of ℓ .

The characteristic classes $s_\omega(\xi)$ satisfy the following properties.

- (1) (Naturality) $s_\omega(f^*\xi) = f^*(s_\omega(\xi))$, for $f : X \longrightarrow Y$.
- (2) (Multiplicativity) $s_\omega(\xi \oplus \eta) = \sum_{\omega=\omega' \cup \omega''} s_{\omega'}(\xi) s_{\omega''}(\eta)$.
- (3) For the canonical complex line bundle ξ_n over the complex projective space CP^n

$$s_\omega(\xi_n) = \begin{cases} x^{2\ell} & \omega = (\ell) \\ 0 & \text{otherwise} \end{cases}$$

where $H^*(CP^n; Z) = Z[x_n]/(x_n^{n+1})$.

Here we see the characteristic class with respect to a formal power series:

$$f(x) = 1 + xt_1 + \cdots + x^n t_n \cdots$$

in the ring $H^*(X; Z_2)[[t_1, t_2, \dots, t_i, \dots]]$ consisting of formal power series of t_i with the coefficients in the Z_2 -cohomology group $H^*(X; Z_2)$. In the same way as the generalized Pontrjagin class, we define the generalized Stiefel–Whitney class $w_t(\xi)$. The class $w_t(\xi)$ satisfies the following properties.

- (1) (Naturality) $w_t(f^*\xi) = f^*(w_t(\xi))$, for $f : X \longrightarrow Y$.
- (2) (Multiplicativity) $w_t(\xi \oplus \eta) = w_t(\xi)w_t(\eta)$.
- (3) For a line bundle ℓ

$$w_t(\ell) = 1 + e_H(\ell)t_1 + \{e_H(\ell)\}^2 t_2 + \cdots + \{e_H(\ell)\}^n t_n + \cdots$$

where $e_H(\ell)$ is the Euler class of ℓ .

Now we define a natural transformation from the oriented cobordism cohomology $\Omega^*(X)$ to the ordinary cohomology

$$\mathcal{P} : \Omega^*(X) \longrightarrow H^*(X; Q)[[t_1, t_2, \dots, t_i, \dots]].$$

Take an element

$$[f] \in \Omega^n(X) = \varinjlim_{\ell} [S^{\ell-n} X^+, MSO(\ell)]_0, \quad f : S^{\ell-n} X^+ \longrightarrow MSO(\ell),$$

where $MSO(\ell)$ is the Thom complex of the ℓ -dimensional universal oriented vector bundle γ_{SO}^ℓ . The homomorphism $\mathcal{P} : \Omega^*(X) \longrightarrow H^*(X; Q)[[t_1, t_2, \dots, t_i, \dots]]$ is defined by

$$\mathcal{P}([f]) = \sigma^{-l+n} f^* \left(\Phi_{\gamma_{SO}^\ell} \left(\frac{1}{p_t(\gamma_{SO}^\ell)} \right) \right),$$

where σ^{-l+n} is the $(-l+n)$ -times suspension isomorphism and $\Phi_{\gamma_{SO}^\ell}$ is the Thom isomorphism for γ_{SO}^ℓ in the ordinary cohomology theory. This transformation satisfies the following propositions.

PROPOSITION 2.1. *We have the following.*

- (1) $\mathcal{P}(f^*(a)) = f^*(\mathcal{P}(a))$, for $f : X \longrightarrow Y$.
- (2) $\mathcal{P}(ab) = \mathcal{P}(a)\mathcal{P}(b)$.
- (3) Let ξ be an n -dimensional oriented vector bundle over X . Then

$$\mathcal{P}(e(\xi)) = e_H(\xi) \frac{1}{p_t(\xi)}.$$

Proof. Here we prove part 3. In the oriented cobordism cohomology theory the Thom class $t(\xi)$ of an n -dimensional oriented vector bundle ξ over X is represented by

$$\tilde{f}_\xi : T(\xi) \longrightarrow MSO(n),$$

that is, the map from the Thom complex $T(\xi)$ to $MSO(n)$ induced by the classifying map for the vector bundle ξ . The Euler class $e(\xi)$ in the oriented cobordism cohomology theory is represented by the composite of the zero section $s : X^+ \longrightarrow T(\xi)$ and $\tilde{f}_\xi$.

