ALGEBRAIC RELATIONS AMONG SOME THETA FUNCTIONS ON THE BOUNDED SYMMETRIC DOMAIN OF TYPE $I_{r,r}$

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Abstract. We give algebraic relations among theta functions on the bounded symmetric domain of type $I_{r,r}$: quadratic relations among those over the ring $\mathbb{Z}[i]$ of Gaussian integers, and over the ring $\mathbb{Z}[\sqrt{-2}]$, and cubic relations among those over the ring $\mathbb{Z}[\omega]$ of Eisenstein integers.

1. Introduction

Theta functions on the bounded symmetric domain $\mathbb{D}^r$ of type $I_{r,r}$ over the ring $\mathbb{Z}[i]$ of Gaussian integers and over the ring $\mathbb{Z}[\omega]$ of Eisenstein integers are utilized for constructions of modular forms on $\mathbb{D}^r$; refer to [DK, F, FS].

In this paper, we give quadratic relations among theta functions on $\mathbb{D}^r$ over $\mathbb{Z}[i]$ and over $\mathbb{Z}[\sqrt{-2}]$, and cubic relations among those over $\mathbb{Z}[\omega]$. Some of quadratic relations among theta functions over $\mathbb{Z}[i]$ were used when the author studied the period map for a four-parameter family of $K3$ -surfaces in [M], and the isomorphism between an octahedron in the 3-dimensional hyperbolic space to that in the 3-dimensional Euclid space in [MY], and the hyperbolic structure on the complement of the Whitehead link in [MNY]. Cubic relations among theta functions over $\mathbb{Z}[\omega]$ will help us to study periods of some families of algebraic varieties and the hyperbolic structure on the complement of the figure-eight knot.

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2. Theta functions on the symmetric domain of type $I_{r,r}$

The bounded symmetric domain $\mathbb{D}^r$ of type $I_{r,r}$ is defined as

$$\mathbb{D}^r = \left\{ \tau \in M_{r,r}(\mathbb{C}) \mid \frac{1}{2i}(\tau - \tau^*) \text{ is positive definite} \right\},$$

where $i = \sqrt{-1}$, $X^* = {}^t\overline{X}$ for a matrix X , and $M_{r,s}(R)$ is the set of $r \times s$ matrices over a ring R . Since

$$\tau = \frac{1}{2}(\tau + \tau^*) + \frac{1}{2i}(\tau - \tau^*)i,$$

any element of $\mathbb{D}^r$ can be expressed as the sum of a hermitian matrix and i times a positive definite hermitian matrix. The unitary group

$$U_{r,r}(\mathbb{C}) = \left\{ g \in GL_{2r}(\mathbb{C}) \mid gJg^* = J = \begin{pmatrix} O & -I_r \\ I_r & O \end{pmatrix} \right\}$$

and an involution T act on $\mathbb{D}^r$ by

$$g \cdot \tau = (g_{11}\tau + g_{12})(g_{21}\tau + g_{22})^{-1}, \quad T \cdot \tau = {}^t\tau,$$

where $\tau \in \mathbb{D}^r$, $g = (g_{ij}) \in U_{r,r}(\mathbb{C})$, and g_{ij} are $r \times r$ matrices.

Let d be a square-free positive integer and $\mathbb{Z}[\delta]$ be the ring of integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, where

$$\delta = \begin{cases} \sqrt{-d} & \text{if } -d \not\equiv 1 \pmod{4}, \\ \frac{-1 + \sqrt{-d}}{2} & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

The lattice $\mathbb{Z}[\delta]^r$ with the inner product

$$\langle m, n \rangle = \begin{cases} \operatorname{Re}(mn^*) & \text{if } -d \not\equiv 1 \pmod{4}, \\ 2 \operatorname{Re}(mn^*) & \text{if } -d \equiv 1 \pmod{4}, \end{cases}$$

is integral and its dual lattice $(\mathbb{Z}[\delta]^r)^*$ is $(1/\sqrt{-d})\mathbb{Z}[\delta]^r$.

Theta functions $\Theta_\delta \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) (\tau)$ on $\mathbb{D}^r$ over $\mathbb{Z}[\delta]$ are defined as

$$\Theta_\delta \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) (\tau) = \begin{cases} \sum_{n \in \mathbb{Z}[\delta]^r} \mathbf{e}[(n+a)\tau(n+a)^* + 2 \operatorname{Re}(nb^*)], & \text{if } -d \not\equiv 1 \pmod{4}, \\ \sum_{n \in \mathbb{Z}[\delta]^r} \mathbf{e}[2(n+a)\tau(n+a)^* + 4 \operatorname{Re}(nb^*)], & \text{if } -d \equiv 1 \pmod{4}, \end{cases}$$

where $\tau \in \mathbb{D}^r$, $a, b \in \mathbb{Q}(\sqrt{-d})^r$ and $\mathbf{e}[x] = \exp[\pi i x]$.

By definition, we have the following.

