OKA'S EXTRA ZERO PROBLEM AND APPLICATION

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Dedicated to the memory of Professor Toshio Nishino

Abstract. We solve the extra zero problem in cylindrical domains in $\mathbb{C}^n$. We apply it to study domains of holomorphy in $\mathbb{C}^n$ which are not rationally convex. This result generalizes Oka's results.

1. Introduction

Let D be a domain in $\mathbb{C}^n$. The collection $(\mathfrak{z}) = \{(f_p, U_p)\}_{p \in D}$, where f_p is a holomorphic function in a neighborhood U_p of p in D , is called a (*holomorphic*) *Cousin II distribution* in D if, for every $p, q \in D$ with $U_p \cap U_q \neq \emptyset$, f_p/f_q is a non-vanishing holomorphic function in $U_p \cap U_q$. For a given $(\mathfrak{z}) = \{(f_p, U_p)\}_{p \in D}$ in D , the *Cousin II problem* is the problem of finding a holomorphic function f in D such that $f = f_p \lambda_p$ in U_p , where λ_p is a non-vanishing holomorphic function in U_p . It is not always solvable (see, for example, [4, 8]). Oka posed the following problem concerning it, in 1943 ([6, No. 2, p. 31, Problem 2]).

EXTRA ZERO PROBLEM. *Given a Cousin II distribution $(\mathfrak{z}) = \{(f_p, U_p)\}_{p \in D}$ in a domain D , find a holomorphic function f in D such that $f = f_p \alpha_p$ in U_p , where α_p is a holomorphic function in U_p and $\{f_p = 0\} \cap \{\alpha_p = 0\} = \emptyset$.*

We call f a solution of the extra zero problem for the given data $(\mathfrak{z})$ in D . A cylindrical domain D in $\mathbb{C}^n$ is the Cartesian product $D_1 \times \cdots \times D_n = \prod_{i=1}^n D_i$ of some domains D_i in $\mathbb{C}_{z_i}$ ($i = 1, \dots, n$).

In this paper, we solve the extra zero problem in cylindrical domains (Theorem 1). As an application, we obtain Theorem 2 which is a generalization of Oka [5] and the example of Stein [7, p. 757].

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THEOREM 1. *We have the following.*

- (i) *Let D be a cylindrical domain D in $\mathbb{C}^2$ and let $(\mathfrak{z})$ be a Cousin II distribution in D . Then for every cylindrical domain $D_0 \Subset D$, the extra zero problem for $(\mathfrak{z})|_{D_0}$ is solvable.*
- (ii) *If $n \geq 3$, there exist a cylindrical domain D in $\mathbb{C}^n$ and a Cousin II distribution $(\mathfrak{z})$ in D such that for some cylindrical domain $D_0 \Subset D$, the extra zero problem for $(\mathfrak{z})|_{D_0}$ is not solvable.*

THEOREM 2. *Let $(\mathfrak{z})$ be a Cousin II distribution in a cylindrical domain D in $\mathbb{C}^n$. Assume that the analytic hypersurface $[(\mathfrak{z})]$ defined by $(\mathfrak{z})$ is irreducible in D . If the Cousin II problem for $(\mathfrak{z})$ is not solvable in D , and if, for every cylindrical domain $D_0 \Subset D$, the extra zero problem for $(\mathfrak{z})|_{D_0}$ is solvable, then $G := D \setminus [(\mathfrak{z})]$ is a domain of holomorphy in $\mathbb{C}^n$ which is not rationally convex in $\mathbb{C}^n$.*

Remark. In the case of $\mathbb{C}^2$, the assumption that, for every cylindrical domain $D_0 \Subset D$, the extra zero problem for $(\mathfrak{z})|_{D_0}$ is solvable, does not require Theorem 1.

2. Preliminaries

In order to prove Theorems 1 and 2, we use Satz 5 of Stein [8] concerning intersection number. (See Proposition 2.4 below.) To apply Stein's Satz 5 to the case of cylindrical domains in $\mathbb{C}^n$, we need the following definition of an idea from Oka's posthumous publication [6, No. 5, p. 111].

Notation. Let $D = \prod_{k=1}^n D_k$ be a cylindrical domain in $\mathbb{C}^n$. We define:

- for $i \in \mathbb{N}$ ($1 \leq i \leq n$), put $D^{(i)} = \prod_{k=1, \dots, n; k \neq i} D_k$, and for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, put $z^{(i)} = (z_1, \dots, \hat{z}_i, \dots, z_n) \in \mathbb{C}^{n-1}$ by omitting z_i ;
- for $i, j \in \mathbb{N}$ ($1 \leq i, j \leq n$), put $D^{(ij)} = \prod_{k=1, \dots, n; k \neq i, j} D_k$, and for $z \in \mathbb{C}^n$, put $z^{(ij)} \in \mathbb{C}^{n-2}$ by omitting z_i and z_j ;
- for $p \in D^{(ij)}$ and an analytic hypersurface S in D , put $H(p) := \{z \in \mathbb{C}^n : z^{(ij)} = p\}$ and put $S(p) := S \cap H(p)$, which is an analytic surface in $D_i \times D_j$.

A Cousin II distribution $(\mathfrak{z})$ in D consists of at most countably many irreducible components (S_v, e_v) ($v \in \mathbb{N}$), where e_v is the vanishing order of the analytic hypersurface S_v in D . We put the zero set $[(\mathfrak{z})] := \bigcup_{v \in \mathbb{N}} S_v \subset D$.

Definition 2.1. (General position) A point $p \in D^{(ij)}$ is in *general position* with respect to $(\mathfrak{z})$ if $H(p) \cap D \not\subset [(\mathfrak{z})]$, and then define the Cousin II distribution $(\mathfrak{z})(p) := (\mathfrak{z}) \cap H(p) = \{(S_v(p), e_v)\}_v$ in $D_i \times D_j$.

We note that the set of all points in general position with respect to $(\mathfrak{z})$ is open and dense in $D^{(ij)}$.

Consider a real two-dimensional surface $\Gamma_{ij} := l_i \times l_j$ in $D_i \times D_j$, where $l_k : [0, 1] \rightarrow D_k$ ($k = i, j$) is an (oriented) smooth simple curve.