$$X^+ \xrightarrow{s} T(\xi) \xrightarrow{\tilde{f}_\xi} MSO(n).$$

Applying $\mathcal{P}$ to the Euler class $e(\xi)$, we calculate

$$\begin{aligned} \mathcal{P}(e(\xi)) &= s^* \tilde{f}_\xi^* \left(\Phi_{\gamma_{SO}^\ell} \left(\frac{1}{p_t(\gamma_{SO}^n)} \right) \right) \\ &= s^* \left(\Phi_\xi \left(\frac{1}{p_t(\xi)} \right) \right) \\ &= \frac{1}{p_t(\xi)} e_H(\xi). \end{aligned} \quad \square$$

Let $f : X \longrightarrow Y$ be a differentiable map between oriented closed manifolds. Let $h : X \longrightarrow S^N$ be an embedding. Then we have the normal bundle v_f for the embedding $f \times h : X \longrightarrow Y \times D^N$, where D^N is the unit disk in the Euclidean space R^N . Let $h^*(\cdot)$ be $\Omega^*(\cdot)$ or $H^*(\cdot)[[t_1, t_2, \dots, t_n, \dots]]$. Then v_f is h^* orientable. Then the Umkehrung homomorphism is the following composite:

$$f_!^h : h^*(X) \xrightarrow{\Phi} \tilde{h}^*(T(v_f)) \xrightarrow{j^*} \tilde{h}^*(Y^+ \wedge S^N) \xrightarrow{\sigma^{-N}} h^*(Y)$$

where Φ is the Thom homomorphism and j is the natural projection as follows:

$$j : Y^+ \wedge S^N = \frac{Y \times D^N}{Y \times S^{n-1}} \longrightarrow \frac{Y \times D^N}{Y \times D^N - \text{Int } D(v_f)} = T(v_f),$$

where $D(v_f)$ denotes the disk bundle associated with the bundle v_f .

PROPOSITION 2.2. *The Umkehrung homomorphisms are combined by $\mathcal{P}$ as follows:*

$$\mathcal{P}(f_!^\Omega(1)) = f_!^H \left(\frac{1}{p_t(v_f)} \right)$$

where v_f is the virtual normal bundle for f .

Proof. The element $f_!^\Omega(1)$ is represented by a map

$$Y^+ \wedge S^N \xrightarrow{j} T(v_f) \xrightarrow{\tilde{f}_{v_f}} MSO(k)$$

where $k = \dim Y + N - \dim X$. Then we have

$$\begin{aligned} \mathcal{P}(f_!^\Omega(1)) &= \sigma^{-N} j^* \tilde{f}_{v_f}^* \left(\Phi_{\gamma_{so}^k} \left(\frac{1}{p_{\mathbf{t}}(\gamma_{so}^k)} \right) \right) \\ &= \sigma^{-N} j^* \left(\Phi \left(\frac{1}{p_{\mathbf{t}}(v_f)} \right) \right) \\ &= f_!^H \left(\frac{1}{p_{\mathbf{t}}(v_f)} \right). \end{aligned} \quad \square$$

By discussing in the same way as $\mathcal{P}$, we use the generalized Stiefel–Whitney class to obtain a multiplicative natural transformation

$$\mathcal{W} : \mathcal{N}^*(X) \longrightarrow H^*(X; \mathbb{Z}_2)[[t_1, t_2, \dots]]$$

where $\mathcal{N}^*(X)$ is the unoriented cobordism cohomology group. This transformation satisfies the following propositions.

PROPOSITION 2.3. *We have the following.*

- (1) $\mathcal{W}(f^*(a)) = f^*(\mathcal{W}(a))$, for $f : X \longrightarrow Y$.
- (2) $\mathcal{W}(ab) = \mathcal{W}(a)\mathcal{W}(b)$.
- (3) *Let ξ be an n -dimensional vector bundle over X . Then*

$$\mathcal{W}(e(\xi)) = e_H(\xi) \frac{1}{w_{\mathbf{t}}(\xi)}.$$

PROPOSITION 2.4. *Let $f : X \longrightarrow Y$ be a differentiable map between closed manifolds. Then the Umkehrung homomorphisms $f_!^{\mathcal{N}}$ and $f_!^{H_2}$ are combined by $\mathcal{W}$ as follows:*

$$\mathcal{W}(f_!^{\mathcal{N}}(1)) = f_!^{H_2} \left(\frac{1}{w_{\mathbf{t}}(v_f)} \right)$$

where v_f is the virtual normal bundle for f and $f_!^{H_2}$ denotes the Umkehrung homomorphism in the ordinary cohomology with the coefficient ring $\mathbb{Z}_2$.