PROPOSITION 1. *Theta functions $\Theta_\delta\left(\frac{a}{b}\right)(\tau)$ satisfy*

- (1) $\Theta_\delta\left(\frac{\varepsilon a}{\varepsilon b}\right)(\tau) = \Theta_\delta\left(\frac{a}{b}\right)(\tau)$ for a unit $\varepsilon \in \mathbb{Z}[\delta]$;
- (2) $\Theta_\delta\left(\frac{a+p}{b+q}\right)(\tau) = \mathbf{e}[-2\langle p, b \rangle] \Theta_\delta\left(\frac{a}{b}\right)(\tau)$ for $p \in \mathbb{Z}[\delta]^r$ and $q \in (1/\sqrt{-d})\mathbb{Z}[\delta]^r \simeq (\mathbb{Z}[\delta]^r)^*$;
- (3) if $b \in (1/2\sqrt{-d})\mathbb{Z}[\delta]^r$, then $\Theta_\delta\left(\frac{a}{-b}\right)(\tau) = \Theta_\delta\left(\frac{a}{b}\right)(\tau)$;
- (4) $\Theta_\delta\left(\frac{a}{b}\right)(g\tau g^*) = \Theta_\delta\left(\frac{a^g}{b(g^*)^{-1}}\right)(\tau)$ for any $g \in GL_r(\mathbb{Z}[\delta])$;
- (5) $\Theta_\delta\left(\frac{a}{b}\right)(T \cdot \tau) = \Theta_\delta\left(\frac{\tilde{a}}{\tilde{b}}\right)(\tau)$.

3. Quadratic relations among theta functions over $\mathbb{Z}[i]$

In this section, we give quadratic relations among theta functions over $\mathbb{Z}[i]$:

$$\Theta\left(\frac{a}{b}\right)(\tau) = \Theta_i\left(\frac{a}{b}\right)(\tau) = \sum_{n \in \mathbb{Z}[i]^r} \mathbf{e}[(n+a)\tau(n+a)^* + 2\operatorname{Re}(nb^*)].$$

THEOREM 1. *We have*

$$\begin{aligned} & 2^r \Theta\left(\frac{a_1 + a_2}{b_1 + b_2}\right)(\tau) \Theta\left(\frac{a_1 - a_2}{b_1 - b_2}\right)(\tau) \\ &= \sum_{e, f \in R_{1+i}^r} \mathbf{e}[2\operatorname{Re}((1+i)(b_1 + b_2)e^*)] \Theta\left(\frac{e + (1+i)a_1}{f + (1+i)b_1}\right)(\tau) \Theta\left(\frac{e + (1+i)a_2}{f + (1+i)b_2}\right)(\tau), \end{aligned}$$

where $R_{1+i} = \frac{\mathbb{Z}[i]}{1+i} / \mathbb{Z}[i]$. In particular,

$$2^r \Theta\left(\frac{a}{b}\right)(\tau)^2 = \sum_{e, f \in R_{1+i}^r} \mathbf{e}[2\operatorname{Re}((1+i)be^*)] \Theta\left(\frac{e + (1+i)a}{f + (1+i)b}\right)(\tau) \Theta\left(\frac{e}{f}\right)(\tau).$$

Proof. For the lattice $L_1 = M_{2,r}(\mathbb{Z}[i])$, we set $L_2 = UL_1$ and $L = L_1 + L_2$, where

$$U = \frac{1+i}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};$$

note that $UU^* = I_2$ and $U^2 = iI_2$. Since

$$L_2 = \left\{ M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \frac{1+i}{2} M_{2,r}(\mathbb{Z}[i]) \mid m_1 + m_2 \in (1+i)\mathbb{Z}[i]^r \right\},$$

$$L = \left\{ M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \frac{1+i}{2} M_{2,r}(\mathbb{Z}[i]) \mid m_1 + m_2 \in \mathbb{Z}[i]^r \right\},$$

we have $[L : L_1] = [L : L_2] = 2^r$,

$$L/L_2 \simeq L_1/(L_1 \cap L_2) = \left\{ P = \begin{pmatrix} p \\ 0 \end{pmatrix} \mid p \in \{0, 1\}^r \right\},$$

$$L/L_1 \simeq L_2/(L_1 \cap L_2) = \left\{ UP = \frac{1+i}{2} \begin{pmatrix} p \\ p \end{pmatrix} \mid p \in \{0, 1\}^r \right\}.$$

For any $Q \in L_1$, the map $\chi_Q : L/L_1 \ni M \mapsto \mathbf{e}[2 \operatorname{Re}(\operatorname{tr}(MQ^*))] \in \mathbb{C}^\times$ is an element of the group $(L/L_1)^\times$ of homomorphisms from L/L_1 to $\mathbb{C}^\times$, where $\operatorname{tr}(X)$ denotes the trace of a square matrix X . It is easy to see that $\chi_Q = 1$ if and only if $Q \in L_1 \cap L_2$. These imply that

$$(L/L_1)^\times = \{\chi_Q \mid Q \in L_1/(L_1 \cap L_2)\}.$$

Apply Lemma 1 of [I, Ch. IV] for L, L_1, L_2 , and for the function

$$\psi(X) = \mathbf{e}[\operatorname{tr}\{(X+A)\tau(X+A)^* + 2 \operatorname{Re}(XB^*)\}],$$

where $\tau \in \mathbb{D}^r$,

$$A = \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix}.$$

Then we have

$$[L : L_1] \sum_{X \in L_1} \psi(X) = \sum_{P, Q \in L/L_2} \left(\sum_{M \in L_2} \chi_Q(M+P) \psi(M+P) \right). \quad (1)$$