Definition 2.2. (Transversality) Γ_{ij} is *transversal* to $(\mathfrak{z})(p)$ if:

- (i) $\#\Gamma_{ij} \cap [(\mathfrak{z})(p)] < \infty$ and $(\Gamma_{ij} \cap [(\mathfrak{z})(p)]) \cap \partial\Gamma_{ij} = \emptyset$;
- (ii) for every $Q = (q_i, q_j) \in \Gamma_{ij} \cap [(\mathfrak{z})(p)]$, some open bi-disk $U_i(Q) \times U_j(Q)$ of Q contains some restriction Γ_{ij}^Q of Γ_{ij} around Q such that $[(\mathfrak{z})(p)]|(U_i(Q) \times U_j(Q))$ is a graph of a univalent $\varphi : U_j(Q) \rightarrow U_i(Q)$;
- (iii) the curve $\varphi(l_j \cap U_j(Q))$ traverses $l_i \cap U_i(Q)$ at q_i .

Then put $s([(\mathfrak{z})(p)], \Gamma_{ij}^Q) := 1$ if $\varphi(l_j \cap U_j(Q))$ traverses $l_i \cap U_i(Q)$ at q_i from the right to the left, and otherwise $s([(\mathfrak{z})(p)], \Gamma_{ij}^Q) := -1$, and also define, when $Q \in S_v(p)$,

$$I((\mathfrak{z})(p), \Gamma_{ij}^Q) := s([(\mathfrak{z})(p)], \Gamma_{ij}^Q) \cdot e_v.$$

Clearly, $I((\mathfrak{z})(p), \Gamma_{ij}^Q)$ is independent of choices of $U_i(Q) \times U_j(Q)$, and hence well defined.

Definition 2.3. (Intersection number) The *intersection number* of $(\mathfrak{z})(p)$ and Γ_{ij} is defined by

$$I((\mathfrak{z})(p), \Gamma_{ij}) := \sum_{Q \in \Gamma_{ij} \cap [(\mathfrak{z})(p)]} I((\mathfrak{z})(p), \Gamma_{ij}^Q).$$

For homotopy classes $[\gamma_k]$ of simple closed curves in D_k ($k = i, j$), we note that there exist smooth representatives γ_k ($k = i, j$) such that $\Gamma_{ij} = \gamma_i \times \gamma_j$ is transversal to $(\mathfrak{z})(p)$ for every $p \in D^{(ij)}$ in general position with respect to $(\mathfrak{z})$, and $I((\mathfrak{z})(p), \Gamma_{ij})$ is independent of choices of such both γ_k ($k = i, j$) and p . Hence, the *intersection number* of $(\mathfrak{z})$ and $([\gamma_i], [\gamma_j])$ is defined by $I((\mathfrak{z}), [\gamma_i], [\gamma_j]) := I((\mathfrak{z})(p), \Gamma_{ij})$. Then we have the following.

PROPOSITION 2.4. *Let $(\mathfrak{z})$ be a Cousin II distribution in a cylindrical domain D in $\mathbb{C}^n$. Then $(\mathfrak{z})$ has a solution of the Cousin II problem in D if and only if $I((\mathfrak{z}), [\gamma_i], [\gamma_j])$ vanishes for each pair i, j ($1 \leq i, j \leq n; i \neq j$), where $[\gamma_k]$ is every homotopy class of a simple closed curve in D_k ($k = i, j$).*

3. Proof of Theorem 1

Let D_1, D_2 be domains bounded by a finite number (at least two) of simple closed curves in $\mathbb{C}_z, \mathbb{C}_w$, respectively. The boundary components of D_1 is $C_0, C_1, \dots, C_m$, where C_0 is the outer boundary; $\partial D_1 = C_0 - \sum_{i=1}^m C_i$. The boundary components of D_2 , as well as D_1 , are $\partial D_2 = \tilde{C}_0 - \sum_{j=1}^n \tilde{C}_j$. For all homotopy classes $[\gamma_i] \times [\tilde{\gamma}_j]$ ($i = 1, \dots, m; j = 1, \dots, n$) in $D := D_1 \times D_2$, where $\gamma_i \sim C_i$ (γ_i is homotopic to C_i) in D_1 and $\tilde{\gamma}_j \sim \tilde{C}_j$ in D_2 , we consider $\Gamma_{ij} := \gamma_i \times \tilde{\gamma}_j$ in D . $\square$

LEMMA 3.1. *Under the above notation, there exist analytic surfaces $\sigma_{ij}^+, \sigma_{ij}^-$ in D such that the intersection number $I(\sigma_{ij}^\pm, [\gamma_i], [\tilde{\gamma}_j]) = \pm 1$, respectively, and $I(\sigma_{ij}^\pm, [\gamma_h], [\tilde{\gamma}_k]) = 0$ for $(h, k) \neq (i, j)$.*

Proof. Fix i, j ($i = 1, \dots, m; j = 1, \dots, n$). By the well-known theorem for conformal mapping theory (see, for example, [9]), there exists a conformal mapping $S_i : D_1 \rightarrow \Delta_i^*$, where Δ_i^* is an annulus $\Delta_i := \{r_i < |x| < 1\}$ with $m - 1$ radial slits, such that $|S_i(C_0)| = 1$, $|S_i(C_i)| = r_i$, and the other contours are flattened into radial slits with two sides, as usual. Similarly, a conformal mapping $\tilde{S}_j : D_2 \rightarrow \tilde{\Delta}_j^*$ also exists, where $\tilde{\Delta}_j^*$ is $\tilde{\Delta}_j := \{\tilde{r}_j < |y| < 1\}$ with $n - 1$ radial slits, such that $|\tilde{S}_j(\tilde{C}_0)| = 1$ and $|\tilde{S}_j(\tilde{C}_j)| = \tilde{r}_j$. Put $\Delta_{ij} := \Delta_i \times \tilde{\Delta}_j$ in $\mathbb{C}_x \times \mathbb{C}_y$, and $\Delta_{ij}^* := \Delta_i^* \times \tilde{\Delta}_j^*$ which is analytically equivalent to D .

