ON QUASI-ABELIAN VARIETIES OF KIND k

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Dedicated to Professor Hideaki Kazama on his sixtieth birthday

Abstract. Gherardelli and Andreotti defined a quasi-abelian variety of kind k . However, this definition is somewhat vague and we do not know the real meaning of the ‘kind’. We give an example of a quasi-abelian variety which is of kind $k > 0$ but not of kind 0, in the sense of Gherardelli and Andreotti. We prove that if a quasi-abelian variety $X = \mathbb{C}^n / \Gamma$ has an ample Riemann form of kind k , then it has an ample Riemann form of kind k' for any k' with $2k \leq 2k' \leq n - m$, where $\text{rank } \Gamma = n + m$. Next we consider the pair (X, L) of a quasi-abelian variety X and a positive line bundle L on it. We characterize an extendable line bundle L to a compactification $\overline{X}$ of X .

1. Introduction

Let $X = \mathbb{C}^n / \Gamma$ be a toroidal group of rank $\Gamma = n + m$. It is called a quasi-abelian variety if it satisfies generalized Riemann conditions [8]. Moreover, Gherardelli and Andreotti defined a quasi-abelian variety of kind k in [8]. There are other works concerning this; see, for example, [1], [7] and [11]. However, we do not completely understand the meaning of the kind even now.

If we consider the kind in the sense of Gherardelli and Andreotti, a quasi-abelian variety may have two different kinds. In Section 3 we give an example of a quasi-abelian variety which is of kind $k > 0$, but not of kind 0. On the other hand, we show that if a quasi-abelian variety $X = \mathbb{C}^n / \Gamma$ is of kind k , then it is of kind k' for any k' with $2k \leq 2k' \leq n - m$ (Theorem 4.1).

We see from the above example and theorem that any quasi-abelian variety has the smallest kind. Then we can give a precise definition of the kind for a quasi-abelian variety.

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We consider the pair (X, L) of a quasi-abelian variety X and a positive line bundle L on it, which is called a polarized quasi-abelian variety. This polarization gives the fibre bundle structure of X on an abelian variety A determined by a hermitian form of L . Replacing the fibres by a product of some number of copies of the one-dimensional complex projective space $\mathbb{P}^1$, we obtain a compactification $\overline{X}$ of X . We study the extendability of L to $\overline{X}$. We prove in Section 6 that L is holomorphically extendable to $\overline{X}$ if and only if L is the pull-back of a theta bundle on A . This was discussed in [4] and also in [6]. However, the proof was incomplete. In this paper we complete the proof using a more general approach to topologically trivial holomorphic line bundles, which is valid for any fibration of a toroidal group.

2. Definitions by Gherardelli and Andreotti

Let $X = \mathbb{C}^n / \Gamma$ be a quotient group of $\mathbb{C}^n$ by a discrete subgroup Γ . It is called a toroidal group if $H^0(X, \mathcal{O}) = \mathbb{C}$. In this case we have $\text{rank } \Gamma = n + m$ ($1 \leq m \leq n$). Let $\mathbb{R}_\Gamma$ be the real linear subspace of $\mathbb{C}^n$ spanned by Γ . We define

$$\mathbb{C}_\Gamma := \mathbb{R}_\Gamma \cap i\mathbb{R}_\Gamma.$$

Then $\mathbb{C}_\Gamma$ is the maximal complex linear subspace contained in $\mathbb{R}_\Gamma$, and we have $\dim_{\mathbb{C}} \mathbb{C}_\Gamma = m$. Such a toroidal group X is said to be of type m . The following definition of quasi-abelian varieties was first given in [8].

Definition 2.1. A toroidal group $X = \mathbb{C}^n / \Gamma$ is said to be a quasi-abelian variety if there exists a hermitian form $\mathcal{H}$ on $\mathbb{C}^n$ such that:

- (1) $\mathcal{H}$ is positive definite on $\mathbb{C}_\Gamma$;
- (2) the imaginary part $\mathcal{A} := \text{Im } \mathcal{H}$ of $\mathcal{H}$ is $\mathbb{Z}$ -valued on $\Gamma \times \Gamma$.

We call a hermitian form $\mathcal{H}$ in the above definition an ample Riemann form for Γ or for X . Let $\mathcal{A}_\Gamma := \mathcal{A}|_{\mathbb{R}_\Gamma \times \mathbb{R}_\Gamma}$. Gherardelli and Andreotti [8] also gave the following definition.