The left-hand side of (1) is equal to $2^r \Theta\left(\frac{a_1+a_2}{b_1+b_2}\right)(\tau) \Theta\left(\frac{a_1-a_2}{b_1-b_2}\right)(\tau)$. The right-hand side of (1) is equal to

$$\begin{aligned} & \sum_{P, Q \in L/L_2} \sum_{M \in L_2} \mathbf{e}[2 \operatorname{Re}\{\operatorname{tr}((M+P)Q^*)\}] \\ & \quad \times \mathbf{e}[\operatorname{tr}\{(M+P+A)\tau(M+P+A)^* + 2 \operatorname{Re}((M+P)B^*)\}] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{P, Q \in L/L_2} \sum_{N \in L_1} \mathbf{e}[2 \operatorname{Re}\{\operatorname{tr}(U^* N Q^*)\}] \mathbf{e}[2 \operatorname{Re}\{\operatorname{tr}(P Q^*)\}] \mathbf{e}[2 \operatorname{Re}\{\operatorname{tr}(P B^*)\}] \\
 &\quad \times \mathbf{e}[\operatorname{tr}\{(U^* N + P + A)\tau(U^* N + P + A)^* + 2 \operatorname{Re}(U^* N B^*)\}] \\
 &= \sum_{P, Q \in L/L_2} \sum_{N \in L_1} \mathbf{e}[2 \operatorname{Re}\{\operatorname{tr}(P B^*)\}] \\
 &\quad \times \mathbf{e}[\operatorname{tr}\{(N + UP + UA)\tau(N + UP + UA)^* + 2 \operatorname{Re}(N(UB + UQ)^*)\}],
 \end{aligned}$$

since $\operatorname{tr}(UXU^*) = \operatorname{tr}(X)$ for any (2×2) -matrix X , and $\operatorname{tr}(PQ^*) \in \mathbb{Z}[i]$. Put

$$UP = \begin{pmatrix} e \\ e \end{pmatrix} \quad \text{and} \quad UQ = \begin{pmatrix} f \\ f \end{pmatrix}, \quad \text{where } e, f \in \frac{1+i}{2}\mathbb{Z}[i]^r.$$

Note that

$$UA = (1+i) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad UB = (1+i) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and that

$$\operatorname{Re}(\operatorname{tr}(PB^*)) = \operatorname{Re}(\operatorname{tr}(BP^*)) = \operatorname{Re}((1+i)(b_1 + b_2)e^*).$$

Thus, the right-hand side of (1) is equal to

$$\sum_{e, f \in R_{1+i}^r} \mathbf{e}[2 \operatorname{Re}((1+i)(b_1 + b_2)e^*)] \Theta \begin{pmatrix} e + (1+i)a_1 \\ f + (1+i)b_1 \end{pmatrix} (\tau) \Theta \begin{pmatrix} e + (1+i)a_2 \\ f + (1+i)b_2 \end{pmatrix} (\tau).$$

By putting $a_1 = a$, $b_1 = b$ and $a_2 = b_2 = 0$, we get the second equality of the theorem. $\square$

Remark 1. The lattices L and $L_1 \cap L_2$ are isomorphic to $(D_4^*)^r$ and $(D_4)^r$, respectively, where D_4 is the root lattice of type D_4 , and D_4^* is its dual.

Remark 2. Refer to [M, MNY, MY] for applications of this theorem.

4. Quadratic relations among theta functions over $\mathbb{Z}[\sqrt{-2}]$

In this section, we give quadratic relations among theta functions over $\mathbb{Z}[\sqrt{-2}]$:

$$\Theta \begin{pmatrix} a \\ b \end{pmatrix} (\tau) = \Theta_{\sqrt{-2}} \begin{pmatrix} a \\ b \end{pmatrix} (\tau) = \sum_{n \in \mathbb{Z}[\sqrt{-2}]^r} \mathbf{e}[(n+a)\tau(n+a)^* + 2 \operatorname{Re}(nb^*)].$$

THEOREM 2. *We have*

$$\begin{aligned} & 2^r \Theta \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} (\tau) \Theta \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix} (\tau) \\ &= \sum_{e, f \in R'_{\sqrt{-2}}} \mathbf{e}[-2 \operatorname{Re}(\sqrt{-2}e(b_1 + b_2)^*)] \prod_{j=1}^2 \Theta \left(\frac{e + \sqrt{-2}a_j}{\sqrt{-2}} + \sqrt{-2}b_j \right) (\tau), \end{aligned}$$

where $R_{\sqrt{-2}} = \frac{\mathbb{Z}[\sqrt{-2}]}{\sqrt{-2}} / \mathbb{Z}[-2]$. In particular,

$$2^r \Theta \begin{pmatrix} a \\ b \end{pmatrix} (\tau)^2 = \sum_{e, f \in R'_{\sqrt{-2}}} \mathbf{e}[-2 \operatorname{Re}(\sqrt{-2}eb^*)] \Theta \left(\frac{e + \sqrt{-2}a}{\sqrt{-2}} + \sqrt{-2}b \right) (\tau) \Theta \left(\frac{e}{\sqrt{-2}} \right) (\tau).$$