Let $l_i := \{re^{i\theta} \in \mathbb{C}_x \mid r_i \leq r \leq 1, \theta = 0\}$ be a radial segment from $|x| = r_i$ to $|x| = 1$ of Δ_i . We can assume that $m - 1$ and $n - 1$ radial slits do not lie on l_i and $\{\tilde{r}_j \leq y \leq 1\}$, respectively. Fix an argument α such that $0 < \alpha < K$, where $K = \min\{\arg\{\tilde{S}_j(\tilde{C}_k)\} \mid k \neq 0, j\} > 0$. Let $\tilde{l}_j := \{\rho e^{i\varphi} \in \mathbb{C}_y \mid \rho = \tilde{r}_j, 0 \leq \varphi \leq \alpha\}$ be a circular arc (clockwise) in the positive direction with respect to $\tilde{\Delta}_j$ and let $\tilde{\delta}_j := \{\rho e^{i\varphi} \in \mathbb{C}_y \mid \tilde{r}_j < \rho < 1, 0 < \varphi < \alpha\}$ be a domain of sector in $\tilde{\Delta}_j$. We then have a one-to-one conformal mapping $\Phi : \tilde{\delta}_j \rightarrow \mathcal{R}$, where $\mathcal{R}$ is a covering Riemann surface over Δ_i with relative boundaries: $\mathcal{R} = \{re^{i\theta} \in \mathbb{C}_x \mid r_i < r < 1, 0 < \theta < \beta\}$, where $\beta = (\log r_i)(\log \tilde{r}_j)/\alpha$ and $l_i = \Phi(\tilde{l}_j)$; in fact, $x = \Phi(y), y \in \tilde{\delta}_j$ is obtained by composing functions $v = \log y, u = \sqrt{-1}(-v \log r_i/\alpha + \beta), x = e^u$. Set $\Sigma_{ij}^+ := \{(x, y) \in \Delta_{ij} \mid x - \Phi(y) = 0\}$. Since $\partial \Sigma_{ij}^+ \subset \partial \Delta_{ij}$, Σ_{ij}^+ is an analytic surface in Δ_{ij} . Hence, we define an analytic surface pulled back $\Sigma_{ij}^+|_{\Delta_{ij}^*}$ to D as σ_{ij}^+ , that is,

$$\sigma_{ij}^+ := \{(z, w) \in D \mid S_i(z) - \Phi(\tilde{S}_j(w)) = 0\}.$$

We first show that $I(\sigma_{ij}^+, \Gamma_{ij}) = +1$: Pick up an oriented closed $\tilde{\gamma}_j$ which is sufficiently close to $\tilde{C}_j$, so that $\tilde{S}_j(\tilde{\gamma}_j)$ is close to $|y| = \tilde{r}_j$. $\Phi(\tilde{S}_j(\tilde{\gamma}_j) \cap \tilde{\delta}_j)$

is a transversal curve from $|x| = 1$ to $|x| = r_i$ in Δ_i . Then the oriented curve $S_i^{-1}(\Phi(\tilde{S}_j(w)))$, $w \in \tilde{\gamma}_j \cap \tilde{S}_j^{-1}(\tilde{\delta}_j)$, traverses γ_i from the right to the left. Therefore, $I(\sigma_{ij}^+, \Gamma_{ij}) = +1$. We next show that $I(\sigma_{ij}^+, \Gamma_{hk}) = 0$. In the case when $\gamma_h \sim C_h$ ($\neq C_i$), the intersection number of the closed curve $S_i(\gamma_h)$ and the transversal curve $\Phi(\tilde{S}_j(\tilde{\gamma}_j) \cap \tilde{\delta}_j)$ is canceled in total. Thus, $I(\sigma_{ij}^+, \Gamma_{hj}) = 0$. In the case when $\tilde{\gamma}_k \sim \tilde{C}_k$ ($\neq \tilde{C}_j$), since we can take $\tilde{\gamma}_k$ which is close to $\tilde{C}_k$ such that $\tilde{S}_j(\tilde{\gamma}_k) \subset \tilde{\Delta}_j \setminus \tilde{\delta}_j$, $\sigma_{ij}^+ \cap \Gamma_{ik} = \emptyset$. Hence, $I(\sigma_{ij}^+, \Gamma_{ik}) = 0$.

Similarly, let $m_j := \{\rho e^{i\varphi} \in \mathbb{C}_y \mid \rho = 1, 0 \leq \varphi \leq \alpha\}$ be a circular arc in the positive direction with respect to $\tilde{\Delta}_j$, and consider a one-to-one conformal mapping $\hat{\Phi} : \tilde{\delta}_j \rightarrow \hat{\mathcal{R}}$ with $l_i = \hat{\Phi}(m_j)$. Then $\sigma_{ij}^- := \{(z, w) \in D \mid S_i(z) - \hat{\Phi}(\tilde{S}_j(w)) = 0\}$ satisfies the lemma's conditions. $\square$

LEMMA 3.2. *Let $G \subset \mathbb{C}_z \times \mathbb{C}_w$ be a domain and let $G_0 \Subset G$. Let $(\mathfrak{z}) = \{(f_p, U_p)\}_{p \in G}$ be a Cousin II distribution in G . Then there exists a continuous function in some neighborhood of $[(\mathfrak{z})]G_0$ with prescribed zero $(\mathfrak{z})|G_0$, where $[(\mathfrak{z})]$ is the analytic surfaces defined by $(\mathfrak{z})$.*

Proof. If necessary, make a coordinates transformation in $\mathbb{C}^2$. Then we may assume that $[(\mathfrak{z})]$ satisfies the Weierstrass condition for w at any point on $[(\mathfrak{z})]$ (cf. [2]). Take G_1 such that $G_0 \Subset G_1 \Subset G$. We simply write anew $(\mathfrak{z}) := (\mathfrak{z})|G_1$. Then $(\mathfrak{z})$ consists of a finite number of pairs (S_v, e_v) ($v = 1, \dots, n$), where e_v is the vanishing order of the irreducible analytic surface S_v in G_1 , and $[(\mathfrak{z})] := \bigcup_{v=1}^n S_v \subset G_1$. Let $p_i = (z_i, w_i)$ ($i = 1, \dots, m; m < \infty$) be the set of singular points at which $[(\mathfrak{z})]$ is locally reducible. Put $[(\mathfrak{z})]^* = [(\mathfrak{z})] \setminus \{p_i\}_{i=1}^m$ and consider the (analytic) projection π from $[(\mathfrak{z})]^*$ to $\mathbb{C}_z$, so that $\pi([(\mathfrak{z})]^*)$ defines a finitely sheeted Riemann surface $\mathcal{R}$ which might have a finite number of branch points $a_1, \dots, a_{m'}$. Note that $[(\mathfrak{z})]^*$ and $\mathcal{R}$ are one to one through π , and $\mathcal{R}$ consists of disjoint connected non-compact Riemann surfaces $\mathcal{R}_v$ ($v = 1, \dots, n$). We have, for each $v = 1, \dots, n$,

$$[(\mathfrak{z})]^* : w = \xi_v(z), \quad z \in \mathcal{R}_v, \quad (3.1)$$

where ξ_v is a single-valued holomorphic function on $\mathcal{R}_v$. Let $p'_j \in [(\mathfrak{z})]^*$ be the point corresponding to the branch point $a_j \in \mathcal{R}$, i.e. $\pi(p'_j) = a_j$ ($j = 1, \dots, m'$). For the sake of convenience, we write $p'_j = p_{m+j}$ ($j = 1, \dots, m'$) and $m + m' = m''$.