Definition 2.2. A quasi-abelian variety $X = \mathbb{C}^n / \Gamma$ of type m is said to be of kind k if there exists an ample Riemann form $\mathcal{H}$ for Γ with

$$\text{rank } \mathcal{A}_\Gamma = 2(m + k).$$

We remark that a quasi-abelian variety X may have two different kinds. We can construct such examples. Furthermore, we have a general statement (see Theorem 4.1). Therefore the kind defined by Gherardelli and Andreotti is not

determined uniquely. We think that it is not only a property of a quasi-abelian variety $X = \mathbb{C}^n / \Gamma$ but also that of an ample Riemann form $\mathcal{H}$ for Γ . We give the following definition.

Definition 2.3. An ample Riemann form $\mathcal{H}$ for Γ is said to be of kind k if $\text{rank } \mathcal{A}_\Gamma = 2(m + k)$, where $\text{rank } \Gamma = n + m$.

3. An example not of kind 0

We have fibration theorems for quasi-abelian varieties; see [1], [7] and [8]. The second author studied these arguments more precisely to characterize quasi-abelian varieties. The following theorem is the characterization of quasi-abelian varieties with ample Riemann forms of kind k .

THEOREM 3.1. [11, Theorem 4.2] *Let X be an n -dimensional toroidal group of type m . Then X is a quasi-abelian variety with an ample Riemann form of kind k if and only if there exists an abelian variety A of dimension $m + k$ with an exact sequence*

$$0 \longrightarrow \mathbb{C}^k \times (\mathbb{C}^*)^{n-m-2k} \longrightarrow X \longrightarrow A \longrightarrow 0.$$

We can restate the generalized Riemann conditions (Definition 2.1) more concretely. Let $\mathbb{C}^n / \Gamma$ be a toroidal group of type m with a period matrix

$$P = (\gamma_1, \dots, \gamma_{n+m}) = (I_n \ V),$$

where I_n is the unit matrix of degree n and V is an $n \times m$ matrix with $\text{rank}(\text{Im } V) = m$. We consider a hermitian form $\mathcal{H}$ on $\mathbb{C}^n$ with $\mathcal{A} := \text{Im } \mathcal{H}$ $\mathbb{Z}$ -valued on $\Gamma \times \Gamma$. Then $\mathcal{H}$ has the representative matrix

$$H = (\mathcal{H}(e_i, e_j))_{i,j=1}^n,$$

where $I_n = (e_1, \dots, e_n)$. We set

$$E = (\mathcal{A}(\gamma_i, \gamma_j))_{i,j=1}^{n+m},$$

which is a skew-symmetric matrix with integral coefficients. We write

$$E = \begin{pmatrix} E_1 & E_2 \\ -{}^t E_2 & E_3 \end{pmatrix},$$

where E_1 is an $n \times n$ matrix and E_3 is an $m \times m$ matrix.

THEOREM 3.2. [11, Theorem 3.1] *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type m with a period matrix $P = (\gamma_1, \dots, \gamma_{n+m}) = (I_n \ V)$.*

- (1) *If $\mathbb{C}^n/\Gamma$ is a quasi-abelian variety with an ample Riemann form $\mathcal{H}$, then the skew-symmetric matrix E corresponding to $\mathcal{A} = \text{Im } \mathcal{H}$ satisfies the following conditions:*

$${}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0; \quad (3.1)$$

$$\frac{i}{2}({}^t\overline{V}E_1V + {}^tE_2V - {}^t\overline{V}E_2 + E_3) > 0. \quad (3.2)$$

- (2) *Conversely, if we have a skew-symmetric $(n+m) \times (n+m)$ matrix E with integral coefficients which satisfies (3.1) and (3.2), then $\mathbb{C}^n/\Gamma$ is a quasi-abelian variety with an ample Riemann form $\mathcal{H}$ whose imaginary part has the representative matrix E on $\Gamma \times \Gamma$.*

Example. We consider a toroidal group $\mathbb{C}^4/\Gamma$ of type 2 with period matrix $P = (I_4 \ V)$, where

$$V = \begin{pmatrix} \sqrt{2}i & \sqrt{5}i \\ \sqrt{3}i & \sqrt{7}i \\ \sqrt{5}i & -\sqrt{2} \\ \sqrt{7} & -5\sqrt{3}i \end{pmatrix}.$$