Proof. Set

$$\begin{aligned} L_1 &= M_{2,r}(\mathbb{Z}[\sqrt{-2}]), \quad U = \frac{1}{\sqrt{-2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ L_2 &= UL_1 = \left\{ M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \frac{1}{\sqrt{-2}} L_1 \mid m_1 + m_2 \in \sqrt{-2}\mathbb{Z}[\sqrt{-2}]^r \right\}, \\ L &= L_1 + L_2 = \left\{ M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \frac{1}{\sqrt{-2}} L_1 \mid m_1 + m_2 \in \mathbb{Z}[\sqrt{-2}]^r \right\}, \\ L/L_2 &\simeq L_1/(L_1 \cap L_2) = \left\{ P = \begin{pmatrix} p \\ 0 \end{pmatrix} \mid p \in \{0, 1\}^r \right\}, \\ L/L_1 &\simeq L_2/(L_1 \cap L_2) = \left\{ UP = \frac{1}{\sqrt{-2}} \begin{pmatrix} p \\ p \end{pmatrix} \mid p \in \{0, 1\}^r \right\}, \\ (L/L_1)^\times &= \left\{ \chi_Q : L/L_1 \ni M \mapsto \mathbf{e} \left[2 \operatorname{Re} \left(\frac{\operatorname{tr}(MQ^*)}{\sqrt{-2}} \right) \right] \in \mathbb{C}^\times \mid Q \in L_1/(L_1 \cap L_2) \right\}, \\ \psi(X) &= \mathbf{e}[\operatorname{tr}\{(X + A)\tau(X + A)^* + 2 \operatorname{Re}(XB^*)\}], \\ A &= \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix}, \end{aligned}$$

and follow the proof of Theorem 1. □

To simplify quadratic relations among $\Theta_{\sqrt{-2}}\left(\frac{a}{b}\right)(\tau)$, we put

$$\begin{aligned}\Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}(\tau) &= \Theta_{\sqrt{-2}}\begin{pmatrix} \frac{a}{\sqrt{-2}} \\ \frac{b}{2} \end{pmatrix}(\tau) \\ &= \sum_{n \in \mathbb{Z}[\sqrt{-2}]^r} \mathbf{e}\left[\left(n + \frac{a}{\sqrt{-2}}\right)\tau\left(n + \frac{a}{\sqrt{-2}}\right)^* + \operatorname{Re}(nb^*)\right].\end{aligned}$$

They satisfy:

- (1) $\Theta_{\sqrt{-2}}\begin{bmatrix} -a \\ -b \end{bmatrix}(\tau) = \Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}(\tau)$;
- (2) $\Theta_{\sqrt{-2}}\begin{bmatrix} a + \sqrt{-2}p \\ b + \sqrt{-2}q \end{bmatrix}(\tau) = \mathbf{e}[-\operatorname{Re}(pb^*)]\Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}(\tau)$ for $p, q \in \mathbb{Z}[\sqrt{-2}]^r$;
- (3) if $b \in \frac{\mathbb{Z}[\sqrt{-2}]^r}{\sqrt{-2}}$, then $\Theta_{\sqrt{-2}}\begin{bmatrix} a \\ -b \end{bmatrix}(\tau) = \Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}(\tau)$;
- (4) $\Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}(T \cdot \tau) = \Theta_{\sqrt{-2}}\begin{bmatrix} -\bar{a} \\ \bar{b} \end{bmatrix}(\tau) = \Theta_{\sqrt{-2}}\begin{bmatrix} \bar{a} \\ -\bar{b} \end{bmatrix}(\tau)$.

COROLLARY 1. For $a, b \in \mathbb{Z}[\sqrt{-2}]^r$, we have

$$2^r \Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}(\tau) = \sum_{p, q \in \{0, 1\}^r} (-1)^{\operatorname{Re}(pb^* + aq^*)} \Theta_{\sqrt{-2}}^2 \begin{bmatrix} p \\ q \end{bmatrix}(\tau).$$

Proof. Theorem 2 implies that

$$\begin{aligned}& 2^r \Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}(\tau) 2^r \Theta \begin{pmatrix} \frac{a}{\sqrt{-2}} \\ \frac{b}{2} \end{pmatrix}(\tau)^2 \\ &= \sum_{e, f \in R_{\sqrt{-2}}^r} \mathbf{e}[-\operatorname{Re}(\sqrt{-2}eb^*)] \Theta \left(\frac{e+a}{\sqrt{-2}} - \frac{b}{\sqrt{-2}} \right)(\tau) \Theta \left(\frac{e}{\sqrt{-2}} \right)(\tau) \\ &= \sum_{e, f \in R_{\sqrt{-2}}^r} \mathbf{e}[-\operatorname{Re}(\sqrt{-2}eb^*)] \Theta_{\sqrt{-2}} \begin{bmatrix} \sqrt{-2}(e+a) \\ -\sqrt{-2}(f-b) \end{bmatrix}(\tau) \Theta_{\sqrt{-2}} \begin{bmatrix} \sqrt{-2}e \\ -\sqrt{-2}f \end{bmatrix}(\tau) \\ &= \sum_{p, q \in \{0, \pm 1\}^r} \mathbf{e}[-\operatorname{Re}(pb^* + aq^*)] \Theta_{\sqrt{-2}}^2 \begin{bmatrix} p \\ q \end{bmatrix}(\tau),\end{aligned}$$

where we put $p = \sqrt{-2}e$ and $q = -\sqrt{-2}f$. $\square$

The domain $\mathbb{D}^r$ is nothing but the upper half plane $\mathbb{H}$ for $r = 1$. There are four $\Theta_{\sqrt{-2}}\begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbb{H}$ over $\mathbb{Z}[\sqrt{-2}]$ for $a, b \in \{0, 1\}$:

$$\Theta_{\sqrt{-2}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Theta_{\sqrt{-2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Theta_{\sqrt{-2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Theta_{\sqrt{-2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Corollary 1 implies the quadratic relation among them:

$$\Theta_{\sqrt{-2}}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}(\tau) = \Theta_{\sqrt{-2}}^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) - \Theta_{\sqrt{-2}}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) - \Theta_{\sqrt{-2}}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau).$$