We first give a solution of the given $(\mathfrak{z})$ near p_i ($i = 1, \dots, m''$) in G_1 . At each point $p_i = (z_i, w_i)$, take a small bi-disk

$$\Delta_i := \Delta_i \times \Gamma_i = \{|z - z_i| < 2r_i\} \times \{|w - w_i| < \rho_i\} \Subset G_1$$

such that those m'' bi-disks $\{\Lambda_i\}_{i=1,\dots,m''}$ are disjoint and $(\Delta_i \times \partial\Gamma_i) \cap [(\mathfrak{z})] = \emptyset$, $(\{z_i\} \times \Gamma_i) \cap [(\mathfrak{z})] = \{p_i\}$. Then Weierstrass' preparation theorem implies

$$f_{p_i} = \omega_i \prod_{h=1}^{l_i} [P_h]^{e_v},$$

where each P_h is an irreducible distinguished pseudopolynomial in w at (z_i, w_i) and ω_i is a non-vanishing holomorphic function for (z, w) in Λ_i . Here, ω_i, l_i, P_h, e_v depend on i . Moreover, $P_h = \prod_{\mu=1}^k (w - \xi_v^\mu)$, where each ξ_v^μ is some branch of the same $w = \xi_v$ in (3.1). Put $\mathfrak{P}_i = \prod_{h=1}^{l_i} [P_h]^{e_v}$ and consider the pairs $(\mathfrak{P}_i, \Lambda_i)$ ($i = 1, \dots, m''$). Then they become a solution of $(\mathfrak{z})$ in the neighborhood $\Lambda := \bigcup_{i=1}^{m''} \Lambda_i$ of $\bigcup_{i=1}^{m''} \{p_i\}$.

Putting for each $i = 1, \dots, m''$,

$$\delta_i := \{|z - z_i| < r_i\} \subset \Delta_i, \quad \lambda_i := \delta_i \times \Gamma_i \subset \Lambda_i, \quad \lambda := \bigcup_{i=1}^{m''} \lambda_i \subset \Lambda.$$

We next give a solution of $(\mathfrak{z})$ in a neighborhood of $[(\mathfrak{z})] \cap (G_0 \setminus \lambda)$ in G_1 . We simply set $[(\mathfrak{z})]^e := [(\mathfrak{z})] \cap (G_1 \setminus \lambda)$, which consists of n disjoint irreducible analytic surfaces $[(\mathfrak{z})]_v^e$ ($v = 1, \dots, n$). Since $[(\mathfrak{z})]^e \subset [(\mathfrak{z})]^*$, we put $\mathcal{R}^e = \pi([(\mathfrak{z})]^e)$, which consists of n disjoint connected non-compact Riemann surfaces $\mathcal{R}_v^e := \pi([(\mathfrak{z})]_v^e) \subset \mathcal{R}_v$ ($v = 1, \dots, n$) without branch points over $\mathbb{C}_z$. From (3.1), we also have

$$[(\mathfrak{z})]_v^e : \quad w = \xi_v(z), \quad z \in \mathcal{R}_v^e.$$

Take G' such that $G_0 \Subset G' \Subset G_1$ and $(G_1 \setminus G') \cap \Lambda = \emptyset$. Put $(\mathcal{R}_v^e)' = \pi([(\mathfrak{z})]_v^e \cap G')$ $\subset \mathcal{R}_v^e$. We take a sufficiently thin tubular neighborhood $V_v \subset G_1$ of each $[(\mathfrak{z})]_v^e \cap G_0$ ($v = 1, \dots, n$):

$$V_v = \bigcup_{z \in (\mathcal{R}_v^e)'} (z, \{|w - \xi_v(z)| < \varepsilon\})$$

such that those n tubular neighborhoods $\{V_v\}_{v=1,\dots,n}$ are disjoint in G_1 . Therefore, if we put

$$f_v = (w - \xi_v)^{e_v} \quad \text{in } V_v$$

and consider the pairs (f_v, V_v) ($v = 1, \dots, n$), then they become a solution of $(\mathfrak{z})$ in the neighborhood $\mathcal{V} := \bigcup_{v=1}^n V_v$ of $[(\mathfrak{z})] \cap (G_0 \setminus \lambda)$.

We consider, for each $i = 1, \dots, m''$, $A_i := \Delta_i \setminus \delta_i = \{z \in \Delta_i \mid r_i < |z - z_i| < 2r_i\}$ and $\mathcal{A}_i := A_i \times \Gamma_i$. If necessary, take a smaller $\varepsilon > 0$. Then we may assume that $V_v \cap (A_i \times \partial\Gamma_i) = \emptyset$, so that $V_v \cap (A_i \times \mathbb{C}_w) \subset \mathcal{A}_i$. Hence, for each V_v and Λ_i

such that $V_{v,i} := V_v \cap \Lambda_i = V_v \cap \mathcal{A}_i \neq \emptyset$ ($v = 1, \dots, n; i = 1, \dots, m''$), we see that $\mathfrak{P}_i/f_v$ is a non-zero holomorphic function on $V_{v,i}$. Let $h_{v,i}$ be the restriction of $\mathfrak{P}_i/f_v$ onto $[(\mathfrak{z})]_{v,i}^e := [(\mathfrak{z})]_v^e \cap V_{v,i}$. We define $h_v := h_{v,i}$ in $[(\mathfrak{z})]_{v,i}^e$ ($i = 1, \dots, m''$). Put $\mathcal{E}_v := \pi(\bigcup_{i=1}^{m''} [(\mathfrak{z})]_{v,i}^e) \subset \mathcal{R}_v^e$ and regard h_v as a non-zero holomorphic function on $\mathcal{E}_v$. We remark that, since $\mathcal{R}_v^e$ is non-compact Riemann surface and $\mathcal{R}_v^e \setminus \mathcal{E}_v$, which consists of one component, is not relatively compact in $\mathcal{R}_v^e$, it follows that the pair $(\mathcal{R}_v^e, \mathcal{E}_v)$ satisfies Runge's theorem. Moreover, there exists a *non-zero* holomorphic function $\tilde{h}_v$ on $\mathcal{R}_v^e$ (which is regarded as a function on $[(\mathfrak{z})]_v^e$) such that $\tilde{h}_v \approx h_v$ on $\mathcal{E}_v$ (approximates sufficiently close).