Put

$$E = \begin{pmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{pmatrix},$$

where

$$E_1 = \begin{pmatrix} 0 & -p & -a & -b \\ p & 0 & -c & -d \\ a & c & 0 & -q \\ b & d & q & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -e & -i^* \\ -f & -j \\ -g & -k \\ -h & -\ell \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}$$

and $a, b, c, d, e, f, g, h, i^*, j, k, \ell, q, r \in \mathbb{Z}$. Further, put

$$R_1 := {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = \begin{pmatrix} 0 & -r_1 \\ r_1 & 0 \end{pmatrix}$$

and

$$R_2 := \frac{i}{2}({}^t\overline{V}E_1V + {}^tE_2V - {}^t\overline{V}E_2 + E_3).$$

By a straight calculation we have

$$\begin{aligned} r_1 = & 5a + 5\sqrt{6}b + \sqrt{35}c + 15d - \sqrt{2}g - \sqrt{7}\ell + (\sqrt{15} - \sqrt{14})p \\ & + (\sqrt{14} + 5\sqrt{15})q + r + i(-2a - \sqrt{35}b - \sqrt{6}c - 7d \\ & + \sqrt{5}e + \sqrt{7}f - 5\sqrt{3}h - \sqrt{2}i^* - \sqrt{3}j - \sqrt{5}k). \end{aligned}$$

Suppose that condition (3.1) in Theorem 3.2 holds. Then $r_1 = 0$, hence

$$\begin{aligned} b = c = f = g = i^* = \ell = p = q = 0, \\ a = 7m, \quad d = -2m, \quad r = -5m, \quad j = -5h, \quad k = e \quad \text{with } m \in \mathbb{Z}. \end{aligned}$$

In this case we have

$$R_2 = \begin{pmatrix} \sqrt{2}e + 2\sqrt{21}m & \sqrt{5}e - 5\sqrt{3}h + (14 + 5i)m \\ \sqrt{5}e - 5\sqrt{3}h + (14 - 5i)m & -5\sqrt{7}h + 7\sqrt{10}m \end{pmatrix}$$

and

$$\det(E) = m^2(2e^2 - 35h^2 + 70m^2)^2.$$

Put

$$\begin{aligned} D(e, h, m) &:= \det(R_2) \\ &= -5e^2 + (-5\sqrt{14} + 10\sqrt{15})eh - 75h^2 - 14\sqrt{5}em + 70\sqrt{3}hm \\ &\quad + (-221 + 14\sqrt{210})m^2 \end{aligned}$$

and

$$D_1(e, m) := \sqrt{2}e + 2\sqrt{21}m.$$

Since $2e^2 - 35h^2 + 70m^2 \neq 0$ for any $(e, h, m) \in \mathbb{Z}^3 \setminus \{0\}$, we see that $\mathbb{C}^4/\Gamma$ has an ample Riemann form of kind 1 if and only if

$$m \neq 0, \quad D > 0 \quad \text{and} \quad D_1 > 0.$$

We note that

$$D(-1, 0, 1) = 8.18423 \dots \quad \text{and} \quad D_1(-1, 1) = 7.75094 \dots$$

Hence, $\mathbb{C}^4/\Gamma$ is a quasi-abelian variety with an ample Riemann form of kind 1, as the above conditions are satisfied when $e = -1$, $h = 0$ and $m = 1$.

Next we show that $\mathbb{C}^4/\Gamma$ does not have an ample Riemann form of kind 0. If it has such a form, then $\det(E) = 0$, $D > 0$ and $D_1 > 0$. In this case $m = 0$. However,

$$D(e, h, 0) = -5e^2 + (-5\sqrt{14} + 10\sqrt{15})eh - 75h^2 < 0$$

for any $e, h \in \mathbb{Z}$ with $(e, h) \neq (0, 0)$, because

$$(\sqrt{14} - 2\sqrt{15})^2 - 60 < 0.$$

Therefore $\mathbb{C}^4/\Gamma$ has no ample Riemann form of kind 0.