3. The characteristic numbers of the multiple point manifold of a self-transverse immersion

Let $f : V^n \longrightarrow W^{n+k}$ be an immersion of an n -dimensional smooth manifold V^n in an $(n+k)$ -dimensional smooth manifold W^{n+k} . If $f^{-1}(y)$ consists of distinct

d -points, then y is said to be d -multiple point. Let $df : \tau(V^n) \longrightarrow \tau(W^{n+k})$ be the tangent map for f that is a bundle homomorphism between the tangent bundles $\tau(V^n)$ and $\tau(W^{n+k})$. For any multiple point $y \in f(V^n)$ with $f^{-1}(y) = \{x_1, x_2, \dots, x_d\}$ the images under the tangent map df of the tangent spaces $T_{x_i}(V^n)$ at x_i are said to be in general position in the tangent space $T_y(W^{n+k})$ if the intersection of $dfT_{x_i}(V^n)$ and the normal spaces $N_{x_i}(V^n)$ of $dfT_{x_i}(V^n)$ span the tangent space $T_y(W^{n+k})$, that is,

$$\bigcap_{i=1}^d dfT_{x_i}(V^n) \oplus \left(\bigoplus_{i=1}^d N_{x_i}(V^n) \right) \cong T_y(W^{n+k}).$$

If for any multiple point y of an immersion $f : V^n \longrightarrow W^{n+k}$ the images under the tangent map of the tangent spaces at the points in $f^{-1}(y)$ are in general position in $T_y(W^{n+k})$, then f is said to be a self-transverse immersion. An immersion is approximated by a self-transverse immersion. Suppose that $f : V^n \longrightarrow W^{n+k}$ be a self-transverse immersion and $n + k - dk \geq 0$. Let

$$M_d = \{(x_1, \dots, x_d) \mid f(x_1) = \dots = f(x_d), x_i \neq x_j (i \neq j)\}.$$

Then M_d is an $(n + k - dk)$ -dimensional submanifold in $V^n \times V^n \times \dots \times V^n$. Let the symmetric group S_d of degree d act on M_d by

$$\sigma(x_1, \dots, x_d) = (x_{\sigma(1)}, \dots, x_{\sigma(d)}), \quad \sigma \in S_d.$$

The orbit space is denoted by $N_d = M_d/S_d$, that is said to be a d -multiple point manifold. Let S_{d-1} act on M_d by

$$\sigma(x_1, \dots, x_d) = (x_1, x_{\sigma(2)}, \dots, x_{\sigma(d)}), \quad \sigma \in S_{d-1},$$

and denote the orbit space by $\tilde{N}_d = M_d/S_{d-1}$. Then we have a d -fold covering $p_d'' : \tilde{N}_d \longrightarrow N_d$. First we recall Ronga's result [6] that plays an important role in this paper. The following pull-back diagram of immersions is said to be a sub-Cartesian:

$$\begin{array}{ccc} \tilde{N}_d \cup \tilde{N}_{d-1} & \xrightarrow{p_d' \cup p_{d-1}''} & N_{d-1} \\ \downarrow f_d \cup f_{d-1} & & \downarrow g_{d-1} \\ V^n & \xrightarrow{f} & W^{n+k} \end{array}$$

where

$$\begin{aligned} p'_d([x_1, \dots, x_d]) &= [x_2, \dots, x_d] \\ p''_{d-1}([x_1, \dots, x_{d-1}]) &= [x_1, \dots, x_{d-1}] \\ g_{d-1}([x_1, \dots, x_{d-1}]) &= f(x_1) \\ f_d([x_1, \dots, x_d]) &= x_1. \end{aligned}$$

This diagram is a generalized example of the clean intersection diagram of the Quillen theorem [5]. Ronga obtained in [6] the following theorem.