For $v = 1, \dots, n$, $i = 1, \dots, m''$, if we put $B_{v,i} := V_v \cup \lambda_i$, $B := \bigcup_{v=1}^n \bigcup_{i=1}^{m''} B_{v,i}$, then $B = (\bigcup_{v=1}^n V_v) \cup (\bigcup_{i=1}^{m''} \lambda_i) = \mathcal{V} \cup \lambda$ is a neighborhood of $[(\mathfrak{z})] \cap G_0$. For each $B_{v,i}$, if we define $F := F_{v,i}$, where

$$F_{v,i} := \begin{cases} \mathfrak{P}_i & \text{in } \lambda_i, \\ \mathfrak{P}_i \cdot \frac{2r_i - |z - z_i|}{r_i} + \sqrt{-1} \cdot f_v \cdot \tilde{h}_v \cdot \frac{|z - z_i| - r_i}{r_i} & \text{in } V_{v,i}, \\ \sqrt{-1} \cdot f_v \cdot \tilde{h}_v & \text{in } V_v \setminus V_{v,i}, \end{cases}$$

then F is a well-defined single-valued continuous function on B with the prescribed zero $(\mathfrak{z})|G_0$; in fact, it is clear that F is a continuous function in B and has the prescribed zero $(\mathfrak{z})|G_0$ in $B \setminus \bigcup V_{v,i}$. It remains to prove that $F = 0$ is $(\mathfrak{z})|G_0$ in each $V_{v,i}$. In $V_{v,i}$, we consider

$$F_{v,i} = f_v \cdot \left(\frac{\mathfrak{P}_i}{f_v} \cdot \frac{2r_i - |z - z_i|}{r_i} + \sqrt{-1} \cdot \tilde{h}_v \cdot \frac{|z - z_i| - r_i}{r_i} \right)$$

and denote the second part by φ , i.e. $F_{v,i} = f_v \cdot \varphi$ in $V_{v,i}$. Since $\mathfrak{P}_i/f_v|[(\mathfrak{z})]_{v,i}^e = h_{v,i}$, $h_v|[(\mathfrak{z})]_{v,i}^e = h_{v,i}$, and $\tilde{h}_v \approx h_v$ on $\mathcal{E}_v (\supset \pi([(\mathfrak{z})]_{v,i}^e))$, it follows that $\tilde{h}_v \approx \mathfrak{P}_i/f_v$ on $[(\mathfrak{z})]_{v,i}^e$. Thus, $\varphi|[(\mathfrak{z})]_{v,i}^e \neq 0$. If we take a smaller $\varepsilon > 0$, then $\varphi \neq 0$ in $V_{v,i}$. Since f_v is the solution of $(\mathfrak{z})$ in $V_v (\supset V_{v,i})$, $F_{v,i}$ is a solution of $(\mathfrak{z})$ in $V_{v,i}$.

Proof of case (i) in Theorem 1. Given a Cousin II distribution $(\mathfrak{z})$ in a cylindrical domain D in $\mathbb{C}^2$ and take a relatively compact cylindrical domain $D_0 = D_1^0 \times D_2^0$ in D . Without loss of generality, we may assume that each component D_k^0 ($k = 1, 2$) of D_0 is bounded by a finite number of piecewise smooth simple closed curves. Take a cylindrical domain D^* ($D_0 \Subset D^* \Subset D$) such that each component of D^* , whose connectivity is the same as that of D_k^0 ($k = 1, 2$), is bounded by a finite number of piecewise smooth simple closed curves. Let γ_i ($i = 1, \dots, m$)

and $\tilde{\gamma}_j$ ($j = 1, \dots, n$) be base cycles in D_1^0 and D_2^0 , respectively. For each $\Gamma_{ij} := \gamma_i \times \tilde{\gamma}_j$ in D_0 , put $e_{ij} := I((\mathfrak{z}), \Gamma_{ij})$ in D . Then we can make a Cousin II distribution $(\mathfrak{z}^*)$ in D^* such that $I((\mathfrak{z}^*), \Gamma_{ij}) = -e_{ij}$ in the following way; if $e_{ij} > 0$, then we consider a Cousin II distribution in D^* consisting of the analytic surface σ_{ij}^- in D^* which satisfies the conditions of Lemma 3.1 and the vanishing order $+e_{ij}$ of σ_{ij}^- . If $e_{ij} < 0$, then we consider a Cousin II distribution in D^* composed of the analytic surface σ_{ij}^+ in D^* which satisfies the conditions of Lemma 3.1 and the vanishing order $-e_{ij}$ of σ_{ij}^+ . If $e_{ij} = 0$, then we consider a constant function 1 in D^* as a Cousin II distribution. Then we define the Cousin II distribution $(\mathfrak{z}^*)$ in D^* which is made up of a finite number of pairs:

$$\{(\sigma_{ij}^\pm, \mp e_{ij})\}_{1 \leq i \leq m, 1 \leq j \leq n} \quad \text{in } D^*,$$

where $\mp e_{ij}$ is the vanishing order of $\sigma_{ij}^\pm$, respectively.

Consider a Cousin II distribution $((\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*))$ in D^* . By applying Lemma 3.2 to the domain $D_0 \subseteq D^*$, there exists a continuous function f in some neighborhood V of $[(\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*)]|D_0$ such that $f = 0$ in D_0 is $((\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*))|D_0$. Note that $I((\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*), \Gamma_{ij}) = -e_{ij}$ in D^* . Since $[(\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*)]$ is analytic surfaces in the cylindrical domain D^* , we see (from [6, No. 1, p. 59]) that there exists a holomorphic polyhedron G in D^* such that $[(\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*)]|D_0 \subseteq G \subseteq V$. Hence, by Oka's principle ([3, p. 94]), there exists a holomorphic function F in G with the given zero $((\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*))|G$.