4. The smallest kind

THEOREM 4.1. *Let $X = \mathbb{C}^n/\Gamma$ be a quasi-abelian variety of type m . Suppose that X has an ample Riemann form of kind k ($0 \leq 2k \leq n - m$). Then for any k' with $2k \leq 2k' \leq n - m$ there exists an ample Riemann form of kind k' for X .*

Proof. Let $\mathcal{H}$ be an ample Riemann form of kind k . We may assume that $\mathcal{H}$ is positive definite on $\mathbb{C}^n$. Suppose $2k < n - m$, and set $\ell := n - m - 2k$. We can take a period matrix $P = (\gamma_1, \dots, \gamma_{n+m})$ of Γ such that

$$\text{Im } {}^t P H \overline{P} = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ -D & 0 & 0 \end{pmatrix} \} \ell \text{ rows} \quad \text{with } D = \text{diag}(d_1, \dots, d_{m+k}),$$

where H is the representative matrix of $\mathcal{H}$, $d_1, \dots, d_{m+k}$ are positive integers with $d_1 | d_2 | \dots | d_{m+k}$. Furthermore, there exist a positive integer d_{m+k+1} with $d_{m+k} | d_{m+k+1}$ and $\gamma_{n+m+1} \in \mathbb{C}^n$ with $\gamma_{n+m+1} \notin \mathbb{R}_\Gamma$ such that if we set $P_1 = (P \ \gamma_{n+m+1})$, then

$$\text{Im } {}^t P_1 H \overline{P}_1 = \begin{pmatrix} 0 & 0 & D_1 \\ 0 & 0 & 0 \\ -D_1 & 0 & 0 \end{pmatrix} \} \ell - 1 \text{ rows} \quad \text{with } D_1 = \text{diag}(d_1, \dots, d_{m+k+1})$$

(see the proofs of Lemma 3.1.8 and Theorem 3.1.16 in [6]). We define

$$\Gamma_1 := \Gamma \oplus \gamma_{n+m+1} \mathbb{Z} \quad \text{and} \quad X_1 := \mathbb{C}^n / \Gamma_1.$$

There exists a covering map $\sigma_1 : X \rightarrow X_1$ whose covering transformation group is $\mathbb{Z}$. Then X_1 is a toroidal group of type $m + 1$. By the construction of X_1 we know that $\mathcal{H}$ is an ample Riemann form for X_1 ; consequently X_1 is a quasi-abelian variety.

If $\ell = 1$, then X_1 is an n -dimensional abelian variety and we have nothing to show.

Suppose $\ell \geq 2$. Since

$$\text{rank}((\text{Im } \mathcal{H})|_{\Gamma_1 \times \Gamma_1}) = 2(m + k + 1) = 2(m + 1) + 2k,$$

$\mathcal{H}$ is an ample Riemann form of kind k for X_1 . By Theorem 3.1 there exists an abelian variety A of dimension $m + 1 + k$ with an exact sequence

$$0 \longrightarrow \mathbb{C}^k \times (\mathbb{C}^*)^{n-m-1-2k} \longrightarrow X_1 \xrightarrow{\tau} A \longrightarrow 0.$$

Then $\tau \circ \sigma_1 : X \longrightarrow A$ is an epimorphism. Let

$$S := \sigma_1^{-1}(\ker \tau) = \ker(\tau \circ \sigma_1).$$

Since $\ker \tau = \mathbb{C}^k \times (\mathbb{C}^*)^{n-m-1-2k}$,

$$\sigma_1|_S : S \longrightarrow \mathbb{C}^k \times (\mathbb{C}^*)^{n-m-1-2k}$$

is also a covering map with covering transformation group $\mathbb{Z}$. Then $S = \mathbb{C}^{k+1} \times (\mathbb{C}^*)^{n-m-2(k-1)}$. Thus we obtain an exact sequence

$$0 \longrightarrow \mathbb{C}^{k+1} \times (\mathbb{C}^*)^{n-m-2(k-1)} \longrightarrow X \xrightarrow{\tau \circ \sigma_1} A \longrightarrow 0.$$

By Theorem 3.2, X has an ample Riemann form of kind $k + 1$. $\square$

By Theorem 4.1 and the example in the previous section, we can give a precise definition of ‘kind’ for any quasi-abelian variety.

Definition 4.2. Let $X = \mathbb{C}^n / \Gamma$ be a quasi-abelian variety of type m . Then the kind of X is the smallest integer k with $0 \leq 2k \leq n - m$ such that there exists an ample Riemann form of kind k for X .