For $r = 2$, there are 16 non-zero $\Theta_{\sqrt{-2}} \begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbb{D}^2$ over $\mathbb{Z}[\sqrt{-2}]$ for $a, b \in \{0, 1\}^2$. Corollary 1 implies linear relations among their squares. The ten $\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}$ satisfying $\operatorname{Re}(ab^*) \in 2\mathbb{Z}$ are linearly independent in these relations. The six $\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}$ satisfying $\operatorname{Re}(ab^*) \notin 2\mathbb{Z}$ are expressed as

$$\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2} \sum_{\substack{pq^* \in 2\mathbb{Z} \\ p, q \in \{0, 1\}^2}} (-1)^{pb^* + aq^*} \Theta_{\sqrt{-2}}^2 \begin{bmatrix} p \\ q \end{bmatrix}.$$

For $r = 3$, there are 64 non-zero $\Theta_{\sqrt{-2}} \begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbb{D}^3$ over $\mathbb{Z}[\sqrt{-2}]$ for $a, b \in \{0, 1\}^3$. Corollary 1 implies linear relations among their squares. The 36 $\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}$ satisfying $\operatorname{Re}(ab^*) \in 2\mathbb{Z}$ are linearly independent in these relations. The 28 $\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix}$ satisfying $\operatorname{Re}(ab^*) \notin 2\mathbb{Z}$ are expressed as

$$\Theta_{\sqrt{-2}}^2 \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2^2} \sum_{\substack{pq^* \in 2\mathbb{Z} \\ p, q \in \{0, 1\}^3}} (-1)^{pb^* + aq^*} \Theta_{\sqrt{-2}}^2 \begin{bmatrix} p \\ q \end{bmatrix}.$$

5. Cubic relations among theta functions over $\mathbb{Z}[\omega]$

In this section, we give cubic relations among theta functions over $\mathbb{Z}[\omega]$:

$$\Theta \begin{pmatrix} a \\ b \end{pmatrix}(\tau) = \Theta_{\omega} \begin{pmatrix} a \\ b \end{pmatrix}(\tau) = \sum_{n \in \mathbb{Z}[\omega]^r} \mathbf{e}[2(n+a)\tau(n+a)^* + 4\operatorname{Re}(nb^*)].$$

THEOREM 3. *We have*

$$\begin{aligned} & 3^r \Theta \begin{pmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \end{pmatrix}(\tau) \Theta \begin{pmatrix} a_1 + \omega a_2 + \omega^2 a_3 \\ b_1 + \omega b_2 + \omega^2 b_3 \end{pmatrix}(\tau) \Theta \begin{pmatrix} a_1 + \omega^2 a_2 + \omega a_3 \\ b_1 + \omega^2 b_2 + \omega b_3 \end{pmatrix}(\tau) \\ &= \sum_{e, f \in R_{\sqrt{-3}}^r} \mathbf{e}[-4\operatorname{Re}(\sqrt{-3}e(b_1 + b_2 + b_3)^*)] \prod_{j=1}^3 \Theta \left(\frac{e + \sqrt{-3}a_j}{\sqrt{-3}} + \sqrt{-3}b_j \right)(\tau), \end{aligned}$$

where $R_{\sqrt{-3}} = \frac{\mathbb{Z}[\omega]}{\sqrt{-3}} / \mathbb{Z}[\omega]$. In particular,

$$\begin{aligned} & 3^r \Theta \begin{pmatrix} a \\ b \end{pmatrix} (\tau)^3 \\ &= \sum_{e, f \in R'_{\sqrt{-3}}} \mathbf{e}[-4 \operatorname{Re}(\sqrt{-3}eb^*)] \Theta \left(\frac{e + \sqrt{-3}a}{\sqrt{-3}} + \sqrt{-3}b \right) (\tau) \Theta \left(\frac{e}{\sqrt{-3}} \right) (\tau)^2. \end{aligned}$$

Proof. Let L_1 be $M_{3,r}(\mathbb{Z}[\omega])$ and L_2 be VL_1 , where

$$V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix};$$

note that $VV^* = I_3$, $V^2 = -\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \in GL_3(\mathbb{Z}[\omega])$. Note also that

$$\begin{aligned} L_2 &= \left\{ M = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in \frac{1}{\sqrt{-3}} M_{3,r}(\mathbb{Z}[\omega]) \left| \begin{array}{l} m_j - m_k \in \mathbb{Z}[\omega]^r \ (1 \leq j, k \leq 3) \\ m_1 + m_2 + m_3 \in \sqrt{-3}\mathbb{Z}[\omega]^r \end{array} \right. \right\}, \\ L_1 \cap L_2 &= \left\{ N = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \in M_{3,r}(\mathbb{Z}[\omega]) \left| n_1 + n_2 + n_3 \in \sqrt{-3}\mathbb{Z}[\omega]^r \right. \right\}, \end{aligned}$$

and that the lattice L generated by L_1 and L_2 is

$$L = \left\{ M = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in \frac{1}{\sqrt{-3}} M_{3,r}(\mathbb{Z}[\omega]) \left| m_j - m_k \in \mathbb{Z}[\omega]^r \ (1 \leq j, k \leq 3) \right. \right\}.$$

Thus, we have

$$[L : L_1] = [L : L_2] = [L_1 : L_1 \cap L_2] = [L_2 : L_1 \cap L_2] = 3^r,$$

and so the representatives of $L/L_2 \simeq L_1/(L_1 \cap L_2)$ and those of $L/L_1 \simeq L_2/(L_1 \cap L_2)$ are given by

$$P = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} \in M_{3,r}(\mathbb{Z}[\omega]), \quad VP = \frac{1}{\sqrt{-3}} \begin{pmatrix} p \\ p \\ p \end{pmatrix} \in \frac{1}{\sqrt{-3}} M_{3,r}(\mathbb{Z}[\omega]), \quad p \in \{0, \pm 1\}^r,$$

respectively.