We define a Cousin II distribution $(\tilde{\mathfrak{z}}) = \{(F - t, G), (1, D_0 \setminus G)\}$ in D_0 , where $0 < t \ll 1$. Then $[(\tilde{\mathfrak{z}})] \cap [((\mathfrak{z}) \cdot (\mathfrak{z}^*) \cdot (\mathfrak{z}^*))] = \emptyset$ by $0 < t$, and $I((\tilde{\mathfrak{z}}), \Gamma_{ij}) = -e_{ij}$ in D_0 by $t \ll 1$. Since $I((\tilde{\mathfrak{z}}) \cdot (\mathfrak{z})), \Gamma_{ij}) = 0$ in D_0 , it follows from the sufficient condition of Proposition 2.4 that $((\tilde{\mathfrak{z}}) \cdot (\mathfrak{z}))$ has a solution of the Cousin II problem in D_0 . Moreover, the result of the construction is $[(\tilde{\mathfrak{z}})] \cap [(\mathfrak{z})] = \emptyset$. Thus, we have proved case (i) in Theorem 1. $\square$

We shall construct a specific example for case (ii) in Theorem 1 by modifying Oka's example ([4, p. 10]).

Example. In $\mathbb{C}^3 = \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_z$, we consider the cylindrical domain $D := D_1 \times D_2 \times D_3$;

$$D_1 : r_1 < |x| < 1, \quad D_2 : r_2 < |y| < 1, \quad D_3 : |z| < 1,$$

where $r_1 + r_2 > 1$. Let σ be an analytic surface $x - y - 1 = 0$ in D . Since $\sigma \cap \{\Re x = 0\} = \emptyset$, σ is divided into two disjoint parts σ^+ and σ^- such

that $\sigma^+ = \sigma \cap \{(x, y, z) \in D \mid \Re x > 0\}$ and $\sigma^- = \sigma \cap \{(x, y, z) \in D \mid \Re x < 0\}$. Put $U_1 := D \setminus \sigma^-$ and $U_2 := D \setminus \sigma^+$. If we set

$$\begin{aligned} f_1(x, y, z) &= (x - y - 1) \cdot z \quad \text{in } U_1, \\ f_2(x, y, z) &= z \quad \text{in } U_2, \end{aligned}$$

then $(\mathfrak{z}) = \{(f_1, U_1), (f_2, U_2)\}$ defines a Cousin II distribution in D (because $D = U_1 \cup U_2$ and $f_1/f_2 = x - y - 1 \neq 0$ in $U_1 \cap U_2$). Thus, $[(\mathfrak{z})] = \sigma^+ \cup \{z = 0\}$ in D . Then there is no solution of the extra zero problem for $(\mathfrak{z})$ in D .

Proof. Assume that a solution F of the extra zero problem does exist for $(\mathfrak{z})$ in D . That is, the holomorphic function F in D has an extra zero $(\mathfrak{z}^*)$ such that $[(\mathfrak{z}^*)] \cap [(\mathfrak{z})] = \emptyset$. Take a closed cylindrical domain $D' := \{r'_1 \leq |x| \leq r''_1\} \times \{r'_2 \leq |y| \leq r''_2\} \times \{|z| \leq r_3\}$ in D , where $r_i < r'_i < r''_i < 1$ ($i = 1, 2$) and $r_3 < 1$. Since $[(\mathfrak{z}^*)] \cap \{z = 0\} = \emptyset$, we can find a small positive number $\varepsilon > 0$ such that $[(\mathfrak{z}^*)][D'] \cap \{|z| \leq \varepsilon\} = \emptyset$. Let γ_1 and γ_2 be circles (counterclockwise) $|x| = c_1$ and $|y| = c_2$ ($r'_i < c_i < r''_i$; $i = 1, 2$), respectively, and put $\Gamma_{12} := \gamma_1 \times \gamma_2$. Fix $z = \varepsilon$. The set $[(\mathfrak{z})(\varepsilon)] \cap \Gamma_{12}$ consists of one point in $\mathbb{C}_x \times \mathbb{C}_y$, and the real one-dimensional curve $\gamma_2 + 1$ in $\mathbb{C}_x$ traverses γ_1 at the point from the right to the left. Thus, $I((\mathfrak{z})(\varepsilon), \Gamma_{12}) = +1$. On the other hand, from the necessary condition in Proposition 2.4, $I((\mathfrak{z}) \cdot (\mathfrak{z}^*)(\varepsilon), \Gamma_{12})$ must be 0. Thus, $I((\mathfrak{z}^*)(\varepsilon), \Gamma_{12}) = -1$ and, hence, $[(\mathfrak{z}^*)][D'] \cap \{z = \varepsilon\} \neq \emptyset$. This is a contradiction. $\square$

In the case $n \geq 4$, we can also make a counterexample of the extra zero problem in cylindrical domains in $\mathbb{C}^n$ in a similar way.

4. Proof of Theorem 2

Proof. Let D and $(\mathfrak{z})$ satisfy the conditions in Theorem 2. Then $G = D \setminus [(\mathfrak{z})]$ is a domain of holomorphy (see [1, Satz 1]).

We shall prove that G is not rationally convex in $\mathbb{C}^n$. Assume that G is rationally convex. By the assumption and Proposition 2.4, for some homotopy classes $[\gamma_k]$ of simple closed curves in D_k ($k = i, j$), there exist some smooth representatives γ_k ($k = i, j$) such that $\Gamma_{ij} = \gamma_i \times \gamma_j$ is transversal to $(\mathfrak{z})(p)$ for some $p \in D^{(ij)}$ in general position with respect to $(\mathfrak{z})$, and $I((\mathfrak{z}), [\gamma_i], [\gamma_j]) \neq 0$. Let $Q^\mu = (z_i^\mu, z_j^\mu)$ ($\mu = 1, \dots, l$) be the intersection points of Γ_{ij} and $[(\mathfrak{z})(p)]$, and let e be the vanishing order of irreducible $[(\mathfrak{z})]$ in D . Then, from Definition 2.2, $I((\mathfrak{z}), [\gamma_i], [\gamma_j]) = I((\mathfrak{z})(p), \Gamma_{ij}) = \sum_{1 \leq \mu \leq l} s([(\mathfrak{z})(p)], \Gamma_{ij}^{Q^\mu}) \cdot e$. Thus, $m := \sum_{1 \leq \mu \leq l} s([(\mathfrak{z})(p)], \Gamma_{ij}^{Q^\mu}) \neq 0$.