5. Polarized quasi-abelian varieties

Let $X = \mathbb{C}^n / \Gamma$ be a quasi-abelian variety of type m . We consider a positive line bundle L on X . We have $L \cong L_0 \otimes L_\rho$, where L_0 is a topologically trivial holomorphic line bundle and L_ρ is the line bundle given by a theta factor $\rho : \Gamma \times \mathbb{C}^n \longrightarrow \mathbb{C}^*$. We may assume that ρ is a reduced theta factor of type $(\mathcal{H}, \psi)$. Since L is positive, $\mathcal{H}$ is an ample Riemann form for Γ (see [2] and [6, Theorem 3.2.13]). We also assume that $\mathcal{H}$ is of kind k . In this case the pair (X, L) is called a polarized quasi-abelian variety of kind k .

By a suitable change of variables and generators of Γ , we can take a period matrix P of Γ as

$$P = \left(\begin{pmatrix} w_{11} & \dots & w_{1,m+k} \\ \dots & \dots & \dots \\ w_{n,1} & \dots & w_{n,m+k} \end{pmatrix} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{n-k} \\ & & & 0 \end{pmatrix} \right), \quad (5.1)$$

where d_i ($i = 1, \dots, n - k$) are positive integers with $d_1 | \dots | d_{n-k}$, and

$$W' := \begin{pmatrix} w_{11} & \dots & w_{1,m+k} \\ \dots & \dots & \dots \\ w_{m+k,1} & \dots & w_{m+k,m+k} \end{pmatrix}$$

belongs to the Siegel upper-half space $\mathfrak{S}_{m+k}$ of degree $m + k$. If we set

$$W'' := \begin{pmatrix} w_{m+k+1,1} & \dots & w_{m+k+1,m+k} \\ \dots & \dots & \dots \\ w_{n,1} & \dots & w_{n,m+k} \end{pmatrix},$$

$$D' := \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{m+k} \end{pmatrix}, \quad D'' := \begin{pmatrix} d_{m+k+1} & & \\ & \ddots & \\ & & d_{n-k} \end{pmatrix},$$

$$P' := \begin{pmatrix} W' \\ W'' \end{pmatrix} \quad \text{and} \quad P'' := \begin{pmatrix} D' & 0 \\ 0 & \begin{pmatrix} D'' \\ 0 \end{pmatrix} \end{pmatrix},$$

then we have

$$P = (P' \ P'').$$

Any $\gamma \in \Gamma$ can be represented uniquely as

$$\gamma = P'a + P''b, \quad a \in \mathbb{Z}^{m+k}, \quad b \in \mathbb{Z}^{n-k}.$$

Furthermore, we may assume that ρ has the following form with respect to this period matrix P :

$$\rho(\gamma, x) = \mathbf{e}(-{}^t a z - \frac{1}{2} {}^t a W' a) \quad (5.2)$$

for $\gamma = P'a + P''b$ and $x = (z, w) \in \mathbb{C}^{m+k} \times \mathbb{C}^{n-m-k}$, where $\mathbf{e}(\cdot) = \exp(2\pi i \cdot)$. For these facts we refer to [4], [5] or [6] (originally in [3]).

6. Compactifications and extendable line bundles

A toroidal group $X = \mathbb{C}^n / \Gamma$ of type m has the standard period matrix P in toroidal coordinates as

$$P = \begin{pmatrix} 0 & I_m & T \\ I_{n-m} & R_1 & R_2 \end{pmatrix},$$

where $(I_m \ T)$ is a period matrix of an m -dimensional complex torus and $(R_1 \ R_2)$ is a real matrix. Let $(z_1, \dots, z_n)$ be coordinates of $\mathbb{C}^n$. The projection $pr :$

$\mathbb{C}^n \longrightarrow \mathbb{C}^m, (z_1, \dots, z_n) \longmapsto (z_1, \dots, z_m)$ induces the $(\mathbb{C}^*)^{n-m}$ -principal bundle $\sigma : X \longrightarrow \mathbb{T}$, where $\mathbb{T}$ is the m -dimensional complex torus with period matrix $(I_m \ T)$. Compactifying the fibres $(\mathbb{C}^*)^{n-m}$ by $(\mathbb{P}^1)^{n-m}$, we obtain the associated $(\mathbb{P}^1)^{n-m}$ -bundle $\bar{\sigma} : \bar{X} \longrightarrow \mathbb{T}$. Stein studied the extendability of any line bundle L on X to $\bar{X}$ in [10].