THEOREM 3.1. (Ronga [6]) *Let h^* be a generalized cohomology theory. If each normal bundle in the above sub-Cartesian pull-back diagram of immersions is h^* -orientable, then it follows that*

$$f^* g_{(d-1)!}(1) = f_d!(1) + e(v_f) f_{d-1}!(1),$$

where $e(v_f)$ is the Euler class of the normal bundle for f .

We recall that if $f : V^n \longrightarrow W^{n+k}$ is a self-transverse immersion of an oriented manifold V^n in an oriented manifold W^{n+k} and k is even, then $\widetilde{N}_d$, N_d and the normal bundle are orientable (cf. [2]). As $\Omega^i(\{a \text{ point}\}) = 0$ for $i > 0$, we use Theorem 3.1 to prove the following corollary.

COROLLARY 3.2. *If $f : V^n \longrightarrow S^{n+2k}$ is a self-transverse immersion of an oriented manifold V^n in the $(n+k)$ -sphere S^{n+2k} , then it follows that*

$$f_d^\Omega(1) = (-1)^{d-1} \{e(v_f)\}^{d-1}.$$

We use Corollary 3.2 to obtain the relation between Pontrjagin numbers of $\widetilde{N}_d$ and V^n for a self-transverse immersion of an oriented manifold in the sphere $f : V^n \longrightarrow S^{n+2k}$.

THEOREM 3.3. *Let $f : V^n \longrightarrow S^{n+2k}$ be a self-transverse immersion of an oriented manifold V^n . Then,*

$$\langle p_{\mathbf{t}}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = (-1)^{d-1} \langle \{e_H(v_f)\}^{d-1} \{p_{\mathbf{t}}(\tau(V^n))\}^d, [V^n] \rangle$$

where v_f is the normal bundle of f .

Proof. Apply $\mathcal{P}$ to the equation $f_d^\Omega(1) = (-1)^{d-1} e(v_f)^{d-1}$ in Corollary 3.2, and we use Proposition 2.1(3) and Proposition 2.2 to obtain

$$f_d^H\left(\frac{1}{p_{\mathbf{t}}(v_{f_d})}\right) = (-1)^{d-1} \left(e_H(v_f) \frac{1}{p_{\mathbf{t}}(v_f)}\right)^{d-1}.$$

Note that $\tau(\widetilde{N}_d) \oplus \nu_{f_d} \cong f_d^*(\tau(V^n))$, and $\tau(V^n) \oplus \nu_f$ is the trivial vector bundle. Then we have

$$f_{d!}^H(p_t(\tau(\widetilde{N}_d))) = (-1)^{d-1} \{e_H(\nu_f)\}^{d-1} \{p_t(\tau(V^n))\}^d. \quad (*)$$

We note that the Umkehrung homomorphism $g_!^H$ for a map $g : M \rightarrow N$ between orientable manifolds M and N coincides with the following composite:

$$H^*(M) \xrightarrow{D} H_*(M) \xrightarrow{g_*} H_*(N) \xrightarrow{D^{-1}} H^*(N),$$

where D denotes the Poincaré isomorphism. Evaluate both sides of $(*)$ on the fundamental class $[V^n]$, and we then complete the proof. $\square$

Applying the natural transformation $\mathcal{W}$ to Ronga's formula of Corollary 3.2 for the unoriented cobordism cohomology, we have

$$f_!^{\mathcal{N}}(1) = \{e(\nu_f)\}^{d-1}$$

for a self-transverse immersion $f : V^n \rightarrow S^{n+k}$. Then we have the following.

THEOREM 3.4. *If $f : V^n \rightarrow S^{n+k}$ is a self-transverse immersion, then*

$$\langle w_t(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = \langle \{e(\nu_f)\}^{d-1} \{w_t(\tau(V^n))\}^d, [V^n] \rangle.$$

COROLLARY 3.5. *If $f : V^n \rightarrow S^{n+k}$ is a self-transverse immersion, then the Euler characteristic modulo 2 of the covering space $\widetilde{N}_d$ for the d -multiple point manifold is described as*

$$\chi_2(\widetilde{N}_d) = \langle \{e(\nu_f)\}^{d-1} \{w(\tau(V^n))\}^d, [V^n] \rangle.$$

Proof. Let $t_1 = 1$ and $t_i = 0$ for $i > 1$ in the formula of Theorem 3.4. Then we see that the left-hand side is the Euler characteristic modulo 2 of $\widetilde{N}_d$. $\square$

This corollary implies the following result obtained in [4].