We divide the proof into two steps.

(1) We may assume that $z_j^\mu \neq z_j^\nu$ ($\mu \neq \nu$; $\mu, \nu = 1, \dots, l$). Let $U_i^\mu \times U_j^\mu$ and $\varphi^\mu : U_j^\mu \rightarrow U_i^\mu$ satisfy the conditions in Definition 2.2. We may also assume that $U_j^\mu \cap U_j^\nu = \emptyset$ ($\mu \neq \nu$). Given $0 < \eta \ll 1$, we consider an η -tubular neighborhood $A_i \times A_j$ of Γ_{ij} , where $A_k = \{z_k \in D_k \mid \text{dist}(z_k, \gamma_k) \leq \eta\}$ ($k = i, j$). Choose $\eta > 0$ so small that $\mathbf{s} := [(\zeta)(p)] \cap (A_i \times A_j)$ is the analytic surfaces in $A_i \times A_j$ which consist of l disjoint irreducible components $\mathbf{s}^\mu$ ($\mu = 1, \dots, l$) such that $\mathbf{s}^\mu \subseteq \varphi^\mu(U_j^\mu) \times U_j^\mu \subset U_i^\mu \times U_j^\mu$ and the projection of $\mathbf{s}^\mu$ to $\mathbb{C}_{z_j}$ is diffeomorphic to a rectangle whose facing two sides are contained in $(\partial A_j) \cap U_j^\mu$ and other two sides are mapped by φ^μ onto two curves in $(\partial A_i) \cap U_i^\mu$. We simply put $A_i/2 = \{z_i \in D_i \mid \text{dist}(z_i, \gamma_i) \leq \eta/2\}$ and $\mathbf{s}/2 := [(\zeta)(p)] \cap (A_i/2 \times A_j)$. Let $\gamma_j^\mu := \{z_j \in \gamma_j \cap U_j^\mu \mid (\varphi^\mu(z_j), z_j) \in \mathbf{s}\}$, $\gamma_j^\mu/2 := \{z_j \in \gamma_j \cap U_j^\mu \mid (\varphi^\mu(z_j), z_j) \in \mathbf{s}/2\}$ and $C_j^\mu := \{(\varphi^\mu(z_j), z_j) \in \mathbf{s} \mid z_j \in \gamma_j^\mu\}$ for $\mu = 1, \dots, l$. By the assumption, (ζ) necessarily satisfies Weierstrass' condition for z_i at any point of $[(\zeta)]$. We thus let $\pi : [(\zeta)] \rightarrow \mathbb{C}^{(i)} = \mathbb{C}_{z_1} \times \dots \times \hat{\mathbb{C}}_{z_i} \times \dots \times \mathbb{C}_{z_n}$ be the analytic projection, whose image $\mathcal{R} := \pi([(\zeta)])$ is a Riemann domain over $D^{(i)}$ with branch surfaces. We connect l points (Q^μ, p) ($\mu = 1, \dots, l$) with a simple smooth curve $\mathcal{L}$ in $[(\zeta)]$, and put $L := \pi(\mathcal{L})$. If necessary, move γ_i, γ_j and $\mathcal{L}$ slightly, we may assume that, for each $\mu = 1, \dots, l$:

- (i) $\mathcal{L} \cap (C_j^\mu \times p) = \{(Q^\mu, p)\}$;
- (ii) any $C_j^\mu \times p$ and $\mathcal{L}$ do not pass the singular sets of $[(\zeta)]$;
- (iii) any $\gamma_j^\mu \times p$ and L do not pass the branch surfaces of $\mathcal{R}$.

Put $K := L \cup (\bigcup_{\mu=1}^l \gamma_j^\mu \times p) \subset \mathcal{R}$. From (i)–(iii), we uniquely have a continuous function $\psi : K \rightarrow \mathbb{C}_{z_i}$ such that $(\psi(z^{(i)}), z^{(i)}) \in [(\zeta)]$ for $z^{(i)} \in K$ and such that $\psi(\zeta_1) \neq \psi(\zeta_2)$ if $\zeta_1 \neq \zeta_2$ as points in $\mathcal{R}$.

Fix two cylindrical domains $D' \subseteq D'' \subseteq D$ with $(A_i \times A_j \times p) \cup \mathcal{L} \subseteq D'$. Put $D' = \prod_{k=1}^n D'_k$ and $D'' = \prod_{k=1}^n D''_k$. By the assumption, we obtain a meromorphic function g in D'' whose pole is only on $[(\zeta)]$. Consider a continuous function $h = 1/|g|$ in D'' . Then h is class C^ω on $D'' \setminus [(\zeta)]$ such that $h \geq 0$ on D'' and $h = 0$ if and only if $z \in [(\zeta)] \cap D''$. Given $\varepsilon > 0$, we define the following tubular neighborhood of $[(\zeta)] \cap D''$ in D'' :

$$V_\varepsilon := \{z \in D'' \mid h(z) < \varepsilon\} \quad (\supset [(\zeta)] \cap D'').$$

For each $z^{(i)} \in K$, we define $V_\varepsilon(z^{(i)}) := \{z_i \in D''_i \mid (z_i, z^{(i)}) \in V_\varepsilon\}$. We denote by $v_\varepsilon(z^{(i)})$ the connected component of $V_\varepsilon(z^{(i)})$ containing $\psi(z^{(i)})$, and put $r_\varepsilon(z^{(i)}) := \max\{|z_i - \psi(z^{(i)})| \mid z_i \in \partial v_\varepsilon(z^{(i)})\} > 0$ so that $\lim_{\varepsilon \rightarrow 0} r_\varepsilon(z^{(i)}) = 0$. We note that,

for $z^{(i)} = z_j \times p \in K$ where $z_j \in \gamma_j^\mu/2$, the set $\{z_i \in D_i'' \mid (z_i, z^{(i)}) \in [(\mathfrak{z})]\}$ consists of finite points: $\psi(z^{(i)}) (= \varphi^\mu(z_j))$ and z_{iv} ($v = 1, \dots, N$) such that $z_{iv} \notin \gamma_i$.