On the contrary, a fibration of a quasi-abelian variety on an abelian variety depends on the kind of an ample Riemann form (Gherardelli–Andreotti Fibration Theorem). Then we consider a polarized quasi-abelian variety (X, L) of kind k . We are interested in the extendability of L . This problem was discussed in [4] and in [6], but the argument was incomplete. We complete the proof in this section.

We assume that a polarized quasi-abelian variety (X, L) is as in the previous section. Let $x = (x_1, \dots, x_n)$ be complex coordinates of $\mathbb{C}^n$ in expressions (5.1) and (5.2). The projection $\tau_0 : \mathbb{C}^n \longrightarrow \mathbb{C}^{m+k}, (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{m+k})$ gives a $(\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$ -bundle $\tau : X \longrightarrow A$ on an $(m+k)$ -dimensional abelian variety A with period matrix $(W' \ D')$. Compactifying the fibres, we obtain the associated $(\mathbb{P}^1)^{n-m-k}$ -bundle $\bar{\tau} : \bar{X} \longrightarrow A$. We study the extendability of L to $\bar{X}$.

To formulate the problem more precisely, we define a group homomorphism $p : \mathbb{C}^n \longrightarrow \mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$ by

$$p(x) := (x_1, \dots, x_{m+k}, \mathbf{e}(x_{m+k+1}/d_{m+k+1}), \dots, \mathbf{e}(x_{n-k}/d_{n-k}), x_{n-k+1}, \dots, x_n).$$

Any $\eta \in p(\Gamma)$ acts on $\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$ by

$$\eta \cdot (y_1, \dots, y_n) := (y_1 + \eta_1, \dots, y_{m+k} + \eta_{m+k}, y_{m+k+1} \cdot \eta_{m+k+1}, \dots, y_{n-k} \cdot \eta_{n-k}, y_{n-k+1} + \eta_{n-k+1}, \dots, y_n + \eta_n),$$

where $(y_1, \dots, y_n)$ are complex coordinates of $\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k$. This action of $p(\Gamma)$ is properly discontinuous and fixed point free. It is extendable to an action on $\mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k}$. Then we have

$$X \cong (\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k) / p(\Gamma)$$

and $\bar{X} \cong \hat{X}$, where

$$\hat{X} := (\mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k}) / p(\Gamma).$$

There exists an embedding $\iota : X \longrightarrow \hat{X}$ with the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{p} & \mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ X & \xrightarrow{\iota} & \hat{X} \end{array}$$

where π and $\widehat{\pi}$ are the canonical projections. We see that

$$\iota(X) = \widehat{\pi}(\mathbb{C}^{m+k} \times (\mathbb{C}^*)^{n-m-2k} \times \mathbb{C}^k).$$

By the projection

$$\tau_1 : \mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k} \longrightarrow \mathbb{C}^{m+k}, \quad (y_1, \dots, y_n) \longmapsto (y_1, \dots, y_{m+k}),$$

we obtain a $(\mathbb{P}^1)^{n-m-k}$ -bundle $\widehat{\tau} : \widehat{X} \longrightarrow A$. It is isomorphic to $\overline{\tau} : \overline{X} \longrightarrow A$ as fibre bundles on A . Then $L \longrightarrow X$ extends holomorphically to $\overline{X}$ if and only if there exists a holomorphic line bundle $L_1 \longrightarrow \widehat{X}$ such that $(\iota^{-1})^*L \cong L_1|_{\iota(X)}$.

Let $\Gamma^* := \tau_0(\Gamma)$. Then $A = \mathbb{C}^{m+k}/\Gamma^*$. Since ρ has the representation (5.2), we have a theta factor $\rho_0 : \Gamma^* \times \mathbb{C}^{m+k} \longrightarrow \mathbb{C}^*$ with $\rho(\gamma, x) = \rho_0(\tau_0(\gamma), \tau_0(x))$. Therefore $L_\rho \cong \tau^*L_{\rho_0}$, where L_{ρ_0} is the theta bundle given by ρ_0 . Hence L_ρ is extendable to $\overline{X}$.

Next we treat the topologically trivial holomorphic line bundle L_0 . We suppose that L_0 extends holomorphically to $\overline{X}$. Then there exists a holomorphic line bundle $\widehat{L} \longrightarrow \widehat{X}$ such that $(\iota^{-1})^*L_0 \cong \widehat{L}|_{\iota(X)}$.