For any $Q \in L_1$, the map $\chi_Q : L/L_1 \ni N \mapsto \mathbf{e}[4 \operatorname{Re}((1/\sqrt{-3})\operatorname{tr}(NQ^*))] \in \mathbb{C}^\times$ is an element of the group $(L/L_1)^\times$ of homomorphisms from L/L_1 to $\mathbb{C}^\times$. Since any element L can be expressed as $N_1 + VN_2$ for $N_1, N_2 \in L_1$, and an element Q of L_1 belongs to $L_1 \cap L_2$ if and only if V^*Q belongs to L_1 , we have

$$\begin{aligned} \mathbf{e}\left[4 \operatorname{Re}\left(\frac{1}{\sqrt{-3}}\operatorname{tr}((N_1 + VN_2)Q^*)\right)\right] &= \mathbf{e}\left[4 \operatorname{Re}\left(\frac{1}{\sqrt{-3}}\operatorname{tr}(VN_2Q^*)\right)\right] \\ &= \mathbf{e}\left[4 \operatorname{Re}\left(\frac{1}{\sqrt{-3}}\operatorname{tr}(N_2Q^*V)\right)\right] = \mathbf{e}\left[4 \operatorname{Re}\left(\frac{1}{\sqrt{-3}}\operatorname{tr}(N_2(V^*Q)^*)\right)\right] = 1 \end{aligned}$$

for any $Q \in L_1 \cap L_2$. Thus, $\chi_Q = 1$ if and only if $Q \in L_1 \cap L_2$; this implies that

$$(L/L_1)^\times = \{\chi_Q \mid Q \in L_1/(L_1 \cap L_2)\}.$$

We follow the proof of Theorem 1 for L, L_1, L_2 and for the function

$$\psi(X) = \mathbf{e}[\operatorname{tr}\{(X+A)\tau(X+A)^* + 4 \operatorname{Re}(XB^*)\}],$$

where

$$\begin{aligned} \tau \in \mathbb{D}^r, \quad A &= \begin{pmatrix} a_1 + a_2 + a_3 \\ a_1 + \omega^2 a_2 + \omega a_3 \\ a_1 + \omega a_2 + \omega^2 a_3 \end{pmatrix}, \\ B &= \begin{pmatrix} b_1 + b_2 + b_3 \\ b_1 + \omega^2 b_2 + \omega b_3 \\ b_1 + \omega b_2 + \omega^2 b_3 \end{pmatrix}, \quad a_j, b_j \in \mathbb{Q}(\omega)^r. \end{aligned}$$

Then we have

$$[L : L_1] \sum_{N \in L_1} \psi(N) = \sum_{P, Q \in L/L_2} \sum_{M \in L_2} \chi_Q(M+P) \psi(M+P). \quad (2)$$

It is easy to see that the left-hand side of (2) becomes

$$3^r \Theta \begin{pmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \end{pmatrix} (\tau) \Theta \begin{pmatrix} a_1 + \omega a_2 + \omega^2 a_3 \\ b_1 + \omega b_2 + \omega^2 b_3 \end{pmatrix} (\tau) \Theta \begin{pmatrix} a_1 + \omega^2 a_2 + \omega a_3 \\ b_1 + \omega^2 b_2 + \omega b_3 \end{pmatrix} (\tau).$$

For any $M \in L_2$, there exists $N' \in L_1$ such that $M = VN'$. Since $M = VN' = V^*(V^2N')$ and $V^2 \in GL_3(\mathbb{Z}[\omega])$, there also exists $N \in L_1$ such that $M = V^*N$.

The right-hand side of (2) is

$$\begin{aligned}
& \sum_{P, Q \in L/L_2} \sum_{M \in L_2} \mathbf{e} \left[4 \operatorname{Re} \left\{ \frac{1}{\sqrt{-3}} \operatorname{tr}((M+P)Q^*) \right\} \right] \\
& \quad \times \mathbf{e}[\operatorname{tr}\{(M+P+A)\tau(M+P+A)^* + 4 \operatorname{Re}((M+P)B^*)\}] \\
&= \sum_{P, Q \in L_1/(L_1 \cap L_2)} \sum_{N \in L_1} \mathbf{e} \left[4 \operatorname{Re} \left\{ \frac{1}{\sqrt{-3}} \operatorname{tr}((V^*N+P)Q^*) \right\} \right] \\
& \quad \times \mathbf{e}[\operatorname{tr}\{(V^*N+P+A)\tau(V^*N+P+A)^* + 4 \operatorname{Re}((V^*N+P)B^*)\}] \\
&= \sum_{P, Q \in L_1/(L_1 \cap L_2)} \sum_{N \in L_1} \mathbf{e}[4 \operatorname{Re}\{\operatorname{tr}(PB^*)\}] \mathbf{e} \left[4 \operatorname{Re} \left\{ \frac{1}{\sqrt{-3}} \operatorname{tr}(NQ^*V^*) \right\} \right] \\
& \quad \times \mathbf{e}[\operatorname{tr}\{(N+VP+VA)\tau(N+VP+VA)^* + 4 \operatorname{Re}(NB^*V^*)\}] \\
&= \sum_{P, Q \in L_1/(L_1 \cap L_2)} \mathbf{e}[4 \operatorname{Re}\{\operatorname{tr}(PB^*)\}] \\
& \quad \times \sum_{N \in L_1} \mathbf{e} \left[\operatorname{tr} \left\{ (N+VP+VA)\tau(N+VP+VA)^* + 4 \operatorname{Re} \left(N \left(VB - \frac{VQ}{\sqrt{-3}} \right)^* \right) \right\} \right].
\end{aligned}$$