Then we find a sufficiently small $\varepsilon > 0$ which satisfies the following conditions:

- (e1) $r_\varepsilon(z^{(i)}) < \eta/4$ for each $z^{(i)} \in K$;
- (e2) $\min\{h(z_i, z_j, p) \mid (z_i, z_j) \in \gamma_i \times (\gamma_j \setminus \bigcup_{\mu=1}^l \gamma_j^\mu/2)\} > \varepsilon$;
- (e3) each $v_\varepsilon(z^{(i)})$ ($z^{(i)} \in K$) is bounded by a simple smooth closed curve $\partial v_\varepsilon(z^{(i)})$ in D_i' and $\partial v_\varepsilon(z^{(i)})$ varies continuously in D_i' with parameter $z^{(i)} \in K$;
- (e4) $(V_\varepsilon(z^{(i)}) \setminus v_\varepsilon(z^{(i)})) \cap \gamma_i = \emptyset$ for $z^{(i)} = z_j \times p$ where $z_j \in \gamma_j^\mu/2$ ($\mu = 1, \dots, l$);
- (e5) we fix one point $(\psi(z_0^{(i)}), z_0^{(i)})$, where $z_0^{(i)} \in K$, and then

$$\min\{|g(z_i, z_0^{(i)})| \mid z_i \in \partial v_\varepsilon(z_0^{(i)})\} > \max\{|g(z_i, z_0^{(i)})| \mid z_i \in \partial D_i'\}.$$

Since $D'' \setminus [(\mathfrak{z})]$ is rationally convex in $\mathbb{C}^n$, it follows from $D' \setminus V_\varepsilon \Subset D'' \setminus [(\mathfrak{z})]$ that, given $0 < \delta \ll 1$, there exists a rational function $\mathfrak{P}/\Omega$ in $\mathbb{C}^n$ (where $\mathfrak{P}, \Omega$ are relatively prime) such that $|g - \mathfrak{P}/\Omega| < \delta$ on $D' \setminus V_\varepsilon$. By the usual argument about the points of indeterminacy,

$$\Omega(z) \text{ does not vanish in } D' \setminus V_\varepsilon. \quad (4.1)$$

If we take $\delta : 0 < \delta \ll 1$ so small, then (e5) implies $\min\{|\mathfrak{P}/\Omega(z_i, z_0^{(i)})| \mid z_i \in \partial v_\varepsilon(z_0^{(i)})\} > \max\{|\mathfrak{P}/\Omega(z_i, z_0^{(i)})| \mid z_i \in \partial D_i'\}$. It follows from the maximum modulus principle that $\Omega(z_i, z_0^{(i)})$ (as a function of $z_i \in \mathbb{C}_{z_i}$) has at least one zero in $v_\varepsilon(z_0^{(i)})$, say λ , the number of zeros with multiplicity of $\Omega(z_i, z_0^{(i)})$. For any fixed $z^{(i)} \in K$, we have $\Omega(z_i, z^{(i)}) \neq 0$ on $\partial v_\varepsilon(z^{(i)})$ by (4.1). Under this condition and (e3), Weierstrass preparation theorem implies that the number of zeros of $\Omega(z_i, z^{(i)})$ in $v_\varepsilon(z^{(i)})$ is independent as long as $z^{(i)}$ runs along K , i.e. is always equal to λ .

(2) We simply put $\Omega_0(z_i, z_j) := \Omega(z_i, z_j, p)$ and $(Z_0) := \{(z_i, z_j) \in A_i \times A_j \mid \Omega_0(z_i, z_j) = 0\}$. Then we prove that $I((Z_0), \gamma_i \times \gamma_j^\mu/2) = s([(\mathfrak{z})](p), \Gamma_{ij}^{Q^\mu}) \cdot \lambda$ for $\mu = 1, \dots, l$. In fact, for every $z_j \in \gamma_j^\mu$ ($\mu = 1, \dots, l$), we put $v_\varepsilon(z_j) := v_\varepsilon(z_j, p)$. Then (4.1) and (e2) imply that $\Omega_0(z_i, z_j) \neq 0$ on $\bigcup_{z_j \in \gamma_j^\mu/2} (\gamma_i \setminus v_\varepsilon(z_j), z_j)$. We see from (e1) that $\gamma_i \cap v_\varepsilon(z_j) = \emptyset$ for every $z_j \in \gamma_j^\mu \setminus \gamma_j^\mu/2$. Hence, $[(Z_0)] \cap (\gamma_i \times (\gamma_j^\mu \setminus (\gamma_j^\mu/2))) = \emptyset$.

Now consider the case when $s([(\mathfrak{z})](p), \Gamma_{ij}^{Q^\mu}) = +1$: We have $\psi(z_j, p) = \varphi^\mu(z_j)$ for $z_j \in \gamma_j^\mu$. Let the subarc $\gamma_j^\mu/2$ of γ_j (with same orientation as γ_j) have initial point a^μ and terminal point b^μ . Note that $v_\varepsilon(a^\mu) (\subset A_i)$ contains λ zeros of $\Omega_0(z_i, a^\mu)$, all of these λ zeros lie in the right-hand side of γ_i from (e1). On the other hand, $v_\varepsilon(b^\mu) (\subset A_i)$ contains λ zeros of $\Omega_0(z_i, b^\mu)$, all of which lie in the left-hand

side of γ_i . If z_j moves along the oriented arc $\gamma_j^\mu/2$, then the center $\varphi^\mu(z_j)$ of $v_\varepsilon(z_j)$ crosses γ_i from the right to the left passing through $z_i^\mu (= \varphi^\mu(z_j^\mu))$. This and (e4) imply that $I((\mathcal{Z}_0), \gamma_i \times \gamma_j^\mu/2) = +\lambda$. In the case when $s([\mathfrak{z}](p)], \Gamma_{ij}^{Q^\mu}) = -1$, we similarly have $I((\mathcal{Z}_0), \gamma_i \times \gamma_j^\mu/2) = -\lambda$.

Since $[(\mathcal{Z}_0)] \cap (\gamma_i \times (\gamma_j \setminus \bigcup_{\mu=1}^l \gamma_j^\mu/2)) = \emptyset$, we finally get

$$I((\mathcal{Z}_0), \gamma_i \times \gamma_j) = \sum_{\mu=1}^l I\left((\mathcal{Z}_0), \gamma_i \times \frac{\gamma_j^\mu}{2}\right) = \lambda \sum_{\mu=1}^l s([\mathfrak{z}](p)], \Gamma_{ij}^{Q^\mu}) = \lambda m.$$

This contradicts Proposition 2.4, so that G is not rationally convex. $\square$

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