COROLLARY 3.6. *Suppose that $f : V^n \rightarrow S^{n+k}$ is a self-transverse immersion. If V^n is the boundary of a compact manifold and d is odd, then the Euler characteristic of the d -multiple point manifold is even.*

THEOREM 3.7. *Suppose that $f : V^n \rightarrow S^{n+2k}$ is a self-transverse immersion of an oriented closed manifold V^n in the sphere S^{n+2k} . If V^n is a boundary of an oriented compact manifold, then $\widetilde{N}_d$ is a boundary of an oriented compact manifold and, furthermore, if d is odd, then N_d is a boundary of an oriented compact manifold.*

Proof. Theorems 3.3 and 3.4 imply that if V^n is a boundary of a compact manifold, then all of the Pontrjagin numbers and the Stiefel–Whitney numbers of $\widetilde{N}_d$ are zero. Therefore, $\widetilde{N}_d$ is a boundary of an oriented compact manifold [8]. By making use of the d -fold covering $p'' : \widetilde{N}_d \longrightarrow N_d$, we have

$$\langle p_{\mathbf{t}}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = d \langle p_{\mathbf{t}}(\tau(N_d)), [N_d] \rangle$$

and if d is odd, then

$$\langle w_{\mathbf{t}}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = d \langle w_{\mathbf{t}}(\tau(N_d)), [N_d] \rangle$$

and so if d is odd, then all Pontrjagin numbers and Stiefel–Whitney numbers of N_d are zero. Therefore, N_d is a boundary of an oriented compact manifold [8]. $\square$

4. On the cobordism classes of the multiple point manifolds

Here we consider the symmetric polynomial of the indeterminate variables α_i

$$\begin{aligned} & s(\underbrace{1, \dots, 1}_{i_1}, \dots, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})(\alpha) \\ &= \Sigma \alpha_1 \alpha_2 \cdots \alpha_{i_1} \alpha_{i_1+1}^2 \cdots \alpha_{i_1+i_2}^2 \cdots \alpha_{i_1+\dots+i_{\ell-1}+1}^\ell \cdots \alpha_{i_1+\dots+i_\ell}^\ell. \end{aligned}$$

We describe $s_\omega(\alpha)$ as the polynomial of elementary symmetric polynomials σ_j with respect to α_i that is denoted by $s_\omega(\sigma_1, \sigma_2, \dots)$. For a vector bundle ξ , let

$$s_\omega(\xi) = s_\omega(p_1(\xi), p_2(\xi), \dots),$$

where $p_i(\xi)$ is the i th Pontrjagin class of ξ . Let CP^n be the complex projective n -space.

We calculate the characteristic class $s_\omega(\tau(CP^n))$ for the tangent bundle over the projective space CP^n . The tangent bundle $\tau(CP^n) \oplus 1$ is isomorphic to the Whitney sum $(n+1)\xi_n$ of $n+1$ canonical line bundles ξ_n . We then have

$$s(\underbrace{1, \dots, 1}_{i_1}, \dots, \underbrace{2, \dots, 2}_{i_2}, \dots, \underbrace{\ell, \dots, \ell}_{i_\ell})(\tau(CP^n)) = \frac{(n+1)!}{i_1! \cdots i_\ell! (n+1-i_1-\dots-i_\ell)!} x_n^{2(i_1+2i_2+\dots+\ell i_\ell)}$$

and

$$s_{(k)}(\tau(CP^n)) = (n+1)x_n^{2k},$$

where $H^*(CP^n; Z) \cong Z[x_n]/(x_n^{n+1})$. As $\Omega_* \otimes Q \cong Q[[CP^2], [CP^4], \dots, [CP^{2n}], \dots]$, the cobordism class of N_d is described as

$$[N_d] = \sum_{i_1+\dots+i_\ell=2n-(d-1)k} a_{i_1 \dots i_\ell} [CP^{2i_1}] \cdots [CP^{2i_\ell}].$$

We now calculate the coefficient of the indecomposable class in the right-hand side.