LEMMA 6.1. *The line bundle $\widehat{L} \longrightarrow \widehat{X}$ is topologically trivial.*

Proof. Let $\widehat{\mathcal{U}} = \{\widehat{U}_i\}$ be a covering of $\widehat{X}$ consisting of simply connected open sets $\widehat{U}_i$. We assume that $\widehat{L}$ is defined by a system of transition functions $\{\widehat{g}_{ij}\}$ with respect to $\widehat{\mathcal{U}}$. Define

$$U_i := \widehat{U}_i \cap \iota(X) \quad \text{and} \quad g_{ij} := \widehat{g}_{ij}|_{U_i \cap U_j}.$$

Then $\widehat{L}|_{\iota(X)}$ is given by a system of transition functions $\{g_{ij}\}$ with respect to a covering $\mathcal{U} := \{U_i\}$ of $\iota(X)$. The Chern class $c_1(\widehat{L})$ of $\widehat{L}$ is given by a 2-cocycle $\{\widehat{z}_{ijk}\} \in Z^2(\widehat{\mathcal{U}}, \mathbb{Z})$ defined by

$$\widehat{z}_{ijk} := \frac{1}{2\pi i}(\log \widehat{g}_{ij} + \log \widehat{g}_{jk} - \log \widehat{g}_{ik}).$$

If we define

$$z_{ijk} := \frac{1}{2\pi i}(\log g_{ij} + \log g_{jk} - \log g_{ik}),$$

then $\{z_{ijk}\}$ determines $c_1(\widehat{L}|_{\iota(X)})$. Since an automorphic factor defining L_0 is given by an automorphic summand, L_0 has a system of transition functions which defines the null 2-cocycle (cf. [6, Lemma 2.1.9]). We have $(\iota^{-1})^*L_0 \cong \widehat{L}|_{\iota(X)}$. Then $\{z_{ijk}\} = 0$. Hence we have $\{\widehat{z}_{ijk}\} = 0$. $\square$

LEMMA 6.2. *For any non-negative integers r, s and t with $r + s + t > 0$, we have*

$$H^q(\mathbb{C}^r \times (\mathbb{C}^*)^s \times (\mathbb{P}^1)^t, \mathcal{O}) = 0, \quad q > 0.$$

Proof. By induction on t we can prove the assertion applying the argument used to show $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ (cf. [9, pp. 47–48]). $\square$

Since $\widehat{L}$ is topologically trivial (Lemma 6.1), the pull-back $\widehat{\pi}^*\widehat{L}$ is also topologically trivial. By Lemma 6.2 we have

$$H^q(\mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k}, \mathcal{O}) = 0, \quad q > 0.$$

Then $\widehat{\pi}^*\widehat{L}$ is analytically trivial. Therefore $\widehat{L} \rightarrow \widehat{X}$ is defined by an automorphic factor

$$\widehat{\alpha} : p(\Gamma) \times (\mathbb{C}^{m+k} \times (\mathbb{P}^1)^{n-m-k}) \rightarrow \mathbb{C}^*.$$

For any $\eta \in p(\Gamma)$, $\widehat{\alpha}(\eta, y)$ does not depend on $y_{m+k+1}, \dots, y_n$. Since

$$\Gamma^* \cong (W' D') \mathbb{Z}^{2(m+k)},$$

there exists an automorphic factor $\alpha_0 : \Gamma^* \times \mathbb{C}^{m+k} \rightarrow \mathbb{C}^*$ such that

$$\widehat{\alpha}(\eta, y) = \alpha_0(\tau_1(\eta), \tau_1(y)).$$

α_0 is equivalent to a homomorphism $\varphi : \Gamma^* \rightarrow \mathbb{C}^*$ as $A = \mathbb{C}^{m+k} / \Gamma^*$ is an abelian variety. Hence we obtain the following theorem.

THEOREM 6.3. *Let (X, L) be a polarized quasi-abelian variety of kind k , where X is of type m . Let $\tau : X \rightarrow A$ and $\bar{\tau} : \bar{X} \rightarrow A$ be as above. Then L extends holomorphically to $\bar{X}$ if and only if there exists a theta bundle $L_\theta \rightarrow A$ such that $L \cong \tau^* L_\theta$.*

Remark. The treatment of topologically trivial holomorphic line bundles in this section is valid for compactification of any fibration of a toroidal group. Using it, we can simplify the argument in [10] and also in [6].

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