We put $e = (-1/\sqrt{-3})p$, $f = (1/\sqrt{-3})q$ for $p, q \in \{0, \pm 1\}^r$, then $\operatorname{tr}(PB^*)$ is expressed as $= -\sqrt{-3}e(b_1 + b_2 + b_3)^*$. Since

$$VA = -\sqrt{-3} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad VB = -\sqrt{-3} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad VP = - \begin{pmatrix} e \\ e \\ e \end{pmatrix}, \quad VQ = \begin{pmatrix} f \\ f \\ f \end{pmatrix},$$

the last summation

$$\sum_{N \in L_1} \mathbf{e} \left[\operatorname{tr} \left\{ (N+VP+VA)\tau(N+VP+VA)^* + 4 \operatorname{Re} \left(N \left(VB - \frac{VQ}{\sqrt{-3}} \right)^* \right) \right\} \right]$$

becomes

$$\prod_{j=1}^3 \Theta \left(-\frac{f}{\sqrt{-3}} - \sqrt{-3}b_j \right) (\tau) = \prod_{j=1}^3 \Theta \left(\frac{e + \sqrt{-3}a_j}{\sqrt{-3}} + \sqrt{-3}b_j \right) (\tau).$$

When $a_1 = a$, $b_1 = b$, $a_2 = a_3 = b_2 = b_3 = 0$, the first equality of this theorem reduces to the second equality. $\square$

Remark 3. The lattices $\sqrt{-3}L$ and $L_1 \cap L_2$ are isomorphic to $(E_6)^r$ and $(E_6^*)^r$, respectively, where E_6 is the root lattice of type E_6 , and E_6^* is its dual.

To simplify cubic relations among $\Theta_\omega\left(\frac{a}{b}\right)(\tau)$, we put

$$\begin{aligned}\Theta_\omega\left[\frac{a}{b}\right](\tau) &= \Theta_\omega\left(\frac{\frac{a}{\sqrt{-3}}}{\frac{b}{3}}\right)(\tau) \\ &= \sum_{n \in \mathbb{Z}[\omega]^r} \mathbf{e}\left[2\left(n + \frac{a}{\sqrt{-3}}\right)\tau\left(n + \frac{a}{\sqrt{-3}}\right)^* + \frac{4}{3}\operatorname{Re}(nb^*)\right].\end{aligned}$$

They satisfy:

- (1) $\Theta_\omega\left[\frac{\varepsilon a}{\varepsilon b}\right](\tau) = \Theta_\omega\left[\frac{a}{b}\right](\tau)$ for a unit $\varepsilon \in \mathbb{Z}[\omega]$;
- (2) $\Theta_\omega\left[\frac{a+\sqrt{-3}p}{b+\sqrt{-3}q}\right](\tau) = \mathbf{e}\left[-\frac{4}{3}\operatorname{Re}(pb^*)\right]\Theta_\omega\left[\frac{a}{b}\right](\tau)$ for $p, q \in \mathbb{Z}[\omega]^r$;
- (3) $\Theta_\omega\left[\frac{a}{b}\right](T \cdot \tau) = \Theta_\omega\left[\frac{-\bar{a}}{\bar{b}}\right](\tau) = \Theta_\omega\left[\frac{\bar{a}}{-\bar{b}}\right](\tau)$.

COROLLARY 2. For $a, b \in \mathbb{Z}[\omega]^r$, we have

$$3^r \Theta_\omega^3\left[\frac{a}{b}\right](\tau) = \sum_{p, q \in \{0, \pm 1\}^r} \mathbf{e}\left[-\frac{4}{3}\operatorname{Re}(pb^* + aq^*)\right] \Theta_\omega^3\left[\frac{p}{q}\right](\tau).$$

Proof. Theorem 3 implies that

$$\begin{aligned}3^r \Theta_\omega^3\left[\frac{a}{b}\right](\tau) &= 3^r \Theta\left(\frac{\frac{a}{\sqrt{-3}}}{\frac{b}{3}}\right)(\tau)^3 \\ &= \sum_{e, f \in R_{\sqrt{-3}}^r} \mathbf{e}\left[-\frac{4}{3}\operatorname{Re}(\sqrt{-3}eb^*)\right] \Theta\left(\frac{\frac{e+a}{\sqrt{-3}} - \frac{b}{\sqrt{-3}}}{\frac{f}{\sqrt{-3}}}\right)(\tau) \Theta\left(\frac{\frac{e}{\sqrt{-3}}}{\frac{f}{\sqrt{-3}}}\right)(\tau)^2 \\ &= \sum_{e, f \in R_{\sqrt{-3}}^r} \mathbf{e}\left[-\frac{4}{3}\operatorname{Re}(\sqrt{-3}eb^*)\right] \Theta_\omega\left[\frac{\sqrt{-3}(e+a)}{-\sqrt{-3}(f-b)}\right](\tau) \Theta_\omega\left[\frac{\sqrt{-3}e}{-\sqrt{-3}f}\right](\tau)^2 \\ &= \sum_{p, q \in \{0, \pm 1\}^r} \mathbf{e}\left[-\frac{4}{3}\operatorname{Re}(pb^* + aq^*)\right] \Theta_\omega^3\left[\frac{p}{q}\right](\tau),\end{aligned}$$