THEOREM 4.1. *Suppose that $f : V^{4n} \longrightarrow S^{4n+2k}$ is a self-transverse immersion of an oriented closed manifold V^n in the sphere S^{n+2k} . If $(d-1)k$ is even, then*

$$[N_d] = \frac{(-1)^{d-1}}{2n - (d-1)k + 1} \langle \{e_H(v_f)\}^{d-1} s_{((2n-(d-1)k)/2)}(\tau(V^{4n})), [V^{4n}] \rangle \\ \times [CP^{2n-(d-1)k}] + \text{decomposable elements in } \Omega_* \otimes Q$$

where $e_H(v_f)$ is the Euler class of the normal bundle of f in the ordinary cohomology and $[V^{4n}]$ is the fundamental class of V^n .

Proof. We observe that the coefficient of $t_{(2n-(d-1)k)/2}$ of the equation given in Theorem 3.3 has

$$\langle s_{((2n-(d-1)k)/2)}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle \\ = (-1)^{d-1} d \langle e_H(v_f)^{d-1} s_{((2n-(d-1)k)/2)}(\tau(V^{4n})), [V^{4n}] \rangle.$$

As

$$\langle s_{((2n-(d-1)k)/2)}(\tau(CP^{2n-(d-1)k})), [CP^{2n-(d-1)k}] \rangle = 2n - (d-1)k + 1$$

and

$$\langle s_{((2n-(d-1)k)/2)}(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = d \langle s_{((2n-(d-1)k)/2)}(\tau(N_d)), [N_d] \rangle,$$

it follows that

$$a_{2n-(d-1)k} \\ = (-1)^{d-1} \frac{1}{2n - (d-1)k + 1} \langle e_H(v_f)^{d-1} s_{((2n-(d-1)k)/2)}(\tau(V^{4n})), [V^{4n}] \rangle.$$

This completes the proof. $\square$

By using the method in the proof of Theorem 4.1, we obtain the following results (cf. Szűcs [7]).

PROPOSITION 4.2. *In the rational oriented cobordism ring $\Omega \otimes Q$, there are the following relations.*

(1) *If $f : V^{4n} \longrightarrow S^{8n-4}$ is a self-transverse immersion, then*

$$[N_2] = -\frac{1}{3} \langle e(v_f) p_1(\tau(V^{4n})), [V^{4n}] \rangle [CP^2].$$

(2) If $f : V^{4n} \longrightarrow S^{8n-8}$ is a self-transverse immersion, then

$$\begin{aligned} [N_2] &= a[CP^4] + b[CP^2]^2 \\ a &= -\frac{1}{5} \langle e(v_f) \{ \{ p_1(\tau(V^{4n})) \}^2 - 2p_2(\tau(V^{4n})) \} \}, [V^{4n}] \rangle \\ b &= \frac{1}{18} \langle e(v_f) \{ 3 \{ p_1(\tau(V^{4n})) \}^2 - 10p_2(\tau(V^{4n})) \} \}, [V^{4n}] \rangle. \end{aligned}$$

5. A remark on the Euler numbers of multiple point manifolds of a self-transverse immersion

Let $f : V^n \longrightarrow S^{n+k}$ be a self-transverse immersion of a closed n -dimensional manifold V^n in the $(n+k)$ -dimensional sphere. In this section we generalize the result about the total sum of the Euler characteristic modulo 2 of the d -multiple point manifolds N_d that is obtained by Akhmetiev *et al* in [1]. Let v_f and $w_i(v_f)$ be the normal bundle of f and the i th Stiefel–Whitney class, respectively. The Euler characteristic modulo 2 of the covering space $\widetilde{N}_d$ for the d -multiple point manifold N_d is described as follows.

THEOREM 5.1. *We have*

$$\sum_{d \geq 1} \chi_2(\widetilde{N}_d) \equiv \left\langle \frac{1}{1 + w_1(v_f) + w_2(v_f) + \cdots + w_{k-1}(v_f)}, [V^n] \right\rangle.$$

Proof. By making use of Corollary 3.5 we calculate

$$\begin{aligned} \sum_{d \geq 1} \chi_2(\widetilde{N}_d) &= \sum_{d \geq 1} \langle \{e(v_f)\}^{d-1} \{w(\tau(V^n))\}^d, [V^n] \rangle \\ &= \left\langle \frac{w(\tau(V^n))}{1 + e(v_f)w(\tau(V^n))}, [V^n] \right\rangle \\ &= \left\langle \frac{1}{w(v_f) + e(v_f)}, [V^n] \right\rangle \\ &= \left\langle \frac{1}{1 + w_1(v_f) + w_2(v_f) + \cdots + w_{k-1}(v_f)}, [V^n] \right\rangle. \quad \square \end{aligned}$$

We use Theorem 5.1 to describe the sum of the Euler characteristics modulo 2 of the d -multiple point manifold N_d for a self-transverse immersion $f : V^n \longrightarrow S^{n+k}$ as follows.

COROLLARY 5.2. *If $n + k$ is odd, then*

$$\sum_{d \geq 1} \chi_2(N_d) \equiv \left\langle \frac{1}{1 + w_1(v_f) + w_2(v_f) + \cdots + w_{k-1}(v_f)}, [V^n] \right\rangle.$$

Proof. If $n + k$ is odd and d is even, then $\dim \widetilde{N}_d = \dim N_d = n - (d - 1)k$ is odd and

$$\chi_2(\widetilde{N}_d) \equiv \chi_2(N_d) \equiv 0 \pmod{2}.$$

By making use of a d -fold covering $p_d'' : \widetilde{N}_d \rightarrow N_d$, if d is odd then $\chi_2(\widetilde{N}_d) \equiv \chi_2(N_d)$. Therefore, Theorem 5.1 implies this corollary. $\square$

The following result is obtained immediately from Corollary 5.2.

COROLLARY 5.3. *Let $f : V^n \rightarrow S^{n+k}$ be a self-transverse immersion of a closed n -dimensional manifold V^n in the sphere S^{n+k} . Suppose that $n + k$ is odd and*

$$w_1(v_f) = w_2(v_f) = \cdots = w_{k-1}(v_f) = 0.$$

Then we have

$$\sum_{d \geq 1} \chi(N_d) \equiv 0 \pmod{2}.$$

6. On the signature of multiple point manifolds

Let M be an orientable closed $4n$ -dimensional manifold and let $\{\alpha_i\}$ be the basis of $H^{2n}(M; Q)$. Denote by $\sigma(M)$ the signature of M defined by the number of positive eigenvalues minus the number of negative eigenvalues for the symmetric matrix $[(\alpha_i \cup \alpha_j, [M])]$. Let $\{L_k\}$ be the multiplicative sequence of polynomials belonging to the power series

$$\frac{\sqrt{x}}{\tanh \sqrt{x}} = 1 + \frac{1}{3}x + \cdots + (-1)^{k-1} 2^{2k} \frac{B_k}{(2k)!} x^k + \cdots,$$

where B_k denotes the Bernoulli number. Let us consider the power series

$$L(p(\xi)) = \sum L_k(p_1(\xi), \dots, p_k(\xi)).$$

The signature $\sigma(M)$ is described as follows [3]:

$$\sigma(M) = \langle L(p(\tau(M))), [M] \rangle.$$

In the same way as the construction of $\mathcal{P}$ we have a multiplicative natural transformation

$$\mathcal{L} : \Omega^*(X) \longrightarrow H^*(X; \mathcal{Q})$$

that is defined for $f : S^{\ell-n} X^+ \longrightarrow MSO(\ell)$ by

$$\mathcal{L}([f]) = \sigma^{-l+n} f^* \left(\Phi_{\gamma_{SO}^\ell} \left(\frac{1}{L(\gamma_{SO}^\ell)} \right) \right),$$

where σ is the suspension isomorphism and $\Phi_{\gamma_{SO}^\ell}$ is the Thom isomorphism with respect to γ_{SO}^ℓ in the ordinary cohomology theory. Then we have the following proposition.

PROPOSITION 6.1. *The natural transformation $\mathcal{L}$ satisfies the following properties.*

- (1) $\mathcal{L}(f^*(a)) = f^*(\mathcal{L}(a))$, for $f : X \longrightarrow Y$.
- (2) $\mathcal{L}(ab) = \mathcal{L}(a)\mathcal{L}(b)$.
- (3) $\mathcal{L}(e(\xi)) = e_H(\xi)(1/L(\xi))$.
- (4) *Let $f : X \longrightarrow Y$ be a differentiable map between orientable manifolds. Then*

$$\mathcal{L}(f_!(1)) = f_!^H \left(\frac{1}{L(v_f)} \right).$$

Applying the natural transformation $\mathcal{L}$ to the formula in Corollary 3.2, the signature of N_d is described as follows.