where we put $p = \sqrt{-3}e$ and $q = -\sqrt{-3}f$. □

Remark 4. In the equality of Corollary 2, the coefficient of $\Theta_\omega^3 \begin{bmatrix} -p \\ -q \end{bmatrix}(\tau)$ is the complex conjugate of that of $\Theta_\omega^3 \begin{bmatrix} p \\ q \end{bmatrix}(\tau)$. If we eliminate one of $\Theta_\omega^3 \begin{bmatrix} -p \\ -q \end{bmatrix}(\tau)$ and $\Theta_\omega^3 \begin{bmatrix} p \\ q \end{bmatrix}(\tau)$ ($(p, q) \neq (0, \dots, 0)$) in this equality by the relation $\Theta_\omega \begin{bmatrix} -p \\ -q \end{bmatrix}(\tau) = \Theta_\omega \begin{bmatrix} p \\ q \end{bmatrix}(\tau)$, then the coefficient of $\Theta_\omega^3 \begin{bmatrix} p \\ q \end{bmatrix}(\tau)$ for $(p, q) = (0, \dots, 0)$ is 1, and the others become -1 or 2 .

PROPOSITION 2. *If $a, b \in \mathbb{Z}[\omega]^r$ and $ab^* \notin \sqrt{-3}\mathbb{Z}[\omega]$ then $\Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}(\tau) = 0$.*

Proof. Note that

$$1 - \omega = \sqrt{-3}\omega^2, \quad \omega^2(1 - (1 - \omega)) = 1.$$

We have

$$\begin{aligned} \Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}(\tau) &= \Theta_\omega \begin{bmatrix} \omega^2(1 - (1 - \omega))a \\ \omega^2(1 - (1 - \omega))b \end{bmatrix}(\tau) \\ &= \Theta_\omega \begin{bmatrix} a - \sqrt{-3}\omega^2a \\ b - \sqrt{-3}\omega^2b \end{bmatrix}(\tau) = \mathbf{e} \left[\frac{4}{3} \operatorname{Re}(\omega^2 ab^*) \right] \Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}(\tau). \end{aligned}$$

Thus, $\Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}(\tau) = 0$ if $\mathbf{e} \left[\frac{4}{3} \operatorname{Re}(\omega^2 ab^*) \right] \neq 1$, which is equivalent to $ab^* \notin \sqrt{-3}\mathbb{Z}[\omega]$ for $a, b \in \mathbb{Z}[\omega]^r$. $\square$

We give examples of cubic relations among $\Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}(\tau)$ for $a, b \in \mathbb{Z}[\omega]^r$. For $r = 1$, there are three non-zero $\Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbb{D}^1 = \mathbb{H}$ over $\mathbb{Z}[\omega]$ for $a, b \in \{0, \pm 1\}$:

$$\Theta_\omega \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Theta_\omega \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \Theta_\omega \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \Theta_\omega \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \Theta_\omega \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Corollary 2 implies the cubic relation among them:

$$\Theta_\omega^3 \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) - \Theta_\omega^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau) - \Theta_\omega^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau) = 0.$$

For $r = 2$, there are 17 non-zero $\Theta_\omega \begin{bmatrix} a \\ b \end{bmatrix}$ on $\mathbb{D}^2$ over $\mathbb{Z}[\omega]$ for $a, b \in \{0, \pm 1\}^2$. Corollary 2 implies linear relations among their cubes. The seven functions

$$\begin{aligned} &\Theta_\omega^3 \begin{bmatrix} 1, 1 \\ 1, -1 \end{bmatrix}, \quad \Theta_\omega^3 \begin{bmatrix} 1, 1 \\ 0, 0 \end{bmatrix}, \quad \Theta_\omega^3 \begin{bmatrix} 1, 1 \\ -1, 1 \end{bmatrix}, \quad \Theta_\omega^3 \begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix}, \\ &\Theta_\omega^3 \begin{bmatrix} 1, -1 \\ 1, 1 \end{bmatrix}, \quad \Theta_\omega^3 \begin{bmatrix} 0, 1 \\ 1, 0 \end{bmatrix}, \quad \Theta_\omega^3 \begin{bmatrix} 0, 0 \\ 0, 0 \end{bmatrix} \end{aligned}$$

are linearly independent in these relations. The rests can be expressed in terms of them:

$$\begin{pmatrix} \Theta_{\omega}^3 \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,0 \\ 0,-1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,-1 \\ 0,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,-1 \\ -1,-1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,1 \\ 0,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,1 \\ -1,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,0 \\ 1,1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,0 \\ 1,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,0 \\ 1,-1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,0 \\ 0,1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \Theta_{\omega}^3 \begin{bmatrix} 1,1 \\ 1,-1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,1 \\ 0,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,1 \\ -1,1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,0 \\ 0,1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 1,-1 \\ 1,1 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,1 \\ 1,0 \end{bmatrix} \\ \Theta_{\omega}^3 \begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix} \end{pmatrix}.$$

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