THEOREM 6.2. *Suppose that $f : V^{4n} \longrightarrow S^{4n+2k}$ is a self-transverse immersion of an oriented closed manifold V^{4n} in the $(4n + 2k)$ -dimensional sphere S^{4n+2k} . If $(d - 1)k$ is even, then*

$$\sigma(N_d) = \frac{(-1)^{d-1}}{d} \langle \{e_H(v_f)\}^{d-1} \{L(\tau(V^{4n}))\}^d, [V^{4n}] \rangle.$$

Proof. Apply $\mathcal{L}$ to the formula given in Corollary 3.2 and use Proposition 6.1 to obtain

$$\langle L(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = (-1)^{d-1} \langle \{e_H(v_f)\}^{d-1} \{L(\tau(V^{4n}))\}^d, [V^{4n}] \rangle.$$

By making use of the d -fold covering $p'' : \widetilde{N}_d \longrightarrow N_d$ we have

$$\langle L(\tau(\widetilde{N}_d)), [\widetilde{N}_d] \rangle = d \langle L(\tau(N_d)), [N_d] \rangle.$$

Thus, we complete the proof. $\square$

Let $f : CP^{2n} \longrightarrow S^{8n-4k}$ be a self-transverse immersion. We calculate the signature of the double point manifold of the self-transverse immersion. By Theorem 6.2

$$\sigma(\tilde{N}_2) = -\langle e(v_f) L(\tau(CP^{2n}))^2, [CP^{2n}] \rangle.$$

On the other hand, we have

$$e(v_f) = \binom{4n-2k}{2n-2k} x_{2n}^{2n-2k}$$

$$\{L(\tau(CP^{2n}))\}^2 = \left(\frac{x_{2n}}{\tanh x_{2n}} \right)^{4n+2}.$$

$\sigma(\tilde{N}_2)$ is the coefficient of x_{2n}^{2n} in the polynomial

$$\binom{4n-2k}{2n-2k} \left(\frac{x_{2n}}{\tanh x_{2n}} \right)^{4n+2} x_{2n}^{2n-2k}.$$

Now we calculate the coefficient of x_{2n}^{2k} in the polynomial

$$\left(\frac{x_{2n}}{\tanh x_{2n}} \right)^{4n+2}.$$

The coefficient is given by the following integration around the origin:

$$\frac{1}{2\pi i} \oint \frac{1}{z^{2k+1}} \left(\frac{z}{\tanh z} \right)^{4n+2} dz.$$

The substitution $u = \tanh z$, with

$$dz = \frac{du}{1-u^2}$$

$$z = \frac{1}{2} \left(\log(1+u) - \log(1-u) \right)$$

$$= u \left(1 + \frac{u^2}{3} + \frac{u^4}{5} + \cdots + \frac{u^{2k}}{2k+1} + \cdots \right),$$

shows that

$$\frac{1}{2\pi i} \oint \frac{1}{z^{2k+1}} \left(\frac{z}{\tanh z} \right)^{4n+2} dz$$

$$= \sum_{\ell=0}^k \left\{ \sum_{i_1+2i_2+\cdots+\ell i_\ell=\ell} \frac{(4n-2k+1)!}{i_1! i_2! \cdots i_\ell! (4n-2k+1-(i_1+i_2+\cdots+i_\ell))!} \right.$$

$$\left. \times \left(\frac{1}{3} \right)^{i_1} \left(\frac{1}{5} \right)^{i_2} \cdots \left(\frac{1}{2\ell+1} \right)^{i_\ell} \right\}.$$

Let $d(4n, 8n - 4k)$ denote this number. Then we have the following result.

THEOREM 6.3. *If $f : CP^{2n} \longrightarrow S^{8n-4k}$ is a self-transverse immersion, then*

$$\sigma(N_2) = -\frac{1}{2}d(4n, 8n - 4k) \binom{4n - 2k}{2n - 2k}.$$

Example. If $f : CP^{2n} \longrightarrow S^{8n-4}$ is a self-transverse immersion, then

$$\sigma(N_2) = -\frac{2n + 1}{3} \binom{4n - 2}{2n - 2}.$$

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